Nonlinear dynamics of smectic liquid crystals with orientational ordering in the layer

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The dynamics of hexatic smectic B liquid crystals and smectic C liquid crystals with a slight anisotropy in the layer is studied. The mode associated with the orientational degrees of freedom of the molecules is taken into account. A complete system of nonlinear, nondissipative, hydrodynamic equations is found. This system of equations is used to construct an effective action which makes it possible to determine the thermal fluctuation effects. The most important effects are those associated with the displacement of smectic layers. The terms in the effective action which are linked with the displacement of the layers are calculated. These terms are used to renormalize the orientational-mode spectrum. The renormalization-group equations for all these terms are derived and solved. The orientational-mode spectrum is calculated with allowance for the fluctuation contribution.

1. INTRODUCTION

The presence of a layered structure is a feature common to all smectic liquid crystals.[1] Although the smectic liquid crystals have a rigid-body order along the direction perpendicular to the layers, the layers can slip past each other as fluids. The fluctuations in a system of this sort are known[2] to destroy the long-range order, which is absent only in an infinite system. In the case of a finite system or, equivalently, in the case of finite wave vectors, the properties of a smectic liquid crystal can be correctly described as a rigid-body-ordered system.

In the study of nonlinear properties of smectics, the function $W(t, \mathbf{r})$ can be used to describe the displacement of the layers. In this function the equation $W=$ const specifies the position of a layer, and the vector ∇W determines the direction of the normal to the layer.

We will consider primarily the so-called hexatic smectic B liquid crystals.[1] In these systems the layer has a long-range orientational order for the coupling with a sixfold symmetry (from which the name is derived). The hexatic order can be specified by a single parameter—an angle that describes the orientational symmetry in the smectic layer. This angle is measured from an arbitrary axis in the plane of the smectic layer which is specified by a unit vector $\mathbf{n}$.

A similar situation also applies to smectic C liquid crystals. In smectic C liquid crystals, the molecules are tilted with respect to the normal to the layer. The layer thus exhibits an anisotropy which we can describe in terms of the director $\mathbf{n}$ within the layer,

$$\mathbf{n}^2=1, \quad \mathbf{n}\nabla W=0. \tag{1}$$

Because of the constraints (1), the vector $\mathbf{n}$ contains only one independent parameter which can be expressed in variational form as

$$\delta\varphi=\frac{2}{|\nabla W|}[\mathbf{n}\nabla W]\delta\mathbf{n}. \tag{2}$$

The variation of the director $\mathbf{n}$ can now be described by the relation

$$\delta\mathbf{n}=-\frac{\nabla W}{|\nabla W|^2}(\mathbf{n}\nabla\delta W)+\frac{[\mathbf{n}\nabla W]}{2|\nabla W|}\delta\varphi. \tag{3}$$

Using definition (2) and constraints (1), we find the following expression for the commutator of the variations:

$$(\delta_2\delta_1-\delta_1\delta_2)\varphi=-\frac{2}{|\nabla W|^3}\nabla_i\delta_1 W\nabla_k\delta_2 W e_{ikn}\nabla_n W. \tag{4}$$

As we have mentioned above, in hexatic smectic B liquid crystals, the rotational symmetry breaks down in the layer, so that there are restrictions imposed on the relative rotation of molecules in the layer. Because of the presence of a sixfold symmetry axis, we cannot introduce an anisotropy vector in the layer, but we can determine the arbitrary direction with respect to which the rotation of the molecules is specified. This rotation can be described by a nonholonomic angle φ, whose transformation properties are the same as those of the angle for smectic C liquid crystals which was introduced above. Specifically, relation (4) is correct.

The part of the energy density of a hexatic smectic B which is attributed to symmetry breaking can be written to lowest order as

$$E_s={}^1/_8\beta[(\nabla W)^2-l^{-2}]^2+{}^1/_2\varkappa(\nabla^2 W)^2+{}^1/_2\alpha(\nabla\varphi)^2. \tag{5}$$

Here β, $\varkappa$, and α are the elastic moduli, and l is the spacing between the layers. The presence of a sixfold symmetry axis rules out the invariants which are anisotropic in the plane of the layer and which have the order of interest to us. The situation is slightly different in a smectic C. In principle, these smectics have anisotropic terms of the type $(\mathbf{n}\nabla\nabla W)^2$, which are attributed to the presence of two distinct directions ∇W and $\mathbf{n}$. However, anisotropic terms of this sort, like the terms of the type $(\nabla\varphi\nabla W)^2$, are small, because the violation of the displacement invariance and of the rotational invariance in the layer is only slight. The first small term is characterized by the parameter $(c_2^2/c_1^2)\sim 10^{-3}$, where c_2 is the velocity of propagation of the second sound which is associated with the oscillation of the layers, and c_1 is the velocity of propagation of ordinary (first) sound. The second

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term is characterized by a small angle of inclination ($\theta \sim 10^{-1}-10^{-2}$) of the molecules with respect to the normal to the layer. These anisotropic effects are disregarded everywhere below. Within this error margin, our results are equally valid for a smectic C and a hexatic smectic B. For definiteness, we will discuss primarily the hexatic smectic B phase, which we will simply call smectic B.

