Anderson metal-insulator transition in a system of metal granules: existence of a minimum metallic conductivity and a maximum dielectric constant

K. B. Efetov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

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A model of a disordered granulated metal is analyzed. The granules are assumed to form a lattice with the structure of a Cayley tree (a Bethe lattice). A supersymmetry method is used. All the calculations reduce to the solution of an integral equation with two real variables. The occurrence of a metal-insulator transition is proved. The critical point separating the metallic and insulating regions is determined. The dielectric constant is found to increase as the transition point is approached from the side of the insulator, but to remain finite, reaching a maximum value at the transition point. In the metallic region the diffusion coefficient decreases, reaching a nonzero minimum at the transition point, and then vanishing discontinuously.

1. INTRODUCTION

There is no longer any doubt that the Anderson transition[1] exists. The behavior of the kinetic coefficients near the transition, on the other hand, has generated much debate. Mott[2] argues that the conductivity σ of a metal decreases with increasing disorder, to a minimum value $\sigma_{\min} \sim e^2/\hbar a$ (a is the interatomic distance) at the transition point. With a further increase in the impurity concentration, the conductivity vanishes discontinuously. This contradicts the view that the conductivity vanishes smoothly at the transition point, rather than discontinuously.

The hypothesis of the existence of a renormalization group[3] has had a major influence on the development of the theory of disordered metals. Simple arguments based on this hypothesis lead to a power-law decay of the conductivity near the transition point. The same result comes from renormalization-group calculations in a space of dimensionality $2 + \varepsilon$ through the use of the σ model.[4–6] The development of the idea of the renormalization group has undermined faith in the existence of a minimum metallic conductivity, and the opposite point of view has won general acceptance.

At the same time, the power-law behavior of physical quantities near a transition point in ordinary $3D$ space can by no means be regarded as proved, since the analysis in a space of dimensionality $2 + \varepsilon$ assumes in all cases that ε is small. Correspondingly, the question of the behavior of the kinetic coefficients near the transition point cannot be regarded as settled.

In this paper we examine a Cayley-tree model of a disordered metal. Abou-Chacra *et al.*[7] analyzed the Anderson model on a Cayley tree. It was asserted in that paper that a transition exists for certain distributions of the random potential. The behavior of the kinetic coefficients near the transition was not discussed. Jonson and Girvin[8] carried out a numerical study of the equations derived in Ref. 7, and they asserted that there exists a minimum metallic conductivity. The conclusions reached by Jonson and Girvin[8] are contradicted by the result derived by Shapiro,[9] who found a power-law decay of the conductivity toward the transition point. Kunz and Souillard[10] have found a power-law increase in a scale length in the Anderson model on a Cayley tree.

The model which we discuss below differs from those studied in Refs. 7–10. We assume that the granules of a metal with impurities are at the nodes of a Cayley tree. Electrons can tunnel from granule to granule. The metal-insulator transition occurs when the average distance between levels in the granules becomes comparable to the amplitude for tunneling between granules. Our calculations are carried out by the supersymmetry method developed in Ref. 11. This method can be used for calculations at finite frequencies in both the conducting and insulating regions. All the calculations reduce to the solution of an integral equation with two variables. We prove that an Anderson transition exists, and we analyze the behavior of the conductivity and of the dielectric constant. We find that the conductivity decreases with decreasing tunneling amplitude, reaches a minimum value at the transition point, and then vanishes discontinuously. The dielectric constant increases toward the transition point but remains finite. These results were reported in summary form in Ref. 12.

2. CHOICE OF MODEL

Before describing the model we consider just what we need to calculate in order to describe the kinetics of electrons in a disordered metal. The equations which we give here are general, not linked with any specific model.

The kinetics of noninteracting electrons is described completely by a density-density correlation function, which is written in the standard form

$$K(\mathbf{r}, \omega, \varepsilon) = \langle G^A_{\varepsilon-\omega}(\mathbf{r}, 0)\,(G_\varepsilon^R(0, \mathbf{r}) - G_\varepsilon^A(0, \mathbf{r}))\rangle, \tag{1}$$

where $G^R_\varepsilon(\mathbf{r}, \mathbf{r}')$ and $G^A_\varepsilon(\mathbf{r}, \mathbf{r}')$ are the retarded and advanced Green's functions. The angle brackets in (1) mean an average over impurities.

Knowing correlation function (1), we can evaluate the diffusion coefficient in the conducting region and the dielectric constant in the localized region. In the conducting region the correlation function K is

$$K(\mathbf{k},\ \omega)=4\pi\nu/(D\mathbf{k}^2-i\omega), \tag{2}$$

where D is the diffusion coefficient, $\mathbf{k}$ the momentum, and ν the state density. Expression (2) holds for small values of $|\mathbf{k}|$ and ω. In the localized region, in the limit of low frequencies ω, the correlation function $K(\mathbf{k},\omega)$ can be described by the following expression[13]

$$K(\mathbf{k},\ \omega)=4\pi\nu A(\mathbf{k})/(-i\omega),\quad A(0)=1. \tag{3}$$

The momenta $\mathbf{k}$ in (3) are arbitrary. At small values of $|\mathbf{k}|$ we have

$$A(k)=1-\varkappa k^2. \tag{4}$$

The quantity $\varkappa$ is the dielectric constant.

At low frequencies ω, expressions (2) and (3) can be written in a common form which holds for arbitrary $\mathbf{k}$:

$$K(\mathbf{k},\ \omega)=4\pi\nu/[A_1(\mathbf{k})-i\omega A_2(\mathbf{k})],\quad A_1(0)=0,\quad A_2(0)=1. \tag{5}$$

The denominator in (5) actually consists of the first terms of an expansion in ω. The conditions at $\mathbf{k}=0$ follow from the conservation of particles. In the conducting region, $A_1(\mathbf{k})$ is nonzero, but it vanishes in the localized region. Comparing (5) with (3), we conclude that the function $A_1(\mathbf{k})$ vanishes at the transition point, simultaneously for all $\mathbf{k}$. This result means that the Anderson transition does not reduce simply to a vanishing of the diffusion coefficient and is instead more profound in nature.

Equations (2)–(5) are written in momentum space. It is not difficult to write corresponding equations in coordinate space. For example, Eq. (2) in the space-time representation becomes

$$K(\mathbf{r}_1,\ \mathbf{r}_2,\ t)=(4\pi Dt)^{-d/2}\exp(-\mathbf{r}^2/4Dt), \tag{6}$$

where d is the dimensionality of the space, and $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. We are forced to use the equations in coordinate space in those cases in which there is no momentum representation. Cayley tree models are an example.

It is frequently more convenient to calculate the density correlation function $\overline{K}$ at coincident points. Although this correlation function embodies less information than does $K(\mathbf{r},\mathbf{r}')$ at arbitrary $\mathbf{r}'$, it does allow us to distinguish between the conducting and localized regions. It is not difficult to show that the correlation function $\overline{K}$ can be written in the form

$$\overline{K}=(2\pi\nu)^2\begin{cases}1/\overline{D}, & \text{conducting region},\\ 1/(-i\omega\simeq\bar{\varkappa}), & \text{localized region}.\end{cases} \tag{6a}$$

Expression (6a) can be derived from (5) by integrating over all $\mathbf{k}$. The coefficients $\overline{D}$ and $\bar{\varkappa}$ completely characterize the diffusion in the conducting region and the polarizability in the insulating region. The knowledge of $\overline{D}$ and $\bar{\varkappa}$ alone is of course not sufficient for calculating D and $\varkappa$. However, the behavior of D near the transition point should be similar to that of $\overline{D}$, and the behavior of $\varkappa$ should be similar to that of $\varkappa$. The idea is that if D vanishes discontinuously at the transition point then $\overline{D}$ should also be discontinuous. If D instead vanishes smoothly, in accordance with a power law, for example, then we should find the same behavior for $\overline{D}$. Analogously, the quantities $\varkappa$ and $\bar{\varkappa}$ should either simultaneously increase to infinity as the transition point is approached or simultaneously reach finite values. The exponents in the power laws may of course be different.

It is thus sufficient to calculate the density correlation function at a single point in order to resolve the question of the existence of a minimum metallic conductivity and a maximum dielectric constant.

We turn now to the formulation of the model for a disordered metal which we will be analyzing in the following sections of this paper. We consider a system of metal granules which are in contact with each other. We assume that impurities are distributed within each granule. We write the Hamiltonian of the system as

$$H=\sum_{ijpq}T_{ij}{}^{pq}a_{ip}{}^{+}a_{jq}+\sum_{i}H_i, \tag{7}$$

$$H_i=\sum_{p}\varepsilon_p^{(i)}a_{ip}{}^{+}a_{ip}+H_i{}'.$$

Here the operators $a_{ip}^{+}(a_{ip})$ create (annihilate) electrons in the ith granule in the state p, and T_{ij}^{pq} are the amplitudes for jumps of electrons. The energies $\varepsilon^{(i)}(p)$ are the eigenenergies in the isolated granules. The quantities H_i' represent the random potential in the granules. This random potential may incorporate ordinary, magnetic, and spin-orbit interactions. Expression (7) also applies when the system is in an external magnetic field. In this case, the quantities $\varepsilon^{(i)}(p)$ should be understood as the eigenenergies in the field.

