Instability of a periodic chain of two-dimensional solitons

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We consider in this paper the Kadomtsev-Petviashvili (KP) equation for media with a decay-type dispersion law. We show that the exact solution of this equation which is a periodic chain of two-dimensional solitons is unstable against small perturbations. We use the inverse scattering method to construct a perturbation mode which increases with time. We find the maximum growth rate of the perturbation mode.

INTRODUCTION

We consider the Kadomtsev-Petviashvili (KP) equation

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\frac{1}{4}\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}u\frac{\partial u}{\partial x}\right]+\frac{3}{4}\beta^2\frac{\partial^2 u}{\partial y^2}=0, \quad (1)$$

where $\beta^2 = \pm 1$, u is a scalar wave field. Equation (1) describes weakly nonlinear waves in media with a weak dispersion and is a quasi-two-dimensional generalization of the well known Korteweg–de Vries (KdV) equation. The quasi-two-dimensionality consists in the fact that the transverse scale (along the y-axis) of the wave motions is much larger than the longitudinal scale (along the x-axis). The case $\beta^2 = -1$ corresponds to media with a decay-type dispersion law and $\beta^2 = 1$ to media with a decayless dispersion law. The properties of Eq. (1) with $\beta^2 = -1$ differ qualitatively from the properties of the equation with $\beta^2 = 1$, and the first one is therefore customarily called the KP I equation and the second the KP II equation. The KP I equation occurs when one describes capillary gravitational waves on a liquid surface, when one considers magnetoacoustic waves in a plasma, and so on. The KP II equation is, for instance, applied in the theory of long-wavelength gravitational waves on shallow water. Kadomtsev and Petviashvili[1] were the first to obtain Eq. (1). They were interested in studying the problem of the stability of a plane KdV soliton against small two-dimensional perturbations which are long in the transverse direction. They showed in Ref. 1 that in media with a decay-type dispersion law the soliton is unstable, but in media with a non-decay dispersion law it is stable. The interest in Eq. (1) renewed appreciably after it was shown in Ref. 2 (see also Ref. 3) that the inverse scattering method was applicable to Eq. (1). The procedure which was proposed in Ref. 2 for simultaneously constructing integrable equations and their exact solutions was subsequently called the "covering method". Up to the present time a wide class of exact solutions has been constructed for the KP I and KP II equations (see, e.g., Refs. 2,4,5). It has been ascertained that only the KP I equation has two-dimensional localized solutions (two-dimensional solitons). Using direct methods recently in Refs. 6,7 a new solution of the KP I equation has been constructed—a periodic chain of two-dimensional solitons. In Ref. 7 also a hypothesis was expressed that this solution is the result of the development of the instability of a plane soliton of the KP I equation (see Ref. 1).

It is the aim of the present paper to study the stability of the new solution using the covering method. The possibility of applying the inverse scattering method to the KP I equation enables us to solve the problem exactly. We ascertain that the periodic chain of two-dimensional solitons is unstable against small perturbations. The hypothesis in Ref. 7 is thus false, as the process of evolution of one solution of the KP I equation—the plane soliton—cannot asymptotically lead to the state described by another unstable solution—the periodic chain of two-dimensional solitons. The result obtained is unexpected as a single two-dimensional soliton taken separately is stable (see Ref. 8) against two-dimensional perturbations. In § 1 we expound the method used. In § 2 we give the solution of the simplest particular case of the problem (we consider a periodic chain moving along the longitudinal x-axis). The axis of the chain is the same as the y-axis. We shall also represent in § 2 the results of the solution of the general case of the problem: we investigate the stability of an arbitrary periodic chain of two-dimensional solitons.

§ 1. METHOD FOR CONSTRUCTING SOLUTIONS OF THE KP I EQUATION

The solution considered depends in the general case on four real parameters which determine the velocity, slope of the axis and period of the chain of two-dimensional solitons (the initial phase of the chain is assumed to equal zero). For greater clarity we consider initially the simpler special case of the problem in which the axis of the chain is the same as the y-axis and the chain itself moves along the x-axis. In this situation the problem depends solely on two real parameters η_1 and η_2:

$$u_0(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\Delta_0(x,y,t), \quad (2)$$

where

$$\Delta_0(x,y,t) =1-2\cos[(\eta_2^2-\eta_1^2)y]\exp[(\eta_1+\eta_2)x-(\eta_1^3+\eta_2^3)t] -\frac{(\eta_1-\eta_2)^2}{4\eta_1\eta_2}\exp[2(\eta_1+\eta_2)x-2(\eta_1^3+\eta_2^3)t]. \quad (3)$$