As we can see from (5), at equilibrium we have $W_0 = z/l$. This expression corresponds to a system of equidistant layers which are perpendicular to the z axis. The fluctuations of W near this equilibrium value govern[3,4] the logarithmic renormalization of the moduli β and $\varkappa$. Here

$$\beta \propto L^{-4/5} \quad \varkappa \propto L^{2/5}. \tag{6}$$

Here $L = -\ln[\min(k_z l, k^2 l^2)]$ is the major logarithm. The layers of thickness l appear in the argument of this logarithm because smectics can be analyzed macroscopically to within dimensions of $\sim l$. It is quite obvious that the fluctuation corrections to the modulus α are small. In the dynamic process, the fluctuations of W give rise to a strong ($\propto \omega^{-1}$) divergence of the kinetic coefficients which determine the attenuation of both the first and second sound.[5-8] In this study we consider the spectrum of a mode which results from rotational symmetry breaking in the layer and which determines, in particular, the optical properties of the smectic C and B phases.

2. NONLINEAR HYDRODYNAMIC EQUATIONS FOR A SMECTIC *B* LIQUID CRYSTAL

To derive nondissipative hydrodynamic equations for smectic B liquid crystals, we will use the Poisson-bracket method (see the review by Dzyaloshinskiĭ and Volovik[9]) with one important modification. The standard Poisson brackets satisfy the Jacobi identities. The Poisson brackets for the variables which describe the hydrodynamics of smectics B and which satisfy the Jacobi identities cannot be constructed, however, without the use of a φ-conjugate variable such as the intrinsic angular momentum of the nematic liquid crystals. This rapidly relaxing variable, however, should be excluded from the hydrodynamic equations. This exclusion, a very cumbersome operation, is possible only after the introduction of the kinetic terms.[9] It would be much simpler to ignore this variable altogether and to lift the requirement that the Jacobi identity for the Poisson brackets that contain the order parameter be satisfied. As a result, the Poisson brackets would generally acquire reactive coefficients which cannot be determined from general considerations.[10,11]

The complete set of hydrodynamic variables of a smectic B consists of the momentum density $\mathbf{j}$, mass density ρ, specific entropy σ, the smectic variable W, and the orientational degree of freedom φ. The system of Poisson brackets for the first four variables has the standard form

$$\{j_k(\mathbf{r}_1), j_i(\mathbf{r}_2)\} = j_i(\mathbf{r}_1)\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2) + \nabla_i\delta(\mathbf{r}_1-\mathbf{r}_2) j_k(\mathbf{r}_2), \tag{7}$$

$$\{\mathbf{j}(\mathbf{r}_1), \rho(\mathbf{r}_2)\} = \rho(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{8}$$

$$\{\mathbf{j}(\mathbf{r}_1), \sigma(\mathbf{r}_2)\} = -\nabla\sigma\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{9}$$

$$\{\mathbf{j}(\mathbf{r}_1), W(\mathbf{r}_2)\} = -\nabla W\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{10}$$

The construction of a common $\mathbf{j}$ and φ bracket is nontrivial. First, the constraints (1)-(3) should be taken into account and, secondly, the stress-tensor symmetry must be arranged. These two requirements uniquely determine the Poisson bracket

$$\{j_i(\mathbf{r}_1), \varphi(\mathbf{r}_2)\} = -\nabla_i\varphi\delta(\mathbf{r}_1-\mathbf{r}_2) - e_{ikn}\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2)\nabla_n W(\mathbf{r}_2)/|\nabla W|. \tag{11}$$

In the case of a smectic C, Poisson bracket (11) contains an auxiliary term

$$\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2)(n_i l_k + n_k l_i)\mu(\mathbf{r}_2), \tag{12}$$

where the unit vector $\mathbf{l} = [\nabla W\mathbf{n}]/|\nabla W|$, along with $\mathbf{n}$ and ∇W, forms the triplet at the right. In a smectic C, however, the corresponding reactive coefficient μ contains a small term, $\sim\theta$, and within our error margin can be discarded. The Poisson bracket $\{\mathbf{j}, \mathbf{n}\}$ can then be written

$$\{j_i(\mathbf{r}_1), n_k(\mathbf{r}_2)\} = -\nabla_i n_k\delta(\mathbf{r}_1-\mathbf{r}_2) - \nabla_n\delta(\mathbf{r}_1-\mathbf{r}_2) n_n\nabla_i W\nabla_k W/|\nabla W|^2 - e_{inm}\nabla_n\delta(\mathbf{r}_1-\mathbf{r}_2)\nabla_m W[\mathbf{n}\nabla W]_k/2|\nabla W|^2. \tag{13}$$

It is easy to show that constraints (1) are the first integrals of a system of equations which is formulated by making use of Poisson brackets (7)-(13), so that constraints (1) are compatible with (7)-(13).

The energy density $\mathbf{E}$ of a smectic B is a function of the variables

$$E = E(\mathbf{j}, \rho, \sigma, \nabla W, \nabla\nabla W, \nabla\varphi). \tag{14}$$

As was noted above, the anisotropic terms appear only in the sixth- and higher-order gradient terms because of the symmetry of the layer of a smectic B.