To study the metal-insulator transition in the case in which the amplitudes (T_{ij}) for jumps from one granule to another are not very large, it is sufficient to restrict the analysis to the case in which the mean free paths in the granules are much greater than the interatomic distances. The macroscopic conductivity is determined primarily by the probability for jumps from granule to granule, not by the mean free path. If an amplitude T_{ij} exceeds the average distance between levels in the individual granules, which is proportional to $(\nu V_i)^{-1}$, where V_i is the volume of the ith granule, the entire system conducts. In the opposite case, the entire system becomes an insulator.

In the limit of long mean free paths in the granules ($\tau\varepsilon_0\gg 1$, where τ is the mean free time, and ε_0 is the Fermi energy) we can use the supersymmetry method of Refs. 6 and 11 and derive a Lagrangian of the interacting diffusion modes for the system under consideration. The metal-insulator transition falls within the range of applicability of this Lagrangian. The Lagrangian of the diffusion modes was written in Refs. 11 and 14 for isolated granules in order to study the statistics of levels. This Lagrangian contains supermatrices Q which have a definite symmetry, which depends on whether magnetic or spin-orbit interactions operate in the system. The presence of a term describing jumps in Hamiltonian (7) couples the supermatrices Q_i corresponding to different granules. The derivation of this additional term in the Lagrangian is completely analogous to the derivation of the Josephson energy in super-conductors.

Assuming that the amplitudes T_{ij}^{pq} vary only slightly along the Fermi surface, we find

$$
\begin{aligned}
F[Q] &= \sum_{i,j} \frac{1}{2} J_{ij}\, \mathrm{STr}\,(Q_i - Q_j)^2 \\
&\quad - \frac{1}{4} i(\omega + i\delta)\pi\nu \sum_j \mathrm{STr}\,(\Lambda Q_j)\, V_j, \\
J_{ij} &= T_{ij}^2 \nu^2 V_i V_j \equiv T_{ij}^2 / \Delta_i \Delta_j .
\end{aligned}
\tag{8}
$$

The first term in Eq. (8) describes the interaction of granules; the second is the effective Lagrangian of the isolated granules; ω is the frequency; STr is the supertrace; and Δ_i is the average distance between levels in the granules. In deriving Lagrangian (8) we used the condition $\tau^{-1} \gg \Delta$. The supermatrices Q and Λ in (8) are the 8×8 matrices

$$
\begin{aligned}
Q &= U \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix} \bar{U}, \\
U &= \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{9}
$$

The matrices u, v, and $\hat{\theta}$ depend on whether symmetry with respect to time reversal holds and on whether there is central symmetry. The corresponding equations are given in the review in Ref. 11. We will write these equations in the present paper where necessary. Here we simply note that $u\bar{u} = v\bar{v} = 1$.

The supersymmetry method reduces the calculation of the density correlation function $K(\mathbf{r},\mathbf{r}')$ to the calculation of the following integral:

$$
K(\mathbf{r},\mathbf{r}') = -2\pi^2\nu^2 \int (Q_{13}{}^{12})_{\mathbf{r}} (Q_{31}{}^{21})_{\mathbf{r}'} \exp(-F[Q]) \prod_i dQ_i,
\tag{10}
$$

where $F[Q]$ is defined by Eq. (8). The superscripts on the matrices Q in Eq. (10) specify the blocks which are written out explicitly in (9); the subscripts specify the elements within these blocks.

The model described by expression (8) is actually a nonlinear lattice σ model. In the continuum limit, this model becomes the σ model constructed in Ref. 6. The metal-insulator transition can of course occur only at $J \sim 1$, since there are no other parameters. The model described by expression (8) obviously cannot be solved exactly for arbitrary lattices.

Despite the formal analogy between Eq. (8) and spin models in which a phase transition occurs, the methods ordinarily used in phase-transition theory cannot be used for model (8). For example, the self-consistent-field method, which involves an average spin, which vanishes above the transition point and which is nonvanishing below this point, gives us no information of any sort in the present case. The reason is $\langle Q \rangle$ exists for arbitrary J and is equal to the matrix Λ. This quantity determines only the state density and is unrelated to the Anderson transition.[11] It can be shown that an expansion in J corresponding to a high-temperature expansion in phase-transition theory is also inapplicable here, since terms of the type $(J/\omega)^n$ arise even in the first orders of the expansion, and these terms grow with decreasing frequency. We will show below that the reason for this behavior is simply that the problem is nonanalytic in J. In the following sections we will accordingly consider only the one particular case in which the granules form a Cayley tree. In this case, the problem can be solved exactly. In phase-transition theory, an analysis on a Cayley tree always yields a good qualitative description of the transition.

3. REDUCTION TO AN INTEGRAL EQUATION

We thus assume that the granules form a Cayley tree and that the couplings between granules are the branches of this tree. We consider the case of an arbitrary branching number m (each branch divides into m other branches). Figure 1 shows part of the three for the case $m = 2$. We assume that the interaction extends only to nearest neighbors and that all the J_{ij} are identical, equal to J.

The structure of the Cayley tree allows us to reduce the evaluation of integral (10) over all the nodes to the evaluation of a finite integral of the solution of some integral equation. To derive this equation we introduce the function

$$
\Psi(Q_0) = \int \exp(-F'[Q]) \prod_j{}' dQ_j,
\tag{11}
$$

where

$$
F'[Q] = \sum_{ij}{}' J\, \mathrm{STr}\,(Q_i - Q_j)^2 - \frac{i(\omega + i\delta)\pi}{4\Delta} \sum_j{}' \mathrm{STr}\,\Lambda Q_j .
$$

To show the meaning of Σ'_{ij}, Σ'_j and Π'_j in (11), we imagine that we cut one of the $m + 1$ branches going into node 0 in Fig. 1. These symbols then represent the sums and products over all the nodes of the branch which has been cut off. We are assuming here that the indices i and j in the sum Σ'_{ij} specify the nearest nodes and that the jth node is closer than the ith node to the base of the branch. The index j in the sum Σ'_{ij} can take on the value 0.

The function Ψ defined in (11) depends only on Q_0. Now considering the function $\Psi(Q_1)$ at an adjacent node, we can easily write the equation

$$
\begin{aligned}
\Psi(Q) = \int \exp\left\{ -\mathrm{STr}\left[\frac{\alpha}{8}(Q - Q')^2 - \frac{\beta}{4}\Lambda Q' \right]\right\} \\
\Psi^m(Q')\, dQ',
\end{aligned}
\tag{12}
$$

where

$$
\alpha = 8J, \quad \beta = -i(\omega + i\delta)\pi\nu V \equiv -i(\omega + i\delta)\pi/\Delta .
$$

The integration in Eq. (12) is over a supermatrix Q of the type in Eq. (9). In the case $m = 1$, which is the case of a $1D$ chain of granules, Eq. (12) becomes linear. In other cases, it is nonlinear. Knowing the function $\Psi(Q)$ we can immediately find the density correlation function $\bar{K}$ at coincident points:

$$
\bar{K} = K(r,r) = -2\pi^2\nu^2 \int Q_{13}{}^{12} Q_{31}{}^{21} \Psi^{m+1}(Q) \exp\left[\frac{\beta}{4} \mathrm{STr}\,\Lambda Q \right].
\tag{13}
$$

The correlation function for the densities at different points can also be written as an integral over a finite number of variables:

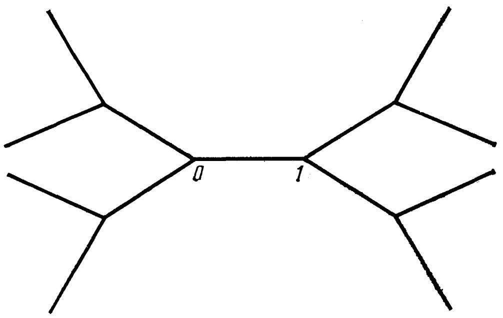

FIG. 1.

$$K(r,r')=-2\pi^2\nu^2\int (Q_{13}{}^{12})_r(Q_{31}{}^{21})_{r'}\,\Psi(Q_r)\Psi(Q_{r'})$$
$$\times\exp\left[-\frac{\alpha}{8}\tilde{\sum_{ij}}\operatorname{STr}(Q_i-Q_j)^2\right]$$
$$\times\tilde{\prod_i}\Psi^{m-1}(Q_i)\exp\left(\frac{\beta}{4}\operatorname{STr}\Lambda Q_i\right)dQ_i. \tag{14}$$

The product $\tilde{\Pi}$ and the $\tilde{\Sigma}$ in Eq. (14) contain the indices of nodes on the path between node r and node r' (the points r and r' are included). In the sum Σ_{ij}, the indices i and j again represent nearest neighbors; the jth node is closer to r, and the ith node is closer to r'. The "path" between nodes r and r' is the broken line along which we can go from node r to r', going through each node only once. The length of the path is the "distance" between nodes r and r'. The correlation function $K(r, r')$ of course depends only on the distance between nodes r and r'. If this distance is equal to R, then (14) contains an integral over $R + 1$ matrices Q. In principle, we could write a recurrence relation for $K(r, r')$.