From the condition that the solution be regular it follows that

$$\eta_1\eta_2<0. \quad (4)$$

Let

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$$\eta_1>0, \quad \eta_2<0, \quad \eta_1+\eta_2<0. \tag{5}$$

If we let the period of the chain $\tau = 2\pi/(\eta_2^2 - \eta_1^2)$ tend to infinity using the limit

$$\eta_1=\eta, \quad \eta_2=-\eta+\varepsilon, \quad \varepsilon\to 0, \tag{6}$$

we get a single two-dimensional soliton:

$$u_0(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\left\{\frac{1}{4\eta^2}+4\eta^2y^2+(x-3\eta^2t)^2\right\}. \tag{7}$$

Recently the covering method was refined in Ref. 9 (for details see Ref. 10) by applying it to a problem of complex analysis—the (nonlocal) $\bar\partial$-problem. ($\bar\partial = \partial/\partial\bar k$, where k is a complex variable and the bar over the k indicates the complex conjugate.) The (local) $\bar\partial$-problem was first used in Ref. 11. We expound briefly this new method for constructing exact solutions of the KP I equation.

We take the function

$$T(k_1,k,x,y,t) = \tilde T(k_1,k)\exp[i(k_1-k)x+i(k^2-k_1^2)y+i(k_1^3-k^3)t]$$

and, for all x, y, and t, we solve the equation

$$\frac{\partial}{\partial\bar k}\chi(k,x,y,t)=\frac{1}{2i}\iint T(k_1,k,x,y,t)\chi(k_1,x,y,t)\,dk_1\,d\bar k_1, \tag{8}$$

where $dk_1 d\bar k_1 = d(\mathrm{Re}\,k_1)d(\mathrm{Im}\,k_1)$. Let

$$\chi(k,x,y,t)=1+\frac{\chi_1(x,y,t)}{k}+O\left(\frac{1}{k^2}\right), \quad |k|\to\infty.$$

The quantity

$$u(x,y,t)=\frac{2}{i}\frac{\partial}{\partial x}\chi_1(x,y,t)$$

is then a solution of the KP I equation. The condition that the function $u(x,y,t)$ be real imposes a restriction on the form of the function $\tilde T$:

$$\tilde T(k_1,k)=\bar{\tilde T}(\bar k,\bar k_1). \tag{9}$$

In particular, the periodic chain (2),(3) is obtained as follows:

$$T(k_1,k,x,y,t)=T_0(k_1,k,x,y,t) =r(x,y,t)\delta(k_1+i\eta_2)\delta(k-i\eta_1) +\bar r(x,y,t)\delta(k_1+i\eta_1)\delta(k-i\eta_2),$$

$$\chi(k,x,y,t)=\chi_0(k,x,y,t)=1+\frac{\chi_{01}(x,y,t)}{k-i\eta_1}+\frac{\chi_{02}(x,y,t)}{k-i\eta_2},$$

where the functions r, χ_{01}, and χ_{02} have the form

$$r=2\pi(\eta_1+\eta_2)\exp[(\eta_1+\eta_2)x+i(\eta_2^2-\eta_1^2)y-(\eta_1^3+\eta_2^3)t],$$

$$\chi_{01}=\frac{1}{\Delta_0}\Big\{-i(\eta_1+\eta_2)\times\exp[(\eta_1+\eta_2)x+i(\eta_2^2-\eta_1^2)y-(\eta_1^3+\eta_2^3)t] -\frac{i}{2\eta_2}(\eta_1^2-\eta_2^2)\exp[2(\eta_1+\eta_2)x-2(\eta_1^3+\eta_2^3)t]\Big\},$$

$$\chi_{02}=\frac{1}{\Delta_0}\Big\{-i(\eta_1+\eta_2)\times\exp[(\eta_1+\eta_2)x+i(\eta_1^2-\eta_2^2)y-(\eta_1^3+\eta_2^3)t] -\frac{i}{2\eta_1}(\eta_2^2-\eta_1^2)\exp[2(\eta_1+\eta_2)x-2(\eta_1^3+\eta_2^3)t]\Big\},$$

$$\delta(k)=\delta(\mathrm{Re}\,k,\mathrm{Im}\,k).$$

The following condition is then satisfied:

$$\frac{\partial}{\partial x}\ln\Delta_0=\frac{1}{i}(\chi_{01}+\chi_{02}).$$

We now proceed to construct a perturbation mode u_1 of the ground state u_0 which increases with time:

$$u=u_0+u_1, \quad u_1\ll u_0,$$

where the function u_1 is described by the KP I equation, linearized on the background u_0:

$$\frac{\partial}{\partial x}\left[\frac{\partial u_1}{\partial t}-v\frac{\partial u_1}{\partial x}+\frac{1}{4}\frac{\partial^3u_1}{\partial x^3}+\frac{3}{2}\frac{\partial}{\partial x}(u_0u_1)\right]-\frac{3}{4}\frac{\partial^2u_1}{\partial y^2}=0. \tag{10}$$