The pressure can be expressed by

$$P = \mathbf{v}\mathbf{j} + \rho\frac{\partial E}{\partial\rho} - E. \tag{15}$$

Here $\mathbf{v} = \partial E/\partial\mathbf{j}$ is the velocity. The hydrodynamic equations derived from (7)-(11) are

$$\frac{\partial\rho}{\partial t} = -\nabla(\rho\mathbf{v}), \quad \frac{\partial\sigma}{\partial t} = -\mathbf{v}\nabla\sigma, \tag{16}$$

$$\frac{\partial W}{\partial t} = -\mathbf{v}\nabla W, \tag{17}$$

$$\frac{\partial\varphi}{\partial t} = -v_i\nabla_i\varphi - \frac{1}{|\nabla W|}e_{jni}\nabla_j W\nabla_n v_i, \tag{18}$$

$$\frac{\partial j_i}{\partial t} = -\nabla_k(T_{ik} - \nabla_j S_{jik}). \tag{19}$$

Equations (19) consists of the following parts:

$$T_{ik} = P\delta_{ik} + \frac{\partial E}{\partial\nabla_k W}\nabla_i W + 2\frac{\partial E}{\partial\nabla_k\nabla_j W}\nabla_j\nabla_i W + \frac{\partial E}{\partial\nabla_k\varphi}\nabla_i\varphi + \frac{\partial E}{\partial\nabla_j\varphi}\nabla_j\varphi\frac{\nabla_i W\nabla_k W}{|\nabla W|^2} - e_{ikj}\frac{\partial E}{\partial\nabla_m W}\frac{\nabla_m\nabla_j W}{|\nabla W|} + e_{ikm}\frac{\partial E}{\partial\nabla_j\varphi}\frac{\nabla_m W\nabla_n W}{|\nabla W|^3}\nabla_m\nabla_j W + 2e_{mkn}\frac{\partial E}{\partial\nabla_j\varphi}\frac{\nabla_i W\nabla_m W}{|\nabla W|^3}\nabla_n\nabla_j W, \tag{20}$$

$$S_{jik} = \frac{\partial E}{\partial\nabla_k\nabla_j W}\nabla_i W + e_{imk}\frac{\partial E}{\partial\nabla_j\varphi}\frac{\nabla_m W}{|\nabla W|}. \tag{21}$$

The right sides of the dissipationless Eqs. (16)-(19), which were derived by means of Poisson brackets, must be supplemented with the kinetic terms which can be written out in a standard fashion.

It is easy to see that expression $\delta\theta_m e_{mik} T_{ik}$ determines the infinitesimal transformation of energy density (14) due to a rotation through an angle $\delta\theta_m$. Consequently, the antisymmetric part of T_{ik} vanishes by virtue of the rotational invariance of E. The second term on the right side of (19) can always be expressed as the divergence of a symmetric tensor:

$$\nabla_k\nabla_j S_{jik}=\nabla_k\nabla_j(S_{jik}+e_{jkm}e_{mrs}S_{isr}+{}^1/_2 e_{ikj}e_{mrs}S_{msr}).$$

Accordingly, if the energy density (14) depends arbitrarily on its arguments, the right side of (19) reduces to the divergence of a symmetric stress tensor. Since the use of (12) does not break the symmetry of the stress tensor, all arguments advanced in this section also apply, within $\theta^2\sim 10^{-2}-10^{-3}$, to a smectic C.

Let us consider those parts of the stress tensor which are related to the order parameter and which arise from energy density (5), when the second term is included in the right side of (19):

$$T_{ik}^{(W)}={}^1/_2\beta[(\nabla W)^2-l^{-2}]\nabla_i W\nabla_k W-\varkappa\nabla_k\nabla^2 W\nabla_i W$$
$$-\varkappa\nabla_i\nabla^2 W\nabla_k W+(P^{(W)}+\nabla_n(\varkappa\nabla^2 W\nabla_n W))\delta_{ik} \quad (22)$$

[$P^{(W)}$ is the W-dependent part of the pressure (15), whose explicit form is unimportant in our case]

$$T_{ik}^{(\varphi)}=\frac{\alpha}{2}\left(\frac{\partial\ln\alpha}{\partial\ln\rho}-1\right)(\nabla\varphi)^2\delta_{ik}+\alpha\nabla_i\varphi\nabla_k\varphi$$
$$+\alpha(\nabla\varphi)^2\frac{\nabla_i W\nabla_k W}{|\nabla W|^2}+\alpha\nabla_j\left(e_{jmk}\nabla_i\varphi\frac{\nabla_m W}{|\nabla W|}+e_{jmi}\nabla_k\varphi\frac{\nabla_m W}{|\nabla W|}\right)$$
$$+\frac{\alpha}{|\nabla W|^3}(e_{mkn}\nabla_i W\nabla_m W+e_{min}\nabla_k W\nabla_m W)\nabla_j\varphi\nabla_n\nabla_j W. \quad (23)$$

Expressions (22) and (23) are now explicitly symmetric.

3. EFFECTIVE ACTION

Having now the nonlinear hydrodynamic equations for a smectic B at our disposal, we can construct, following the procedure outlined in Ref. 12, the effective action I, which allows us to study the dynamic fluctuation effects in the system under consideration. Since the variables ρ, σ, and $\mathbf{j}$ fluctuate only slightly in a smectic B, we can drop them by following the procedure described in Ref. 7. As a result, we find

$$I=\int dt d^3r\mathscr{L}(p_1,p_2,W,\varphi). \quad (24)$$

The reactive part of the Lagrangian is

$$\mathscr{L}_r=p_1\frac{\partial W}{\partial t}+p_2\frac{\partial\varphi}{\partial t}$$
$$-\frac{1}{\rho}\left[p_1\nabla_n W+p_2\left(\nabla_n\varphi+e_{jln}\frac{\nabla_j W}{|\nabla W|}\nabla_l\right)\right]$$
$$\times\left(\frac{\partial}{\partial t}\right)^{-1}\left(\delta_{in}-\frac{\nabla_i\nabla_n}{\nabla^2}\right)\nabla_k(T_{ik}^{(W)}+T_{ik}^{(\varphi)}). \quad (25)$$

Here p_1 and p_2 are the auxiliary Bose fields which are coupled to W and φ, respectively. The dependence of the effective action I on the Fermi fields used in Refs. 7 and 12 can be dropped in this case, since the determinant resulting from integration over these fields is equal to unity in the perturbation theory because of the analytic properties of the Green's functions and because of the regularization.