The problem of evaluating the density correlation functions on a Cayley tree has thus been reduced to one of finding the solutions of an integral equation and of evaluating integrals of these solutions. Equation (12) is still quite complicated, however, since it contains a large number of variables (the elements of supermatrix Q). It is accordingly very useful too simplify Eq. (12) further. This simplification can be carried out by virtue of the symmetry of the original Hamiltonian (8) and of the corresponding kernel in (12), which contain only the one matrix Λ which we have specified. Because of this symmetry, the solution $\Psi(Q)$ can depend only on the variables $\hat{\theta}$ in Eq. (9). This fact is evident from Eqs. (11). We can accordingly integrate over u' and v' immediately in Eq. (12). The result of this integration depends on the symmetry of supermatrix Q. This supermatrix takes its simplest form when the system contains an external magnetic field or magnetic impurities (model II in the classification of Ref. 11). The further modifications of (12) below are carried out for this case in particular.

The supermatrices u, v, and $\hat{\theta}$ in the case of model II are

$$u=u_1u_2,\quad u_1=\begin{pmatrix}1-2\eta\eta^* & 2\eta\\ -2\eta^* & 1-2\eta^*\eta\end{pmatrix},\quad u_2=\begin{pmatrix}e^{i\varphi} & 0\\ 0 & e^{i\chi}\end{pmatrix},$$
$$v=\begin{pmatrix}1+2\varkappa\varkappa^* & 2i\varkappa\\ -2i\varkappa^* & 1+2\varkappa^*\varkappa\end{pmatrix}\quad \hat{\theta}=\begin{pmatrix}\theta & 0\\ 0 & i\theta_1\end{pmatrix}. \tag{15}$$

Here η, η^*, $\varkappa$, and $\varkappa^*$ are Grassmann variables; and φ, χ, θ, and θ_1 are real numbers which satisfy the inequalities $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant\chi\leqslant 2\pi$, $0\leqslant\theta\leqslant\pi$, and $\theta_1\geqslant 0$. The volume element dQ is written in terms of these variables as[11,14]

$$dQ=\frac{1}{2^6\pi^2}\frac{\sin\theta\,\operatorname{sh}\theta_1}{(\operatorname{ch}\theta_1-\cos\theta)^2}\,d\eta\,d\eta^*\,d\varkappa\,d\varkappa^*\,d\varphi\,d\chi\,d\theta\,d\theta_1. \tag{16}$$

Assuming that Ψ in (12) depends only on the variables θ and θ_1, we can carry out the integration over all the other variables. The integral cannot be evaluated directly in this particular parametrization, since an indeterminate expression of the $0\cdot\infty$ type arises (the integral over θ_1 and θ diverges at small values of θ and θ_1, while the integration over the Grassmann variables gives us zero). This difficulty can be avoided if we initially calculate the derivative of the integral with respect to β, in which case there is no indeterminacy of any sort, and then reconstruct the function itself from its derivative. As a result we find

$$\Psi(\boldsymbol{\lambda})=\int_{-1}^{1}\int_{1}^{\infty}L(\boldsymbol{\lambda},\boldsymbol{\lambda}')\exp[\beta(\lambda'-\lambda_1')]\frac{\lambda_1-\lambda}{\lambda_1'-\lambda'}\Psi^m(\boldsymbol{\lambda}')\,d\lambda'\,d\lambda_1'$$
$$+\exp[\alpha(\lambda-\lambda_1)]\Psi^m(1,1), \tag{17}$$

where

$$\bar{L}(\boldsymbol{\lambda},\boldsymbol{\lambda}')={}^1/{}_2\alpha^2\exp[\alpha(x-x_1)][I_0(\alpha y_1)I_0(\alpha y)(x_1+x)$$
$$-I_1(\alpha y_1)I_0(\alpha y)y_1+I_1(\alpha y)I_0(\alpha y_1)y],$$
$$\boldsymbol{\lambda}=(\lambda,\lambda_1),\quad \boldsymbol{\lambda}'=(\lambda',\lambda_1'). \tag{18}$$

In (18), I_0 and I_1 are Bessel functions,

$$x_1=\lambda_1\lambda_1',\quad x=\lambda\lambda',\quad y_1=[(\lambda_1^2-1)(\lambda_1'^2-1)]^{1/2},$$
$$y=[(1-\lambda^2)(1-\lambda'^2)]^{1/2},\quad \lambda=\cos\theta,\quad \lambda'=\cos\theta',$$
$$\lambda_1=\operatorname{ch}\theta_1,\quad \lambda_1'=\operatorname{ch}\theta_1'.$$

Setting λ and λ_1 equal to unity in Eq. (17), we find the relation

$$\Psi(1,1)=\Psi^m(1,1).$$

We thus have

$$\Psi(1,1)=1. \tag{19}$$

For λ_1 and λ approximately equal to unity, $\Psi(\boldsymbol{\lambda})$ becomes

$$\Psi(\boldsymbol{\lambda})=1+(\lambda-\lambda_1)c_0, \tag{19a}$$

where c_0 is a constant. Direct integration verifies the following important property of the kernel L:

$$\int_{-1}^{1}\int_{1}^{\infty}L(\boldsymbol{\lambda},\boldsymbol{\lambda}')\frac{\lambda_1-\lambda}{\lambda_1'-\lambda'}\,d\lambda'\,d\lambda_1'=1. \tag{20}$$

Using our parametrization, (9), (15), we can put the correlation function $\bar{K}$ in Eq. (13) in the following form:

$$\bar{K}=2\pi^2\nu^2\int_{-1}^{1}\int_{1}^{\infty}\frac{\lambda_1+\lambda}{\lambda_1-\lambda}\Psi^{m+1}(\boldsymbol{\lambda})\exp[\beta(\lambda-\lambda_1)]\,d\lambda\,d\lambda_1. \tag{21}$$

Expressions (17), (18), and (21) show that the problem of calculating the correlation function $\bar{K}$ reduces to one of evaluating the integral of the solution of a nonlinear integral equation with two variables. The expression for the kernel L in

(18) is quite lengthy. In many cases it is convenient to use a more compact form of Eqs. (17) and (18):

$$\frac{\Psi(\lambda)-1}{\lambda_1-\lambda}=\int_{-1}^{1}\int_{1}^{\infty}\Gamma(\mathbf{n}\mathbf{n}',\mathbf{n}_1\mathbf{n}_1')\times\frac{\exp[\beta(\lambda'-\lambda_1')]\Psi^m(\lambda')-1}{\lambda_1'-\lambda'}\frac{d\mathbf{n}'\,d\mathbf{n}_1'}{(2\pi)^2}, \qquad (22)$$

where

$$\Gamma(\mathbf{n}\mathbf{n}',\mathbf{n}_1\mathbf{n}_1')=\frac{\alpha^2}{2}\left[\exp(-\alpha\mathbf{n}_1\mathbf{n}_1')\frac{d}{d\alpha}\exp(\alpha\mathbf{n}\mathbf{n}')-\exp(\alpha\mathbf{n}\mathbf{n}')\frac{d}{d\alpha}\exp(-\alpha\mathbf{n}_1\mathbf{n}_1')\right],$$

$$\mathbf{n}=((1-\lambda^2)^{1/2}\cos\varphi,\ (1-\lambda^2)^{1/2}\sin\varphi,\lambda),\ \mathbf{n}'=((1-\lambda'^2)^{1/2}\times\cos\varphi',\ (1-\lambda'^2)^{1/2}\sin\varphi',\ \lambda'), \qquad (23)$$

$$\mathbf{n}_1=((\lambda_1^2-1)^{1/2}\cos\varphi_1,\ (\lambda_1^2-1)^{1/2}\sin\varphi_1,\ \lambda_1),$$

$$\mathbf{n}_1'=((\lambda_1'^2-1)^{1/2}\cos\varphi_1',\ (\lambda_1'^2-1)^{1/2}\sin\varphi_1'\lambda_1').$$

Integrating over φ' and φ_1' in (22), and using Eqs. (19) and (20), we can return to Eq. (17), (18).

4. DIFFUSION IN THE LIMIT OF LARGE INTERGRANULE COUPLING CONSTANTS

A Cayley tree differs in many ways from ordinary d-dimensional lattices. The very concept of a diffusion coefficient on a Cayley tree is not simple. It is thus useful to calculate the density correlation function in the limit of large coupling constants, $J \gg 1$. If such a correlation function is of a diffusive form, then there is no difficulty in determining the diffusion coefficient.

In the limit $J \gg 1$ small deviations of the super-matrix Q from Λ arise. It is simplest to apply expressions (12) and (14) directly. For the supermatrix Q there exists a convenient parametrization[11]:

$$Q=\Lambda(1+iP)(1-iP)^{-1},\qquad P=\begin{pmatrix}0 & B\\ \bar{B} & 0\end{pmatrix},\qquad B=\begin{pmatrix}a & i\sigma\\ \rho & ib\end{pmatrix}. \qquad (24)$$

Here a and b are real numbers, and σ and ρ are Grassmann variables. The Jacobian of the transformation to the variables P is 1.