The treatment which follows is carried out in a coordinate system fixed to the chain moving with a velocity

$$v=\eta_1^2-\eta_1\eta_2+\eta_2^2. \tag{11}$$

To construct solutions of Eq. (10) we use a method first proposed in Ref. 4. One must linearize the basic Eq. (8) about the background of the ground state:

$$T=T_0+T_1, \quad \chi=\chi_0+\chi_1,$$

where

$$\frac{\partial}{\partial\bar k}\chi_1=\frac{1}{2i}\iint(T_1\chi_0+T_0\chi_1)\,dk_1\,d\bar k_1. \tag{12}$$

The linearized Eq. (12) is obtained from (8) when we substitute instead of the scalar functions χ and T the triangular matrices $\begin{bmatrix} a_0 & a_1 \\ 0 & a_2 \end{bmatrix}$.

§ 2. BENDING INSTABILITY OF A PERIODIC CHAIN OF TWO-DIMENSIONAL SOLITONS

The growing perturbation mode is constructed as follows. Let

$$T_1(k_1,k,x,y,t)=f_0\delta(k_1-is)\delta(k-i\eta_2)\exp[\theta(s,x,y,t)], \tag{13}$$

where f_0 is an arbitrary complex constant, s an arbitrary real constant, and

$$\theta(s,x,y,t)=(\eta_2-s)x+i(s^2-\eta_2^2)y+[s^3-\eta_2^3+v(\eta_2-s)]t. \tag{14}$$

We look for a solution of Eq. (12) in the form

$$\chi_1=\chi_{11}/(k-i\eta_1)+\chi_{12}/(k-i\eta_2).$$

The condition

$$u_1(x,y,t)=\frac{2}{i}\frac{\partial}{\partial x}\{\chi_{11}+\chi_{12}\}. \tag{15}$$

is then satisfied. As the function T_1 does not satisfy the reality condition (9), the perturbation (15) will be a complex function. Splitting off either the real or the imaginary parts of u_1 we get a real perturbation which is a solution of Eq. (10) by virtue of the linearity of Eq. (10) and the reality of the function u_0.

Using the following relation from the theory of generalized functions (see Ref. 12):

$$\frac{\partial}{\partial\bar k}\frac{1}{k-k_0}=\pi\delta(k-k_0),$$

we get a set of two linear algebraic equations for the unknown functions χ_{11} and χ_{12}. Solving this set we have

$$\frac{1}{i}(\chi_{11}+\chi_{12})$$
$$=-\frac{1}{2\pi}\frac{1}{\Delta_0}\left[f_0\chi_0(k=is)e^{\theta}+\frac{f_0}{4\pi\eta_2}\frac{\eta_1-\eta_2}{\eta_1+\eta_2}r\chi_0(k=is)e^{\theta}\right]. \tag{16}$$

The conditions that $u_1(x,y,t)$ decrease as $|x|\to\infty$ and that it be finite on the axis of the chain enable us to fix the real parameter s within the range:

$$\eta_2<s<-\eta_1. \tag{17}$$

Here

$$u_1(x,y,t)\propto\exp[(\eta_2-s)x]\to 0,\quad x\to+\infty,$$
$$u_1(x,y,t)\propto\exp[-(\eta_1+s)x]\to 0,\quad x\to-\infty.$$

The growth rate γ of the perturbation mode which we have constructed equals

$$\gamma(s)=(s-\eta_2)(s+\eta_1)(s+\eta_2-\eta_1). \tag{18}$$

The maximum of the growth rate is reached in the point

$$s_0=-3^{-1/2}(\eta_1^2-\eta_1\eta_2+\eta_2^2)^{1/2}, \tag{19}$$

which lies within the range $(\eta_2,-\eta_1)$. If $\eta_1=0$, we have

$$\max_s \gamma=2\cdot 3^{-3/2}v^{3/2}(\eta_1=0,\eta_2)=2\cdot 3^{-3/2}|\eta_2|^3,$$

where the velocity v of the chain is given by Eq. (11).