Expression (25) for the Lagrangian density, which is described by the reactive part of the hydrodynamic equations, must be supplemented with a term which is related to the kinetic terms in the hydrodynamic equations. We can then calculate the correlation functions of the fields W,φ,p_1, and p_2 taking an average over the relevant products with a weight $\exp(iI)$. The nonvanishing pairing expectation values (of the Fourier components) are

$$\langle W(\omega,\mathbf{k})p_1(\omega_1,\mathbf{k}_1)\rangle=-(2\pi)^4\delta(\omega+\omega_1)\delta(\mathbf{k}+\mathbf{k}_1)G_1(\omega,\mathbf{k}),$$
$$\langle W(\omega,\mathbf{k})W(\omega_1,\mathbf{k}_1)\rangle=-(2\pi)^4\delta(\omega+\omega_1)\delta(\mathbf{k}+\mathbf{k}_1)D_1(\omega,\mathbf{k}),$$
$$\langle\varphi(\omega,\mathbf{k})p_2(\omega_1,\mathbf{k}_1)\rangle=-(2\pi)^4\delta(\omega+\omega_1)\delta(\mathbf{k}+\mathbf{k}_1)G_2(\omega,\mathbf{k}), \quad (26)$$
$$\langle\varphi(\omega,\mathbf{k})\varphi(\omega_1,\mathbf{k}_1)\rangle=-(2\pi)^4\delta(\omega+\omega_1)\delta(\mathbf{k}+\mathbf{k}_1)D_2(\omega,\mathbf{k}).$$

Since the variable φ is defined only in terms of its variation, we should define the Fourier component $\varphi(\omega,\mathbf{k})$ more accurately. This component can for example, be defined by

$$\varphi(\omega,\mathbf{k})=\int dt d^3r\frac{i}{\omega}\exp(i\omega t-i\mathbf{kr})\frac{\partial\varphi}{\partial t}. \quad (27)$$

We should emphasize that the expectation values of $\langle pp\rangle$ are zero. The functions D_1 and D_2 determine the pairing correlation functions of the fluctuating quantities and the functions G_1 and G_2 determine the susceptibility of the system to the external forces which should be added to the right sides of Eqs. (17) and (18). Accordingly, $G_1(\omega)$ and $G_2(\omega)$ are analytic functions in the upper half-plane.

Substituting expressions (22) and (23) into Lagrangian density (25), we find the reactive part of the effective action for a smectic B in first approximation. Retaining only those terms that describe the strongest fluctuation effects and simplifying the expression, we find the terms

$$\mathscr{L}_1=p_1\frac{\partial W}{\partial t}-p_1\nabla_i W\left(\delta_{ik}-\frac{\nabla_i\nabla_k}{\nabla^2}\right)$$
$$\times A_k+bp_1\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_\perp^2\nabla^2 W, \quad (28)$$

$$\mathscr{L}_2=p_2\frac{\partial\varphi}{\partial t}+hp_2\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_\perp^2\nabla^2\varphi, \quad (29)$$

$$\mathscr{L}_3=-hp_1\nabla_i W\left(\frac{\partial}{\partial t}\right)^{-1}e_{ikz}\nabla_k\nabla^2\varphi, \quad (30)$$

$$\mathscr{L}_4=-p_2 e_{zji}\nabla_j f_4 A_i. \quad (31)$$

Here and elsewhere in the present work, the subscripts z and $\perp$ refer to the components directed respectively along the z axis and at right angles to it. In (28)-(31) we have $f_4=1$,

$$a=\beta/\rho l^4,\quad b=\varkappa/\rho l^2,\quad h=\alpha/\rho, \quad (32)$$

$$A_i=\frac{1}{2}al^2\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}\{[l^2(\nabla W)^2-1]\nabla_i W\}. \quad (33)$$

Expressions (28)-(31) must be supplemented with that part of the Lagrangian density which is governed by the dissipative terms,

$$\mathscr{L}_{d1}=-2gp_1\nabla_\perp^2 W+i\frac{2T}{\rho}gp_1\nabla_\perp^4\nabla^{-2}\left(\frac{\partial}{\partial t}\right)^{-2}p_1, \quad (34)$$

$$\mathscr{L}_{d2}=-2\Gamma hp_2\nabla^2\varphi+i\frac{2T}{\rho}\Gamma p_2{}^2. \tag{35}$$

Here Γ is the kinetic coefficient.