In the leading approximation in J it is sufficient to retain only the first terms of the expansion of Q in P. In this case the kernel in integral equation (12) becomes Gaussian, and this equation itself can be written

$$\Psi(P)=\int\exp\left\{-\mathrm{STr}\left[\frac{\alpha}{2}(P-P')^2+\frac{\beta}{2}P'^2\right]\right\}\Psi^m(P')\,dP'. \qquad (25)$$

The solution Ψ will of course also be Gaussian:

$$\Psi(P)=\exp(-{}^1/_2 c\,\mathrm{STr}\,P^2). \qquad (26)$$

Substituting (26) into (25), we find c:

$$c_{1,2}=\frac{1}{2m}\{\alpha(m-1)-\beta\pm[\alpha^2(m-1)^2+2\alpha\beta(m+1)+\beta^2]^{1/2}\}. \qquad (27)$$

In the low-frequency limit, $\beta\to 0$, we have

$$c_1=\alpha(m-1)/m,\qquad c_2=-\beta/(m-1). \qquad (28)$$

If $\beta=0$, the coefficient is $c_2=0$. This value corresponds to the solution $\Psi=1$, which is always a solution of Eq. (22) if $\beta=0$. At large J, however, this solution must be discarded, since the integrals in (13) and (14) diverge for all complex frequencies ω in the upper half-plane $\mathrm{Re}\beta>0$. If, on the other hand, we formally evaluate integrals (13) and (14) for $\mathrm{Re}\beta<0$ and then continue into the half-plane $\mathrm{Re}\beta>0$, the result is found to be nonanalytic in the upper frequency half-plane.

Solution c_1 in (27) is analytic in the upper frequency half-plane, and when expression (26) is substituted into integrals (13) and (14), the latter converge. The density correlation functions are thus analytic in the upper half-plane, as would be required by causality. Substituting (24) and (26) into (14), and evaluating the Gaussian integrals, we find the following expression for the density correlation function:

$$K(r,r')=\frac{(2\pi\nu)^2}{c(m+1)+\beta}\left(m+\frac{\beta}{c}\right)^{-R}, \qquad (29)$$

where R is the distance between points r and r'. Expression (29) holds for all symmetries of supermatrix Q—not only for a model with magnetic interactions. In the low-frequency limit this correlation function is, for coincident points,

$$\bar{K}=\frac{(2\pi\nu)^2}{\bar{D}},\qquad \bar{D}=\frac{m^2-1}{m}\alpha. \qquad (30)$$

Recalling the relationship between β and the frequency ω, we find a time representation for $K(r,r')$. In the limit of large distances $R \gg 1$ and long times $\alpha\Delta t \gg 1$ we have

$$K(r,r',t)=\frac{2\nu^2\pi^{1/2}\Delta m}{m-1}\frac{R\exp[-Dt(m^{1/2}-1)^2/m^{1/2}]}{m^{R/2}(Dt)^{3/2}}\times\exp\left(-\frac{R^2}{4Dt}\right), \qquad (31)$$

where $D=m^{1/2}\alpha\Delta/\pi$ is the diffusion coefficient. The last exponential factor in Eq. (31) is characteristic of free diffusion. The coefficient of this exponential function, on the other hand, differs from that for a real d-dimensional lattice, (6). The rapid decay of this coefficient over time makes the model on a Cayley tree similar to models in an infinite-dimension space. The same situation arises in phase-transition theory.

Equation (30) can also be derived from (17), (18), and (21). For this derivation we need to write λ_1 and λ in the form

$$\lambda_1=1+s_1,\quad \lambda=1-s,\quad \lambda_1'=1+s_1',\quad \lambda'=1-s'$$

and, assuming s_1, s, s_1' and s' to be small, we need to carry out expansions in these variables. All the integrals over s_1 and s should be evaluated between the limits of 0 and ∞. We seek a solution Ψ in the form

$$\Psi=\exp[-c(\lambda_1-\lambda)]. \qquad (32)$$

After going through the appropriate calculations, we find Eqs. (28) and (30).

The condition $\alpha \gg 1$ is sufficient only for $m \sim m-1$. We can examine the limit $m\to 1$ formally. In this limit, we need the more stringent condition $\alpha \gg 1/(m-1)$ in order to derive (30) and (31). In the opposite limit $m \gg 1$, expression (32) with the coefficient c from (28) holds up to $\lambda_1 \sim m$, not only at values of λ_1 near unity. Equation (30) is not changed, but the applicability condition becomes $\alpha \gg 1/m^2$.

The results of this section show that in the limit of large intergranule coupling constants the frequencies on a Cayley tree diffuse freely, and the diffusion coefficient is proportional to the coupling constant.

5. LOCALIZATION IN THE LIMIT $\alpha \ll 1$

We show below that at $\alpha < \alpha_c$, where α_c is a critical value, the diffusion disappears, and the system becomes an insulator. The most interesting case is the low-frequency limit $\omega \ll \Delta$. In this limit, Eqs. (17) and (18) and integral (21) simplify significantly in a local region. A simplification of this sort has proved very important in the problem of localization in wires.[15] As in Ref. 15, large values $\lambda_1 \sim \Delta/\omega$ become important. In (17), (18), and (21) we should therefore take the limit of large λ_1 and λ_1' everywhere. Using the change of variables $z = \beta\lambda_1$ and going through some straightforward calculations, we replace (17), (18), and (21) by

$$\Psi(z) = \int_0^\infty \bar{L}\left(\frac{z'}{z}\right) \Psi^m(z') e^{-z'} \frac{dz'}{z'}, \tag{33}$$

where

$$\bar{L}(x) = \alpha\left(\frac{x}{2\pi}\right)^{1/2} \left\{\frac{d}{d\alpha}\left(\frac{\operatorname{sh}\alpha}{\alpha^{1/2}}\right) \exp\left[-\frac{\alpha}{2}\left(x+\frac{1}{x}\right)\right] - \frac{\operatorname{sh}\alpha}{\alpha^{1/2}} \frac{d}{d\alpha} \exp\left[-\frac{\alpha}{2}\left(x+\frac{1}{x}\right)\right]\right\},$$

$$\bar{K} = \frac{(2\pi\nu)^2}{-i\omega\bar{\varkappa}}, \qquad \bar{\varkappa}^{-1} = \frac{\Delta}{\pi}\int_0^\infty \Psi^{m+1}(z) e^{-z}\, dz. \tag{34}$$

Expressions (33) and (34) hold for arbitrary α in a local region and completely determine the problem of finding the integral dielectric constant $\bar{\varkappa}$. Expression (33) was derived under the assumption that the solution Ψ is a function of the variable $\beta\lambda_1$ at small values of β. Such a solution becomes the solution $\Psi = 1$ at $\beta = 0$. Consequently, the simplified equation (33) contains only solutions which become $\Psi = 1$ in the limit $\beta \to 0$. Solutions of the type (32), corresponding to diffusion, are not contained in Eq. (33). To find solutions of that type we need to use the exact equation (17), (18). In the sections which follow we determine the critical value α_c which separates the conducting region from the insulating region. At this point we carry out an expansion in the limit $\alpha \ll \alpha_c$. We write solution Ψ in the form

$$\Psi(z) = 1 + U_1(z) + U_2(z). \tag{35}$$

Here we have written the first terms of the expansion for $\alpha \ll 1$. Substituting (35) into (38), and equating terms of identical order, we find

$$U_1(z) = \int_0^\infty \bar{L}\left(\frac{z'}{z}\right) e^{-z'} \frac{dz'}{z'} - 1,$$

$$U_2(z) = m\int_0^\infty \bar{L}\left(\frac{z'}{z}\right) U_1(z) e^{-z'} \frac{dz'}{z'}. \tag{36}$$

Values $z \sim 1$ are important in the evaluation of integral (34). Carrying out the integration in (36) at $z \sim 1$, we find

$$U_1(z) = -\left(\frac{\alpha}{\alpha+2z}\right)^{1/2} z, \quad U_2(z) = -\frac{\alpha m}{4}\left(\frac{z}{\pi}\right)^{1/2} \ln\frac{1}{\alpha}. \tag{37}$$

Substituting (35) and (37) into (34), and integrating, we find the first terms of the expansion for $\bar{\varkappa}$:

$$\bar{\varkappa} = \frac{\pi}{\Delta}\left[1 + \frac{1}{2}(m+1)\left(\frac{\alpha\pi}{2}\right)^{1/2} + \frac{1}{8}\alpha m(m+1)\ln\frac{1}{\alpha}\right]. \tag{38}$$

With $\alpha = 0$, $\bar{\varkappa}$ corresponds to the integrated dielectric constant of an isolated granule. The functional dependence of $\bar{\varkappa}$ on α in the limit $\alpha \to 0$ is highly nonanalytic, making it impossible to expand the density correlation function in a series in α. This fact is seen formally in the appearance of the divergences mentioned in Sec. 2. With increasing α, the value $\bar{\varkappa}$ increases. Comparing the subsequent terms in expansion (38) with the preceding terms, we see that this expansion holds for $\alpha m^2 \ln(1/\alpha) \ll 1$ with $m \gtrsim 1$.