Setting $s\approx\eta_2$ and $f_0=4\pi\eta_2(\eta_1+\eta_2)(s-\eta_2)$, in (16), we get

$$u_1(x,y,t)\approx\exp[\theta(s,x,y,t)]\left[-\eta_2\frac{\partial u_0}{\partial x}-i\frac{\eta_2}{\eta_2-\eta_1}\frac{\partial u_0}{\partial y}\right], \tag{20}$$

where the function θ is given in (14). The first term on the right-hand side of (20) corresponds to the fact that sections of the chain at right angles to the axis are stretched in opposite directions, while the second term corresponds to compression and extension of these parts in the longitudinal direction along the axis of the chain. This scheme enables us to construct various perturbations, among which are some against which the periodic chain of two-dimensional solitons is stable. Giving

$$T_1(k_1,k,x,y,t)=f_0\delta(k_1+is)\delta(k-i\eta_1)\exp[\tilde\theta(s,x,y,t)],$$

where

$$\tilde\theta(s,x,y,t)=(\eta_1+s)x+i(s^2-\eta_1^2)y-[s^3+\eta_1^3-v(\eta_1+s)]t,$$

we can construct yet another perturbation mode. One can show that the chain is stable against this mode—the perturbation is damped with a damping rate

$$\delta(s)=-\gamma(s),$$

where $\gamma(s)$ is given by (18). As before the parameter s if fixed in the range $(\eta_2,-\eta_1)$.

Using the results obtained we consider the case of a single two-dimensional soliton, (7). To do this we take the limit (6) in Eq. (16). First putting $f_0=2\pi(\eta_1+\eta_2)(s^2-\eta_2^2)$, we get

$$u_1(x,y,t)=2\frac{\partial}{\partial x}L(x,y,t), \tag{21}$$

where

$$L(x,y,t)=\frac{1/2\eta+x-2i\eta y}{1/4\eta^2+4\eta^2y^2+x^2}\left[s^2-\eta^2+\frac{1+4i\eta^2y+2sx}{1/4\eta^2+4\eta^2y^2+x^2}\right]$$
$$\cdot\exp[-(\eta+s)x+i(s^2-\eta^2)y+(s^3-3\eta^2s-2\eta^3)t]. \tag{22}$$

It is clear from Eq. (22) that the simultaneous decrease of the function $u_1(x,y,t)$ as $x\to\pm\infty$ can be obtained provided $s=-\eta$. The function $u_1(x,y,t)$ of (21),(22) then degenerates into a superposition of shifted modes of the two-dimensional soliton:

$$u_1(x,y,t)=\eta\frac{\partial u_0(x,y)}{\partial x}-\frac{i}{2}\frac{\partial u_0(x,y)}{\partial y}. \tag{23}$$

Each term in (23) is an exact solution of Eq. (10).

We now consider an arbitrary chain of two-dimensional solitons: the velocity, slope of the axis, and period are arbitrary. The chain then depends on four real parameters: ξ_1, ξ_2, η_1, and η_2:

$$u_0(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\Delta_0(x,y,t), \tag{24}$$

where

$$\Delta_0=1-2\cos(\Phi_2)e^{\Phi_1}+\left[1+\frac{|\bar\varkappa_2-\varkappa_1|^2}{(\bar\varkappa_2-\varkappa_2)(\bar\varkappa_1-\varkappa_1)}\right]e^{2\Phi_1}, \tag{25}$$

$$\Phi_1(x,y,t)$$
$$=(\eta_1+\eta_2)x-2(\xi_1\eta_1+\xi_2\eta_2)y+(3\xi_2^2\eta_2-\eta_2^3+3\xi_1^2\eta_1-\eta_1^3)t, \tag{26}$$

$$\Phi_2(x,y,t)=(\xi_2-\xi_1)x+(\xi_1^2-\eta_1^2+\eta_2^2-\xi_2^2)y+$$
$$+(\xi_2^3-3\xi_2\eta_2^2+3\xi_1\eta_1^2-\xi_1^3)t, \tag{27}$$

$$\varkappa_{1,2}=\xi_{1,2}+i\eta_{1,2}. \tag{28}$$

The parameters η_1 and η_2 must satisfy the regularity condition (4). Let η_1 and η_2 satisfy Eqs. (5). The simplest chain (2),(3) is obtained from (24)–(28) if $\xi_1=\xi_2=0$. In the limit $\varkappa_2\to\bar\varkappa_1$ we get from (24)–(28) a single two-dimensional soliton:

$$u_0(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\left\{\frac{1}{4\eta_1^2}+4\eta_1^2(y-3\xi_1t)^2+\{x-3(\xi_1^2+\eta_1^2)t\right.$$
$$\left.-2\xi_1(y-3\xi_1t)\}^2\right\}. \tag{29}$$