Expressions (28) and (34) determine the effective action for the smectic variable W, whose renormalization was discussed in detail in Ref. 7. For the coefficients a and b we found

$$a\propto L^{-4/5}\text{ and } b\propto L^{2/5}, \tag{36}$$

consistent with the logarithmic behavior of elastic moduli (6). The coefficient a in expression (31) is renormalized exactly the same way as the coefficient in (28). This coefficient is renormalized by analogy with the renormalization of all coefficients of the factors $(\nabla W)^2 - l^{-2}$ (Ref. 7). We can express the coefficient g in (34) as

$$g^2=b\ln(b/b_0). \tag{37}$$

We should emphasize that the dissipative term (34) is of a purely fluctuation origin. Accordingly, $g = 0$ at the normalization point b_0. Let us also consider the expressions for the Green's functions which are associated with the smectic variable,[7]

$$G_1(\omega,\mathbf{k})=\omega/(\omega^2-\eta^2+2ig\omega k_\perp{}^2), \tag{38}$$

$$D_1(\omega,\mathbf{k})=-\frac{4Tk_\perp{}^4}{\rho k^2l^2}[(\omega^2-\eta^2)^2+4g^2\omega^2k_\perp{}^4]^{-1}. \tag{39}$$

Here

$$\eta^2=(ak_z{}^2+bk^4)k_\perp{}^2/k^2. \tag{40}$$

4. FLUCTUATION CORRECTIONS TO THE ORIENTATIONAL-MODE SPECTRUM

It is quite obvious that the fluctuation effects associated with the orientational mode in a smectic B are weak effects. We will therefore disregard the fluctuations of the variables p_2 and φ. In the smectic B and smectic A phases (Refs. 6 and 7), the smectic variable W is thus the only quantity that fluctuates appreciably. The large fluctuation of W is due to the absence of elastic moduli in the variation of W, which are quadratic in the gradients and which are responsible for the rigidity in the plane perpendicular to the z axis. Accordingly, a typical wave vector $\mathbf{q}$ for a smectic mode, as we can see from expression (40), is estimated to be

$$q_z/q\sim lq\ll 1. \tag{41}$$

At the same time, under actual experimental conditions we find $q_z \sim q$ for the orientational mode. We should therefore first consider several terms in the effective action for the smectic variable. These terms are small with respect to parameter (41) and are therefore irrelevant in the renormalization of the smectic-mode spectrum.

Because of the fluctuations, terms (28) and (30) in the Lagrangian density, which have "tails" $p_1\nabla W$, generate the corrections

$$\mathscr{L}_5=2iB_if_5(\delta_{ik}-\nabla_i\nabla_k/\nabla^2)A_{\perp k}, \tag{42}$$

$$\mathscr{L}_6=2ihB_if_6e_{ikz}\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}\nabla^2\varphi. \tag{43}$$

Here

$$B_i=ig\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_{\perp k}p_1\nabla_{\perp i}\nabla_{\perp k}W. \tag{44}$$

In (42) we have dropped a term proportional to A_z, which we will no longer need. The purely fluctuational origin of the terms in (42) and (43) can be seen in the g-factor in (44), which vanishes at the normalization point. We should emphasize that terms (42) and (43) appear because of the power-law diagrams and are then renormalized in a logarithmic manner. The renormalization of these terms is analyzed in the Appendix, in which we show that the logarithmic behavior of g in B_i is the same as its behavior in (34), so that the coefficients f_5 and f_6 in (42) and (43) are constants. We will show below that $f_5 = f_6 = 1$ follows from the fluctuation-dissipation theorem.

Upon renormalization, the terms in the Lagrangian density generate a series of new terms in a logarithmic manner. The generation (and renormalization) of these terms is due to the following pairings (see the Appendix):

$$\langle A_iB_j\rangle\to dg\delta_{ij}{}^\perp\nabla_z{}^2\left(\frac{\partial}{\partial t}\right)^{-1}, \tag{45}$$

$$\langle A_iA_j\rangle\to\frac{4T}{\rho}dg\delta_{ij}{}^\perp\nabla_z{}^2\left(\frac{\partial}{\partial t}\right)^{-2}. \tag{46}$$

Here dg represents a change in g due to an infinitesimal shift of the ultraviolet cutoff. First, the terms that appear can be represented in the form of expressions (31), (42), and (43), in which f_4, f_5, and f_6 become functions of $g\nabla_z^2(\partial/\partial t)^{-1}$. Secondly, the new terms that appear can be represented in the form

$$\mathscr{L}_7=-{}^1/{}_2iB_if_7(\delta_{ij}-\nabla_i\nabla_j/\nabla^2)B_j, \tag{47}$$

$$\mathscr{L}_8=-ip_2e_{zji}\nabla_jf_8B_i, \tag{48}$$

$$\mathscr{L}_9=-2ihp_2f_9\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_\perp{}^2\nabla^2\varphi, \tag{49}$$

$$\mathscr{L}_{10}=-{}^1/{}_2ip_2f_{10}\nabla_\perp{}^2p_2. \tag{50}$$

Here f_7, f_8, f_9, and f_{10} are also functions of $g\nabla_z^2(\partial/\partial t)^{-1}$. Since the terms (47)-(50) are of a purely fluctuational origin, the initial values of these functions vanish.