This expansion shows that at sufficiently small values of α the system is an insulator. Recalling the results of the preceding section—that the system conducts at large values of α—we conclude that a metal-insulator Anderson transition occurs.

6. GREEN'S FUNCTION OF THE INTEGRAL EQUATION

To analyze the transition from the conducting region into the localized region, we use the basic integral equation written in the form in Eqs. (22) and (23). Introducing

$$U = \frac{1-\Psi}{\lambda_1 - \lambda}, \quad \hat{\Gamma}\Phi(\lambda) = \int \Gamma(\mathbf{n}\mathbf{n}', \mathbf{n}_1\mathbf{n}_1')\,\Phi(\lambda)\frac{d\mathbf{n}'\, d\mathbf{n}_1'}{(2\pi)^2} \tag{39}$$

for arbitrary Φ, we transform Eq. (22) to

$$\{\hat{\Gamma}^{-1} - m\exp[\beta(\lambda-\lambda_1)]\}U = \frac{1-\exp[\beta(\lambda-\lambda_1)]}{\lambda_1-\lambda} - \exp[\beta(\lambda-\lambda_1)]F[U], \tag{40}$$

where

$$F[U] = \{[1-U(\lambda_1-\lambda)]^m + mU(\lambda_1-\lambda) - 1\}/(\lambda_1-\lambda)$$

contains only terms which are nonlinear in U.

Let us examine solutions Ψ which decay in the limit $\lambda_1 \to \infty$. Correspondingly, U should approach a constant value as $\lambda_1, \lambda \to 1$ [see (19a)] and approach the asymptote $U \sim 1/\lambda_1$ as $\lambda_1 \to \infty$. The function $F[U]$ has a very important property: For any $m > 1$, for functions U of the type described here, the condition $F[U] \geqslant 0$ holds.

We can now rewrite Eq. (40) in a final form convenient for analysis of the transition:

$$U(\lambda) = \int_{-1}^{1}\int_{1}^{\infty} G_\beta(\lambda, \lambda')\left\{\frac{1-\exp[\beta(\lambda'-\lambda_1')]}{\lambda_1'-\lambda'} - \exp[\beta(\lambda-\lambda_1')]\right\} \times F[U]\, d\lambda'\, d\lambda_1', \tag{41}$$

where $G_\beta(\lambda, \lambda')$ is the Green's function which satisfies the equation

$$\{\hat{\Gamma}^{-1}-m\exp[\beta(\lambda-\lambda_1)]\}G_\beta(\lambda, \lambda')=\delta(\lambda-\lambda')\delta(\lambda_1-\lambda_1'). \quad (42)$$

As we mentioned earlier, the density correlation function must be an analytic function of the frequency ω in the upper half-plane; this means it must be analytic with respect to β in the right-hand half-plane. The condition of analyticity is of assistance in choosing the appropriate solution of Eq. (41).

The function $G_\beta(\lambda, \lambda')$ can be expanded in the eigenfunctions Φ_E of the operator $\hat{\Gamma}^{-1} - m\exp[\beta(\lambda-\lambda_1)]$ in the usual way:

$$G_\beta(\lambda, \lambda')=\sum_E \frac{\Phi_E(\lambda)\Phi_E(\lambda')}{E}, \quad (43)$$

where the E are the eigenenergies.

Before we take up the properties of the solutions of Eq. (41), let us examine the analytic properties of Green's function G_β in (42); these properties are determined in turn by the singularities of the Green's function at zero frequency, G_0. With $\beta = 0$ in expansion (43), Φ_E and E are the eigenfunctions and eigenenergies of the operator $\hat{\Gamma}^{-1} - m$. Naturally, the functions Φ_E are now also eigenfunctions of the operator $\hat{\Gamma}$. Because of the particular form of the operator $\hat{\Gamma}$ in (23), we can write the eigenfunctions $\Phi_E(\lambda)$ in product form:

$$\Phi_E(\lambda)=\varphi_{E_0}(\lambda)\chi_{E_1}(\lambda_1). \quad (44)$$

The eigenfunctions φ_E in (44) are the Legendre polynomials $P_n(\lambda)$. Substituting into (23), and integrating, using the formulas for combining these polynomials, we test this assertion and evaluate the eigenvalues. In precisely the same way we can verify that the $\chi_E(\lambda_1)$ must also be spherical harmonics. The Legendre polynomials, however, are not suitable here, since the functions χ_E must decay in the limit $\lambda_1 \to \infty$. As a complete set of orthogonal spherical harmonics which decay in the limit $\lambda_1 \to \infty$, we choose the so-called cone functions, which are the Legendre functions P_ν with index $\nu = -1/2 + i\varepsilon$, where ε is a real number.[16] The most important properties—orthogonality and completeness—for these functions can be written

$$\int_1^\infty P_{-1/2+i\varepsilon}(z)P_{-1/2+i\varepsilon'}(z)\,dz=\frac{\delta(\varepsilon-\varepsilon')}{\varepsilon\,\mathrm{th}\,\pi\varepsilon},$$

$$\int_0^\infty P_{-1/2+i\varepsilon}(z)P_{-1/2+i\varepsilon}(z')\,\varepsilon\,\mathrm{th}\,\pi\varepsilon\,d\varepsilon=\delta(z-z'). \quad (45)$$

The functions $P_{-1/2+i\varepsilon}(z)$ are real if ε is real.

Equation (43) with $\beta = 0$ can now be rewritten as

$$G_0(\lambda, \lambda')=\sum_{n=0}^\infty \int_0^\infty G_{0n\varepsilon}P_n(\lambda)P_n(\lambda')P_{-1/2+i\varepsilon}(\lambda_1) \times P_{-1/2+i\varepsilon}(\lambda_1')\,\varepsilon\,\mathrm{th}\,\pi\varepsilon\,d\varepsilon, \quad (46)$$

where $G_{0n\varepsilon}^{-1} = \Gamma_{n\varepsilon}^{-1} - m$ are the eigenvalues of the operator $\hat{\Gamma}$. Using[16]

$$\int_1^\infty e^{-\alpha z}P_\nu(z)\,dz=\left(\frac{2}{\pi\alpha}\right)^{1/2}K_{\nu+1/2}(\alpha),$$

$$\int_{-1}^1 P_n(x)e^{\alpha x}\,dx=\left(\frac{2\pi}{\alpha}\right)^{1/2}I_{n+1/2}(\alpha),$$

where K and I are Bessel functions of imaginary argument, we find

$$\Gamma_{n\varepsilon}(\alpha)=\alpha\left[K_{i\varepsilon}(\alpha)\frac{d}{d\alpha}I_{n+1/2}(\alpha)-I_{n+1/2}(\alpha)\frac{d}{d\alpha}K_{i\varepsilon}(\alpha)\right]. \quad (47)$$

The function $G_0(\lambda, \lambda')$ is well defined if $G_{0n\varepsilon}^{-1}$ does not vanish at any ε. For any α, the function $\Gamma_{n\varepsilon}^{-1}$ is minimal for $n = 0$, $\varepsilon = 0$, and increases monotonically with increasing n and ε. Consequently, $G_{0n\varepsilon}^{-1}$ is always positive if $\Gamma_{00}^{-1}(\alpha) \geqslant m$. The function $\Gamma_{00}^{-1}(\alpha)$ becomes arbitrarily large at small values of α. With increasing α, this function falls off monotonically. A singularity first appears in $G_{0n\varepsilon}$ at $n = 0$, $\varepsilon = 0$, where α reaches the critical value α_c determined by the condition

$$\Gamma_{00}^{-1}(\alpha_c)=m. \quad (48)$$

Singularities appear in $G_{0n\varepsilon}$ with $n \neq 0$ at larger values of α. In the limiting cases of large values of m and of values of m near unity we can find an analytic solution of Eq. (48). For $m \sim 1$, the solution can be found only be numerical methods. The results are

$$\alpha_c=\frac{1}{8(m-1)}\quad(m\to1),\qquad \frac{\alpha_c}{2\pi}\ln^2\frac{2}{\alpha_c}=\frac{1}{m^2}\quad(m\gg1),$$

$$\alpha_c=6.803\cdot10^{-2}\quad(m=2),\qquad \alpha_c=1.884\cdot10^{-2}\quad(m=3). \quad (49)$$

At $m = 1$, no singularity appears in $G_{0n\varepsilon}$ or on the right side of Eq. (46).