We give the expression for the corresponding function T_0:

$$T_0(k_1,k,x,y,t)=r(x,y,t)\delta(k_1-\bar\varkappa_2)\delta(k-\varkappa_1)+\bar r(x,y,t)\delta(k_1$$
$$-\bar\varkappa_1)\delta(k-\varkappa_2), \tag{30}$$

where

$$r_1=2i\pi(\bar\varkappa_2-\varkappa_1)\exp[i(\bar\varkappa_2-\varkappa_1)x+i(\varkappa_1^2-\bar\varkappa_2^2)y+i(\bar\varkappa_2^3-\varkappa_1^3)t]. \tag{31}$$

To solve the problem we must construct only two perturbation modes of the chain. One of them is given by the function

$$T_1(k_1,k,x,y,t)=f_0\delta(k_1-s)\delta(k-\varkappa_2)\exp[\theta(s,x,y,t)], \tag{32}$$

where

$$\theta(s,x,y,t)=i(s-\varkappa_2)x+i(\varkappa_2^2-s^2)y+i[s^3-\varkappa_2^3+v_1(s-\varkappa_2)$$
$$+v_2(\varkappa_2^2-s^2)]t, \tag{33}$$

and the other by the function

$$T_1(k_1, k, x, y, t)=f_0\delta(k_1+s)\delta(k-\varkappa_1)\exp[\tilde{\theta}(s, x, y, t)], \tag{34}$$

where

$$\tilde{\theta}(s, x, y, t)=-i(s+\varkappa_1)x+i(\varkappa_1^2-s^2)y-i[s^3+\varkappa_1^3+v_1(s+\varkappa_1)+v_2(s^2-\varkappa_2^2)]t. \tag{35}$$

We have written Eqs. (24)–(31) in the laboratory system of coordinates and (32)–(35) in the system of coordinates fixed to the periodic chain which moves in the xy-plane with a velocity $v=(v_1,v_2)$. Omitting the calculations we give the main result: any periodic chain is unstable against small perturbations. The maximum growth rate of the perturbation equals

$$\max_s \gamma=(s_0-\eta_2)\left|(s_0+\eta_1)\left[s_0-\eta_1+\eta_2+\frac{3}{s_0}\frac{(\xi_2-\xi_1)^2\eta_1\eta_2}{(\eta_1+\eta_2)^2}\right]\right|, \tag{36}$$

where

$$s_0=-\left\{\frac{1}{6}\frac{\eta_1^3+\eta_2^3}{\eta_1+\eta_2}-\frac{1}{2}\frac{(\xi_2-\xi_1)^2\eta_1\eta_2}{(\eta_1+\eta_2)^2}\pm\left[\left(\frac{1}{6}\frac{\eta_1^3+\eta_2^3}{\eta_1+\eta_2}-\frac{1}{2}\frac{(\xi_2-\xi_1)^2\eta_1\eta_2}{(\eta_1+\eta_2)^2}\right)^2-\frac{(\xi_2-\xi_1)^2\eta_1^2\eta_2^2}{(\eta_1+\eta_2)^2}\right]^{1/2}\right\}^{1/2} \tag{37}$$

We must in (36) substitute from (37) an s_0 which satisfies inequality (17). As before, inequality (17) arises due to the boundedness of the perturbation u_1 on the axis of the chain and its decrease along any other straight line in the xy-plane as $x^2+y^2\to\infty$.

For arbitrary values of the parameters ξ_1, ξ_2, η_1, and η_2 one of the two points s_0 from (37) necessarily lies in the range $(\eta_2, -\eta_1)$. If, however, for some values of the parameters both points fall into that range one must substitute that value of s_0 which gives the maximum of the right-hand side of (36). Putting $\xi_1=\xi_2=0$ in (36) and (37) we get the maximum of the growth rate (18),(19) for the particular case of the simplest chain considered above. When studying the special case $\eta_1+\eta_2=0$ (the axis of the chain is the same as the x-axis) it is sufficient to take in (36),(37) the limit $\eta_1+\eta_2\to 0$. As a result we get

$$\max_s \gamma={}^3/_4\eta(\xi_2-\xi_1)^2.$$

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