In the terms presented above, the functions f are renormalized because of the following pairings:

$$dI_5\leftarrow-{}^1/{}_2i\langle I_5I_5\rangle,\quad dI_7\leftarrow{}^1/{}_2i\langle I_5I_5\rangle+i\langle I_5I_7\rangle,$$
$$dI_4\leftarrow i\langle I_5I_4\rangle,\quad dI_6\leftarrow i\langle I_5I_6\rangle,$$
$$dI_8\leftarrow i\langle I_4I_5\rangle+i\langle I_4I_7\rangle+i\langle I_8I_5\rangle,$$
$$dI_9\leftarrow i\langle I_4I_6\rangle,\quad dI_{10}\leftarrow{}^1/{}_2i\langle I_4I_4\rangle+i\langle I_4I_8\rangle. \tag{51}$$

Using rules (45) and (46) to describe the expectation values (51), we find the following differential equations for the functions f:

$$\frac{df_5}{dg}=2\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1} f_5^2,$$
$$\frac{df_7}{dg}=i\left(\frac{16T}{\rho g}f_5\tilde{f}_5-2if_5f_7+2if_7\tilde{f}_5\right)\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1},$$
$$\frac{df_4}{dg}=2f_4f_5\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1},\quad \frac{df_6}{dg}=2f_5f_6\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1},\tag{52}$$
$$\frac{df_9}{dg}=if_5^2\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1},$$
$$\frac{df_{10}}{dg}=i\left(\frac{4T}{\rho}f_4\tilde{f}_4-f_4f_8+\tilde{f}_4f_8\right)\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1}.$$

Here $\tilde{f}(g)\equiv f(-g)$.

Solving now these differential equations with the initial conditions at $g=0$

$$f_4=f_5=f_6=1,\quad f_7=f_3=f_9=f_{10}=0$$

(they are determined by the initial values), we finally find

$$f_4=f_5=f_6=\frac{\partial/\partial t}{\partial/\partial t-2g\nabla_z^2},\quad f_9=\frac{ig\nabla_z^2}{\partial/\partial t-2g\nabla_z^2},$$
$$\frac{1}{4}f_7=-\frac{i}{2}f_8=f_{10}=\frac{4T}{\rho}\frac{g\nabla_z^2}{(\partial/\partial t)^2-4g^2\nabla_z^4}.\tag{53}$$

5. CONCLUSION

We can now direct our attention to the orientational-mode spectrum. The initial part of the effective action for this mode is given by expressions (29) and (35), the fluctuational part is given by expressions (49) and (50), and the functions f_9 and f_{10} in (49) and (50) are given in (53). In the Gaussian approximation of interest to us, the nonholonomic nature of the variable φ is unimportant, and we find the expressions

$$G_2(\omega,\mathbf{k})=\frac{\omega+2igk_z^2}{\omega^2-hk^2k_\perp^2-4h\Gamma gk^2k_z^2+2igk_z^2\omega+2ih\Gamma\omega k^2},\tag{54}$$

$$D_2(\omega,\mathbf{k})=G_2(\omega,\mathbf{k})\frac{4T}{\rho}\left(\frac{gk_z^2k_\perp^2}{\omega^2+4g^2k_z^2}+\Gamma\right)G_2(-\omega,\mathbf{k}).\tag{55}$$

The relation

$$-\int\frac{d\omega}{2\pi}D(\omega,\mathbf{k})=\frac{T}{\rho hk^2}\tag{56}$$

can easily be verified by direct integration. Using (32), we find that this expression is the same as the single-time correlation function $\langle\varphi\varphi\rangle$ which can be found from the energy density (5). Expression (56) is a consequence of the fluctuation-dissipation theorem.

As we have already mentioned, the orientation-mode spectrum is determined by the poles of the Green's function $G_2(\omega,\mathbf{k})$. It is evident from expression (54) that there are two intervals of wave vectors. In the first interval, which is specified by the inequality $h^{1/2}kk_\perp\gg gk_z^2$, the spectrum is determined by the initial equations for φ,

$$\omega_\pm=-i\Gamma hk^2\pm(hk_\perp^2/k^2-h^2\Gamma^2)^{1/2}k^2.\tag{57}$$

In the second interval, which is determined by the inverse inequality $gk_z^2\gg h^{1/2}kk_\perp$, the spectrum is determined by the fluctuation effects. This spectrum is

$$\omega_1=-2igk_z^2,\quad \omega_2=-2ih\Gamma k^2-ihk^2k_\perp^2/2gk_z^2.\tag{58}$$

There are two branches of the spectrum in (57) and (58) because the orientation mode includes oscillations of two degrees of freedom: the oscillation of the variable φ and the oscillation of one of the parts of the transverse component of the velocity $\mathbf{v}$.

Keeping in mind that smectics C have been studied experimentally much more thoroughly than hexatic smectics B (see Ref. 13, for example), we will estimate the parameters for this particular case in the expressions obtained above. The quantity h is estimated to be

$$h=\alpha/\rho\sim K\theta^2/\rho.\tag{59}$$

Here K is the Frank modulus, and θ is the angle of inclination of the molecules with respect to the normal to the layer. As regards g, at $L\gg1$ we find

$$g=L^{1/5}(^1/_5 b_0\ln L)^{1/2},\tag{60}$$

consistent with (36) and (37). For b_0 we find

$$b_0\sim\frac{K}{\rho}\frac{c_2^2}{c_1^2}.\tag{61}$$

From estimates (59) and (61) we see that both h and g^2, in comparison with K/ρ, contain a small parameter of the same order of magnitude,

$$c_2^2/c_1^2\sim\theta^2\sim10^{-3}.\tag{62}$$

The boundary between the two indicated regions in k space is thus determined only by the propagation direction of the orientation mode, although the fluctuation region increases slowly (on a logarithmic scale). The spectrum is determined by $h^{1/2}\Gamma$ in the region in which the initial spectrum (57) is applicable. In this region we find $k_\perp\approx k$, according to the estimates presented above. At $h^{1/2}\Gamma<1$ spectrum (57) describes the propagation of waves in accordance with the square dispersion law. The damping of these waves is generally of the same order of magnitude as that of the frequency. At $h^{1/2}>1$ spectrum (57) describes two diffusion modes. The fluctuation spectrum (58) describes two diffusion modes. The damping of the first mode is much stronger than that of the second and is devoid of any initializing parameters of the orientation mode.