In the limit of small values of ε and for $\alpha \ll \alpha_c$, Green's function $G_{00\varepsilon}$ takes the simple form

$$G_{00\varepsilon}=a/(b\varepsilon^2+\tau),\qquad \tau=(\alpha_c-\alpha)/\alpha_c, \quad (50)$$

where a and b are coefficients which depend on m. The Green's function $G_{00\varepsilon}$ in (50) has the customary form for Green's functions corresponding to critical modes in phase-transition theory. In contrast with phase-transition theory, however, divergences do not appear in the physical quantities in the limit $\tau \to 0$. The reason is that the right side of (46) has a factor $\varepsilon\tanh\pi\varepsilon$. Because of this factor, the function $G_0(\lambda, \lambda')$ remains finite as $\tau \to 0$. Consequently, divergences should not appear in the solution U of Eq. (41) or in the dielectric constant $\bar{\varkappa}$ in this limit. Despite the absence of divergences from the physical quantities as $\alpha \to \alpha_c$, the point α_c is the boundary of the insulating region, as will be shown below.

We now return to the analytic properties of the function $G_\beta(\lambda, \lambda')$. These properties depend strongly on the spectrum of eigenenergies E in expansion (43). As we have already shown, for each α there exists a minimum eigenvalue $E_{\min}$ of the operator $\hat{\Gamma}^{-1}-m$. The operator $\hat{\Gamma}^{-1} - m\exp[\beta(\lambda-\lambda_1)]$ is found from the operator $\hat{\Gamma}^{-1}-m$

by adding the function $V = m\{1 - \exp[\beta(\lambda - \lambda_1)]\}$. This function is positive for positive real β. The minimum eigenvalue $E_{\min}(\beta)$ of the operator $\hat{\Gamma}^{-1} - m\exp[\beta(\lambda - \lambda_1)]$ therefore increases monotonically from the value $E_{\min}$ at $\beta = 0$ to the value $E_{\min} + m$ in the limit $\beta \to \infty$. The value $E_{\min} + m$ is the same as the minimum eigenvalue of the operator $\hat{\Gamma}^{-1}$ and is positive for arbitrary α. The energy spectrum remains continuous at $\beta \neq 0$.

At $\alpha < \alpha_c$ the value of $E_{\min}$ is positive. We then conclude that $E_{\min}(\beta)$ is positive for all values of β on the real semiaxis, $\beta > 0$. In this case, for any values of β in the right half-plane, the path of the integration in (43) does not intersect a singularity of the integrand. Correspondingly, for $\alpha < \alpha_c$ the function $G\beta(\lambda, \lambda')$ is analytic in the right half plane. This analyticity is preserved at $\alpha = \alpha_c$.

If, on the other hand, we have $\alpha > \alpha_c$, then $E_{\min} < 0$. In the limit $\beta \to \infty$, however, $E_{\min}(\beta)$ increases to $E_{\min} + m > 0$. Consequently, at some real $\beta_c > 0$ the quantity $E_{\min}(\beta)$ vanishes. At this value of β, the integration path in Eq. (43) goes through a singularity for the first time. This result means that the analyticity in β is violated. For $\operatorname{Re}\beta < \beta_c$, the limit of the function G_β as $\operatorname{Im}\beta \to +0$ is different from the limit as $\operatorname{Im}\beta \to -0$. The imaginary part, G''_β, naturally has different limits; for the real part, the limits are identical. As $\beta \to 0$ with $\alpha > \alpha_c$, the imaginary part G''_β does not disappear. It can be written

$$G_0'' = c\,\operatorname{sgn}(\operatorname{Im}\beta)\,P_{-1/2+i\sigma}(\lambda_1)\,P_{-1/2+i\sigma}(\lambda_1'), \qquad (51)$$

where c is a numerical factor. The index σ is the root of the equation $\Gamma_{0\sigma}^{-1}(\alpha) = m$.

Consequently, for $\alpha > \alpha_c$ the function G_β is analytic in the region $\operatorname{Re}\beta > \beta_c$, where β_c is the root of the equation $E_{\min}(\beta_c) = 0$, but it has a cut along the line segment $[0, \beta_c]$. At $\alpha = \alpha_c$ we have $\beta_c = 0$; at $\alpha \leqslant \alpha_c$, the function G_β is analytic over the entire half-plane $\operatorname{Re}\beta > 0$.

7. THE METAL-INSULATOR TRANSITION

The singularity α_c of the Green's function G_β in Eqs. (47) and (48) is the point of the metal-insulator transition. At $\alpha < \alpha_c$, the system is in a localized state, while at $\alpha > \alpha_c$ it is conducting. This assertion follows from an analysis of the solutions of Eq. (41), to which we now turn.

To find U at $\alpha < \alpha_c$ we expand Eq. (41), treating $F[U]$ as a perturbation. We then seek a solution U in the form of a series each term of which is a function of $\lambda, \lambda_1, \beta$, and α. All the terms of the series for $\alpha < \alpha_c$ are analytic in β in the right-hand half-plane—a consequence of the analyticity of the Green's function G_β. The resulting series clearly converges at large $|\beta|$. In this limit we have $U = 1/(\lambda_1 - \lambda)$. The ratio of the terms of the series to the preceding terms increases with decreasing $|\beta|$. But at small values of $|\beta|$ the series converges if $\alpha < \alpha_c$. That this is true can be seen most simply by examining Eq. (33), derived in the limit $\beta \to 0$. If the series had begun to diverge at some α, this circumstance would have given rise to a singularity in $\partial\Psi/\partial\alpha$. For the function $\partial\Psi/\partial\alpha$ we easily find the following equation with the help of (33):

$$\hat{\Gamma}_0 \frac{\partial\Psi}{\partial\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha} \bar{L}\left(\frac{z'}{z}\right) e^{-z'}\,\Psi^m(z')\frac{dz'}{z'}, \qquad (52)$$

where

$$\hat{\Gamma}_0\Phi(z) = \Phi(z) - m\int_0^\infty \bar{L}\left(\frac{z'}{z}\right) e^{-z'}\,\Psi^{m-1}(z')\,\Phi(z')\frac{dz'}{z'}.$$

The solution of $\partial\Psi/\partial\alpha$ can become singular only if zero eigenvalues appear for the operator $\hat{\Gamma}_0$. However, the operator $\hat{\Gamma}_0$ acquires zero eigenvalues no sooner than the operator $\hat{\Gamma}_1$ does:

$$\hat{\Gamma}_1\Phi(z) = \Phi(z) - m\int_0^\infty \bar{L}\left(\frac{z'}{z}\right)\Phi(z')\frac{dz'}{z'},$$

since the kernel $\bar{L}$ is positive definite, and $\Psi(z) < 1$. A zero eigenvalue of the operator $\hat{\Gamma}_1$ first appears at $\alpha = \alpha_c$, where α_c is determined by (47) and (48). At $\alpha < \alpha_c$, therefore, no singularities can appear in $\partial\Psi/\partial\alpha$. It follows that the function Ψ found through the solution of Eq. (41) by successive approximations is analytic with respect to β in the right half-plane if $\alpha < \alpha_c$. This function decays as $\lambda_1 \to \infty$. Consequently, $\bar{\varkappa}$ is also analytic.

At small values of β we can use Eq. (33). For $\alpha \ll \alpha_c$, this equation has already been analyzed in the preceding section. At $\alpha \sim \alpha_c$, numerical methods must be used to solve it. Numerical calculations have been carried out for $m = 2$ and $m = 3$. Equation (33) was solved by an iterative method using the recurrence scheme

$$\Psi_{n+1}(z) = \int_0^\infty \bar{L}\left(\frac{z}{z'}\right) e^{-z'}\,\Psi_n^m(z')\frac{dz'}{z'}. \qquad (53)$$

The integral in (34) was then evaluated. A check verified that the function $\Psi(z)$ falls off monotonically from 1 to 0. Figure 2 shows the dielectric constant $\bar{\varkappa}$ as a function of α. We see that $\bar{\varkappa}$ remains finite. This result confirms the assertion which was made on the basis of qualitative arguments. As we approach the critical point α_c the number of iterations required to find $\bar{\varkappa}$ with some specified accuracy increases by orders of magnitude. In the region $\alpha > \alpha_c$, the iterations diverge. This numerical analysis confirms that a critical point exists and that the dielectric constant is finite at this point.

The quantity $\bar{\varkappa}(\alpha_c)$ can be calculated analytically if we formally take the limit $m \to 1$. According to (48), α_c is large in this limit. At large values of α, the kernel $\bar{L}(z'/z)$ in Eq. (43) has a sharp maximum at $z = z'$. It thus becomes possible to reduce the integral equation (33) to a differential equation. Expanding the function $\Psi(z')$ in gradients, and retaining only the leading terms, we find

$$\frac{z^2}{2\alpha}\frac{d^2\Psi}{dz^2} - z\Psi + (m-1)\,\Psi\ln\Psi = 0. \qquad (54)$$

This equation has been elsewhere derived for the case $m = 1$ in the problem of localization in wires.[15,11] For $m \neq 1$ and arbitrary α, the nonlinear equation (54) cannot be solved analytically, except in the case $\alpha = \alpha_c = 1/8(m-1)^{-1}$. At this value of α, the solution is

$$\Psi = \exp(-2\sqrt{2\alpha z}). \qquad (55)$$

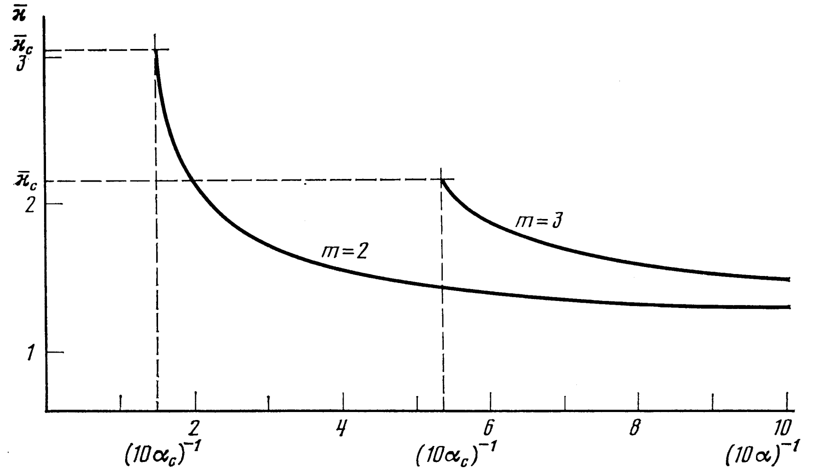

FIG. 2. The integral dielectric constant $\bar{\varkappa}$ as a function of α^{-1}, the reciprocal of the reduced probability for tunneling between granules.

Substituting Eq. (55) into Eq. (34), and integrating, we find

$$\bar{\varkappa}(\alpha_c)=\frac{\pi}{\Delta}\frac{2}{m-1}. \tag{56}$$

Our study shows that at $\alpha<\alpha_c$ the system is an insulator, and the dielectric constant $\bar{\varkappa}$ remains finite as $\alpha\to\alpha_c$. The solution found here cannot be used at $\alpha>\alpha_c$. If we attempt to seek a solution of Eq. (41) in series form, treating $F[U]$ as a perturbation, we immediately find that each term in the series at $\alpha>\alpha_c$ is nonanalytic in β in the right half-plane. This conclusion follows immediately from the nonanalyticity of the Green's function G_β in Eq. (41). This result apparently is related to the absence of nonincreasing solutions of Eq. (33) for $\alpha>\alpha_c$. By analogy with phase-transition theory, where an order parameter appears spontaneously below the critical point, we suggest that for $\alpha>\alpha_c$ a solution U which does not vanish in the limit $\beta\to 0$ appears in Eq. (40) or (41). This solution was found in Section 4 for the region $\alpha\gg\alpha_c$ [Eq. (32)] in a study of diffusion. The solution Ψ falls to zero in this region if $\beta=0$. Correspondingly, U decreases from some constant C_0 [see (19a)], which is reached as λ_1, $\lambda\to 1$, to zero, having the asymptotic behavior $U\sim 1/\lambda_1$ as $\lambda_1\to\infty$. The existence of a solution Ψ which decays for $\beta=0$ means that the system is conducting. This conclusion follows from the convergence of the integral (21) with $\beta=0$; this integral is proportional to the reciprocal of the diffusion coefficient $\bar{D}$ (the solution Ψ must of course fall off quite rapidly if this convergence is to be achieved).

We will now prove, first, that a solution of the diffusion type can exist only for $\alpha>\alpha_c$. For this purpose we multiply both sides of Eq. (40) by the Legendre polynomial $P_\nu(\lambda_1)$, and we integrate over λ_1 and λ. In the limit $\beta=0$ we have

$$(\Gamma^{-1}_{0,-i(\nu+1/2)}(\alpha)-m)\,U_{0\nu}=-F_{0\nu}, \tag{57}$$

where

$$U_{0\nu}=\int_{-1}^{1}\int_{1}^{\infty}UP_\nu(\lambda_1)\,d\lambda\,d\lambda_1,\qquad F_{0\nu}=\int_{-1}^{1}\int_{1}^{\infty}F[U]P_\nu(\lambda_1)\,d\lambda\,d\lambda_1.$$

the quantity $\Gamma_{0,-i(\nu+1/2)}(\alpha)$ can be derived from (47) through the simple replacement $\varepsilon\to -i(\nu+1/2)$.

Let us assume that at $\alpha<\alpha_c$ there exist nonvanishing solutions U which have the asymptotic behavior $U\sim 1/\lambda_1$ as $\lambda_1\to\infty$. This situation corresponds to the existence of solutions Ψ which fall off to zero. Using (57) we easily see that this assumption is contradictory: At $\alpha<\alpha_c$ the quantity $\Gamma^{-1}_{0,-i(\nu+1/2)}(\alpha)-m$ is positive for $\nu=-1/2$. At some $\nu=\nu_c$, $-1/2<\nu_c<0$, this quantity vanishes. As ν increases further, it becomes negative, approaching a limit $1-m$ as $\nu\to 0$. The quantities $U_{0\nu}$ and $F_{0\nu}$ in Eq. (57) remain finite on the line segment $[0,\nu_c]$. However, $F_{0\nu}$ is strictly positive, since the functions $F[U]$ and P_ν are positive. It then follows from Eq. (57) that $U_{0\nu}$ is negative for $-1/2\leqslant\nu\leqslant\nu_c$ and tends toward infinity as $\nu\to\nu_c$. This result contradicts the assumption regarding the function U, since P_ν is a monotonically decreasing positive function with the asymptotic behavior $P_\nu(\lambda_1)\sim\lambda_1^{-\nu}$ as $\lambda_1\to\infty$ with $\nu>-1/2$. Consequently, a solution of the diffusion type exists only if $\alpha>\alpha_c$.

An explicit solution U cannot be found for arbitrary $\alpha>\alpha_c$. We will therefore analyze the properties of this solution at a qualitative level. Particularly interesting is the behavior of the function U (or Ψ) as $\alpha\to\alpha_c$, since this behavior determines how the diffusion coefficient vanishes. If the function Ψ fell off very slowly with increasing λ_1 toward the point of the transition the integral (21) would take on a large value. In this case the diffusion coefficient $\bar{D}$ would vanish. In the opposite case in which the relative change of the function Ψ is of order unity and does not approach infinity as $\alpha\to\alpha_c$, the diffusion coefficient $\bar{D}$ should tend toward a finite limit $\bar{D}_0$ and should vanish abruptly. Mott's arguments regarding the existence of a minimum metallic conductivity would then be correct.

We first examine the asymptotic behavior of the solution Ψ with $\beta=0$ as $\lambda_1\to\infty$, to verify that the integral (21) converges. In this limit Eqs. (17), (18) are rewritten

$$\Psi(\lambda_1)=\int_0^\infty \bar{L}(z)\,\Psi(\lambda,z)\frac{dz}{z}, \tag{58}$$

where $\bar{L}(z)$ is defined in (33). The solution $\Psi(\lambda_1)$ does not depend on λ at $\lambda_1\gg 1$. Equation (58) differs from Eq. (33) in that it lacks an exponential function in the integrand; this function arises only at finite frequencies. It is convenient to transform to the variables $\theta_1=\ln 2\lambda$, $t=\ln 2z$ [cf. (18)]. To show that the asymptotic behavior of the solution of Eq. (58)

is

$$\Psi(\lambda_1)\equiv\Psi(e^{\theta_1}/2)\sim\exp[-a(\theta_1^{1/2}+\gamma)\exp(b\theta_1^{1/2})]. \tag{59}$$

we substitute (59) into (58), transforming everywhere to the variables θ_1, t. The right side of (58) is then expressed in terms of the integral

$$B(\alpha)=\int_{-\infty}^{\infty}\exp\left\{-\left[\alpha\operatorname{ch}t+\frac{t}{2}+am(\theta_1^{1/2}+\gamma)\exp(b\theta_1^{1/2})\right]\right\}dt. \tag{60}$$

At large $\theta_1\gg1$ the integral $B(\alpha)$ can be evaluated by the method of steepest descent. We need to minimize the expression in the exponential function in the usual way and to find its value at the minimum. All the calculations are carried out to the accuracy required for writing terms of order $\theta_1^{1/2}\exp(b\theta_1^{1/2})$ and $\exp(b\theta_1^{1/2})$ in the exponential function. We will not calculate the coefficients of the exponential function. After some straightforward transformations we find that the form of the function in Eq. (59) is reproduced. Comparing the coefficients of $\theta_1^{1/2}\exp(b\theta_1^{1/2})$ and $\exp(b\theta_1^{1/2})$ in the exponential functions on the right and left sides of Eq. (58), we find

$$a=\alpha(2m\ln m)^{-1/2},\quad b=(2\ln m)^{1/2}. \tag{61}$$

The number γ is arbitrary.