Experimental data of Ref. 13 show that the typical relaxation time of the director in a smectic C is 10^{-1}–10^{-2} s for characteristic dimensions $k\sim10^4$ cm^{-1}. Using (57) and (59), we then estimate the coefficient Γ to be

$$\Gamma\sim10^3-10^4\ \text{c/cm}^2$$

On the other hand, estimate (59) yields $h\sim10^{-9}$ cm^4/c^2. Accordingly, $h^{1/2}\Gamma<1$ for the data obtained in Ref. 13.

APPENDIX

We should first recall what a renormalization involves. If the fluctuation effects on the scale k^{-1} result from fluctuations with a wave vector $q\gg k$, we can find an effective distribution function for a characteristic dimension k^{-1} by eliminating the fast variables. In our case, we should break up the variables p_1 and W into slow and fast variables p_1' and W' (the fast variables have a characteristic wave vector q) and then integrate the distribution function $\exp(iI)$ over the fast variables p_1' and W'. We thus obtain logarithmic corrections to I, justifying the use of the inequality $q \gg k$. If we restrict ourselves to a second-order expansion of I in fast variables p_1' and W', then the integral over p_1' and W' will be Gaussian, consistent with the single-loop approximation in perturbation theory (which will be used exclusively below). We recall that a zero-charge state applies to smectics; i.e., L^{-1} is a small real parameter of the expansion in the loops, where L is the major logarithm.

Turning our attention to specific calculations, we first point out that by virtue of estimate (41) we can set the ratio $q_\perp^2/q^2$ equal to unity in all intermediate integrals, and we can set q_i equal to $q_{\perp i}$. We will use the following term of the expansion of A_i in fast variables:

$$A_i' = al^2\nabla_z\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_z W'\nabla_i W'. \tag{A.1}$$

Using Wick's rule for pairing, we find

$$\langle A_i' A_j'\rangle \to 2\int\frac{d\nu d^3q}{(2\pi)^4}a^2l^4q_z^2q_{\perp i}q_{\perp j}D_1^2(\nu,\mathbf{q})\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-2}. \tag{A.2}$$

Here the D functions are generated by the expectation values of the fast variables and are given by expression (39). Because of the logarithmic nature of integral (A.2), we can assume that the arguments of the D functions are the same. Averaging $q_{\perp i}\, q_{\perp j}$ over the angles gives us $1/2g_\perp^2\delta_{ij}$. As a result, we find an integral which was analyzed in the Appendix of Ref. 7 and which allows us to renormalize τ. Recalling the relation $\tau = 4Tg/\rho l^2$ (see Ref. 7) and transforming from an integral over dL to a differential, we find relation (46).

Let us now consider the expectation value $\langle A_i' B_j'\rangle$, where A_i' is given by expression (A.1), and B_j' is found from (44) by substituting the fast variables W' and p_l' for the variables W and p_1. Averaging in accordance with Wick's rule, we find

$$\langle A_i' B_j'\rangle \to i\int\frac{d\nu d^3q}{(2\pi)^4}l^2agq_{\perp i}q_{\perp j}\frac{q_\perp^2}{\nu}$$
$$\times(2q_z+k_z)D_1(\nu,\mathbf{q})G_1(\nu,\mathbf{q}+\mathbf{k})\nabla_z\left(\frac{\partial}{\partial t}\right)^{-1}. \tag{A.3}$$

Here $\mathbf{k} = -i\nabla$, and the expressions for the Greens functions are given in (38) and (39). The zeroth term of the expansion in k/q in expression (A.3) vanishes upon integration, so that the k_z dependence in the integrand in (A.3) remains in force. The dependence on the outer frequency ω and on the component $k_\perp$ of the wave vector in (A.3) may, however, be dropped. Integrating in (A.3) over the frequency, which reduces to taking the residues in the D poles, we find the following expression, in which the terms linear in k_z are retained:

$$\langle A_i' B_j'\rangle \to -\frac{iT}{4\rho}\int\frac{d^3q}{(2\pi)^3}\frac{ag}{\xi}q_\perp^2q_{\perp i}q_{\perp j}(2q_z+k_z)$$
$$\times\left[\frac{1}{\nu_+^*(2igq^2\nu_+^* - aq_zk_z)} - \frac{1}{\nu_-^*(2igq^2\nu_-^* - aq_zk_z)}\right]$$
$$\times\nabla_z\left(\frac{\partial}{\partial t}\right)^{-1}. \tag{A.4}$$

Here

$$\nu_\pm^* = igq^2 \pm \xi,\quad \xi = (\eta^2 - g^2q^4)^{1/2}. \tag{A.5}$$

Decomposing (A.4) to the first order in k_z/q_z, we find the logarithmic integral

$$\langle A_i' B_j'\rangle \to -\frac{T}{4\rho}\int\frac{d^3q}{(2\pi)^3}\frac{a}{g}\frac{q_{\perp i}q_{\perp j}}{q_\perp^2}$$
$$\times\left[\frac{2g^2q^4}{\eta^4} + \frac{aq_z^2}{\eta^6}(\eta^2 - 4g^2q^4)\right]\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1}. \tag{A.6}$$