The form of the asymptotic behavior in Eqs. (59), (61) proves that the integral (21) converges for all α and for all $m>1$. Nevertheless, knowledge of the asymptotic behavior is not sufficient for evaluating the integral (21). The reason is that Eq. (58) is invariant under the change of scale $\lambda_1\to A\lambda_1$. Any function Ψ of the form

$$\Psi(\lambda_1)=\Psi_0(A\lambda_1), \tag{62}$$

where Ψ_0 is some solution of Eq. (58), and A is an arbitrary number, can also be a solution of this equation. The fact that the number γ in the asymptotic solution (59) is arbitrary is a consequence of this invariance. In principle, there might be the possibility that A would approach zero as the transition point α_c was approached. The integral in (21) might then increase without bound, causing the diffusion coefficient $\overline{D}$ to vanish. To determine the coefficient A we need to join the solution (62) with the solution for small λ_1. However, even without going through these complicated calculations, we can see that the coefficient A will always be of order unity and will not approach zero as $\alpha\to\alpha_c$.

Let us assume that $\Psi_0(\lambda_1)$ is some solution of Eq. (58) which varies by $\lambda_1\sim1$. As such a solution we might take, for example, the solution having the asymptotic behavior in (59), (61) with $\gamma=0$. The exact solution of Eq. (17), (18) with $\lambda_1\gg1$ is written in the form in (62). We assume that as $\alpha\to\alpha_c$ the parameter A approaches zero in (62); we see that this assumption is contradictory. Specifically, the function U defined in (39) takes the following form in the region $\lambda_1\gg1$:

$$U(\lambda_1)=AU_0(A\lambda_1),\quad U_0(\lambda_1)=[1-\Psi_0(\lambda_1)]/\lambda_1. \tag{63}$$

We now recall Eq. (57). In the case $A\ll1$, the integrals are dominated by the region $\lambda_1\sim1/A$. Substituting (63) into (57), and introducing the variable $z=A\lambda_1$, we can put (57) in the following form for $A\ll1$ and $\nu=-1/2$:

$$[m-\Gamma_{00}^{-1}(\alpha)]\int_0^{\infty}\frac{U_0(z)}{z^{1/2}}dz=\int_0^{\infty}\frac{F[U_0]dz}{z^{1/2}}. \tag{64}$$

In deriving (64) we used expression (40) for $F[U]$ and the asymptotic behavior $P_{-1/2}(z)\sim\ln z/z^{1/2}$ at $z\gg1$. The integrals in Eq. (64) converge at both large and small values of z. The typical interval for the variation of the function U_0, like that for $\Psi_0(z)$, is $z\sim1$. The integrals in Eq. (64) are thus on the of order unity. We have assumed, however, that α approaches α_c. In this case, $m-\Gamma_{00}^{-1}(\alpha)$ approaches zero. We thus have a contradiction: The left side of (64) is approaching zero, but the right side remains of order unity (we recall that $F[U_0]\geqslant0$). Consequently, the assumption that A tends toward zero as $\alpha\to\alpha_c$ is incorrect. The relative changes in the functions Ψ and U are of order unity, and both depend strongly on the parameters λ_1 and λ. Only in this case can we satisfy Eq. (57) with $\nu=-1/2$ and $\alpha\to\alpha_c$. The integral in (21) must also remain of order unity as $\alpha\to\alpha_c$. This assertion means that the diffusion coefficient $\overline{D}$ tends toward a finite value as the transition point is approached, demonstrating the existence of a minimum metallic conductivity.

8. CONCLUSION

The exact solution of the model on a Cayley tree given above can be used to describe the Anderson metal-insulator transition. We have derived expressions for the total diffusion coefficient in the conducting region and for the dielectric constant in the localized region. We have asserted that the conductivity tends toward a finite value as the transition point is approached. As the transition point is approached from the side of the insulator, the dielectric constant increases, but it remains finite even at the transition point.

Actually, we have calculated not the diffusion coefficient and the susceptibility themselves but some integral quantities which determine the density correlation function for coincident points. To calculate the actual diffusion coefficient we need to calculate the density correlation function for different points. However, the critical behavior which has been found here is a consequence of an abrupt change in the solution Ψ at the transition point. The same function Ψ determines the behavior of the density correlation function at different points, $K(r,r')$ [see (14)]. The correlation function $K(r,r')$ is equal to a finite-dimensional integral of a combination containing Ψ. Accordingly, discontinuities should also occur in the correlation function $K(r,r')$ and thus in the diffusion coefficient. For the same reason, the actual dielectric constant does not become infinite at the transition point. The limitation imposed because we have considered only systems which have a magnetic interaction (model II) is not important. The results which would be obtained for model I (a system with only potential scattering) and for model III (potential and spin-orbit scattering) can be expected to be qualitatively the same.

The conclusion that there exists a minimum metallic conductivity agrees with Mott's assertion[2] but differs from the results of Refs. 3–6, 9, 10, and 17. This is apparently the first assertion of the existence of a maximum dielectric constant. We believe that the existence of a minimum metallic conductivity and of a maximum dielectric constant is not a consequence of the particular features of the model on a Cayley tree. Let us examine some possible reasons for the discrepancy between the results of the present paper and those of Refs. 3–6, 9, 10, and 17.

Götze[17] used a nonrigorous approximation, which may have qualitatively changed the results.

Kunz and Souillard[10] used the Anderson model on a Cayley tree to calculate a correlation length which diverged in a power-law fashion as the transition point was approached. They linked this length with the conductivity on the basis of dimensionality considerations, which were not proved.[10] Critical modes of some sort also exist in the theory set forth in the preceding sections [see (50)]. However, these critical modes do not make a divergent contribution because of the small phase volume. The existence of a diverging length is thus not by itself proof that the conductivity tends toward zero.

Shapiro[9] also examined a model on a Cayley tree. Shapiro's derivation uses some rather strong assumptions. For example, it is assumed that the unaveraged coefficient for reflection from an arbitrary piece of a Cayley tree is approximately unity near the transition point and fluctuates only slightly. Shapiro uses averaging methods whose validity has not been proved. Furthermore, it is assumed that the resistivity behaves in the same way as the total resistances of the entire tree.

The concept of scaling, initially proposed by Thouless[18] for one-dimensional systems, and subsequently generalized to arbitrary dimensionality,[3] has had a major influence on the present understanding. For one-dimensional systems, this concept has led to the prediction of localization for any, arbitrarily slight disorder. An exact calculation shows that the one-dimensional geometry does in fact always lead to localization. This result confirms the applicability of scaling for the description of one-dimensional systems. The results predicted by scaling[3] for a space of dimensionality $2+\varepsilon$ agree with the perturbation-theory results[19,20] and the results found by renormalization-group methods.[4–6] In the scheme proposed by Abrahams *et al.*,[3] there is no minimum metallic conductivity in a space with any dimensionality d. If we assume that models on a Cayley tree correspond to the limit $d\to\infty$, we conclude that the result found above contradicts the scaling hypothesis. However, we are not completely convinced even by the results found by the renormalization-group method in a space of dimensionality $2+\varepsilon$. Our reasoning is that the noncompact σ model used to describe disordered metals differs from ordinary compact σ models. In the calculations, large values of λ_1 turn out to be important. Actually, all the results for the localized region are derived in the limit $\lambda_1 \gg 1$. It is the noncompact nature of the group of matrices Q which is associated with the existence of a minimum metallic conductivity. On the other hand, studies by perturbation theory or by the renormalization-group method consider only the contribution from the regions of λ and λ_1 near unity.

A perturbation theory can also be constructed for Eqs. (17), (18). The specific features of the noncompact group will of course not be seen in any way in the first terms of the expansion. These features may, however, cause an increase in the coefficients in the higher orders of perturbation theory, rendering this theory inapplicable in many regards. Furthermore, in the renormalization-group method in the case of compact σ models the reproducibility of the Lagrangian upon renormalizations is guaranteed by the circumstance that $\mathrm{Tr}(\nabla Q)^2$ is the only possible combination because of the absence of any special direction in the space of the matrices Q. In a noncompact σ model, a special direction of Λ arises, since the term $i\omega S\,\mathrm{Tr}\Lambda Q$ serves as a cutoff for the integrals which may diverge at $\omega=0$. Accordingly, we do not rule out the possibility that additional terms of the type $\gamma_1\mathrm{Tr}(\Lambda\nabla Q)^2$ or $\gamma_2(\mathrm{Tr}\Lambda\nabla Q)^2$ will arise upon renormalizations. The quantities γ_1 and γ_2 would of course have to be exponentially small with respect to the nucleating charge, since these additional terms do not arise in perturbation theory; in the region of large effective charges, however, these terms may become large. Kravtsov and Lerner[21] have recently asserted that the use of a single-parameter renormalization group leads to contradictions. In summary, we believe that at present there are no rigorous results which deny the existence of a minimum metallic conductivity or of a maximum dielectric constant.

In analyzing the experimental situation one must bear in mind the existence of large-scale inhomogeneities, which can be analyzed by perturbation theory. Such inhomogeneities are not dealt with in the theory of localization. Strictly speaking, the minimum metallic conductivity and the maximum dielectric constant can be observed only in systems in which there are no large-scale inhomogeneities. The incorporation of such inhomogeneities in a percolation theory should lead to a power-law decay of the conductivity near the transition point and to a power-law increase in the dielectric constant. Mott and Kaveh[22] believe that the experiments which have already been carried out furnish evidence for the existence of a minimum metallic conductivity, although differing interpretations are possible (see Ref. 23, for example).

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