Let us integrate (A.6) over the angle in the plane $q_\perp$ and transform to the variables η and χ,

$$q_z = \eta a^{-1/2}\sin\chi,\quad q^2 = \eta b^{-1/2}\cos\chi. \tag{A.7}$$

Integrating the resulting expression over the angle χ, we finally find

$$\langle A_i' B_j'\rangle \to -\int\frac{d\eta}{2^7\pi\eta}\frac{a^{1/2}T}{b^{3/2}\rho}\left(1+\frac{b}{g^2}\right)\delta_{ij}^\perp\nabla_z^2\left(\frac{\partial}{\partial t}\right)^{-1}. \tag{A.8}$$

Since (see Ref. 7)

$$dg = \frac{T}{2^7\pi\rho}\frac{a^{1/2}g}{b^{3/2}}\left(1+\frac{b}{g^2}\right)dL, \tag{A.9}$$

we find from (A.8) relation (45) of the text proper, after transforming from an integral to a differential.

We will use the following terms of the expansion of the effective action in the fast variables, which is defined by (28):

$$I_\text{I} = -\int dt\, d^3r\, al\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}p_1\nabla_zW'\nabla_kW', \tag{A.10}$$

$$I_\text{II} = -\int dt\, d^3r\, al\left[\nabla_i\left(\frac{\partial}{\partial t}\right)^{-1}p_1'\nabla_iW\nabla_zW'\right.$$
$$\left.+\nabla_z\left(\frac{\partial}{\partial t}\right)^{-1}p_1'\nabla_iW\nabla_iW'\right]. \tag{A.11}$$

Applying Wick's rule to describe the expectation value of the fast variables, we find the fluctuation correction to B_j

$$-\langle I_\text{I}B_j'I_\text{II}\rangle = \nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}$$
$$\times p_1\int\frac{d\nu\, d^3q}{(2\pi)^4}ga^2l^2q^2q_j(2q_k+k_k)(2q_i+k_i)\cdot$$
$$\times[G_1(\nu,\mathbf{k}+\mathbf{q})G_1(-\nu,\mathbf{q})D_1(\nu,\mathbf{q}) - D_1(\nu,\mathbf{k}+\mathbf{q})G_1^2(-\nu,\mathbf{q})$$
$$+D_1(\nu,\mathbf{q})G_1(\nu,\mathbf{k}+\mathbf{q})G_1(\nu,\mathbf{q})]\nabla_iW. \tag{A.12}$$

Here $\mathbf{k} = -i\nabla$. The zeroth term of expansion (A.12) in k/q vanishes upon integration, so that the k dependence in the integrand of (A.12) remains in force. The dependence of the wave vector on the component k_z in (A.12) can, however, be dropped (as we have done with the dependence of the wave vector on the outer frequency ω). Integrating in (A.12) over the frequency ν, we find the following expression in which the terms linear in k are retained:

$$-\langle I_I B_j' I_{II}\rangle = \frac{T}{\rho}\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}$$

$$\times p_1 \int \frac{d^3q}{(2\pi)^3} ga^2 q_z^2 q^2 q_j (2q_k+k_k)(2q_i+k_i)$$

$$\times\{[\eta^4(\eta^2+4bq^2\mathbf{qk})]^{-1}$$

$$+[16\xi\nu_+^{*2} gq^2(gq^2\nu_+^* - ibq^2\mathbf{qk}+g\nu_+^*\mathbf{qk})]^{-1}$$

$$+(\nu_+\to\nu_-)\}\nabla_i W. \qquad (A.13)$$

Decomposing (A.13) to first order in k/q, we find the logarithmic integral

$$-\langle I_I B_j' I_{II}\rangle = \frac{T}{\rho}\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1} p_1 \int \frac{d^3q}{(2\pi)^3} ga^2 q_z^2 q^2 q_j$$

$$\times\left\{\frac{q_k k_i + q_i k_k}{4g^2q^4\eta^6}(\eta^2+4g^2q^4)\right.$$

$$\left.-\frac{\mathbf{qk}}{2g^2q^6}q_i q_k\left[\frac{1}{\eta^6}(\eta^2-4g^2q^4)+\frac{4bq^4}{\eta^8}(\eta^2+6g^2q^4)\right]\right\}\nabla_i W. \qquad (A.14)$$

Integrating (A.14) over the angle in the plane $q_\perp$, switching to the variables η and χ in (A.7), and then integrating over the angle χ, we find

$$-\langle I_I B_j' I_{II}\rangle = i\nabla_{\perp k}\left(\frac{\partial}{\partial t}\right)^{-1}$$

$$\times p_1 \frac{T}{\rho}\int \frac{d\eta}{2^7\pi\eta}\frac{ga^{1/2}}{b^{3/2}}\left(1+\frac{b}{g^2}\right)\nabla_{\perp j}\nabla_{\perp k} W. \qquad (A.15)$$

Comparing (A.15) with (A.9) and with expression (44), we see that (A.15) gives an exact renormalization of g in expression (44).

Finally, one more comment is in order. At first glance, it may seem that terms like those in (42) and (43) can be found from (28) and (30) by using exactly the same logarithmic method that was used for renormalizing B_j. The factor B'_j in (A.12) in this case is replaced by $p'_\perp \nabla_j W'$. A straightforward calculation shows, however, that the logarithmic part of the corresponding integral vanishes. In summary, the terms (42) and (43) appear only because of the power-law diagrams, as we have pointed out in the body of the paper.

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