The structure of two-dimensional solitons in media with anomalously small dispersion

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We show that situations are possible in a number of cases when the dispersion coefficient in a Taylor series expansion of the frequency in the wavenumber becomes in the long-wavelength approximation nearly zero (for instance, for gravitational-capillary waves in shallow water or oblique magnetosonic waves in a cold plasma). When constructing the evolution equation which describes the dynamics of weakly nonlinear perturbations it is necessary in that case to retain the next dispersion term, which corresponds to taking into account higher powers in the expansion of the dispersion equation in powers of the wavenumber. The most typical equation for such cases when one has multidimensional perturbations in media with a quadratic nonlinearity which move at small angles to a chosen x-axis is the generalization of the well known Kadomtsev-Petviashvili equation[2]. Using numerical calculations we find stationary solutions of this equation in the form of two-dimensional multisolitons which exhibit damped oscillatory asymptotic behavior along the direction of motion at large distances from the peaks. It is shown that the amplitudes of the solitons must be larger than some threshold level determined by the parameters of the equation. We give estimates of the characteristic amplitudes and velocities of the solitons for waves on water.

1. There exists a large number of examples of various media which in the long-wavelength limit have a typical "Korteweg" dispersion: $\omega = c_0 k + \beta k^3$, where ω is the frequency of a monochromatic wave and k the wavenumber. When a quadratic nonlinearity is present in the medium, small-amplitude plane perturbations in this case are described by the Korteweg-de Vries (KdV) equation,[1] and weakly non-one-dimensional perturbations by the Kadomtsev-Petviashvili (KP) equation.[2] It was shown in Refs. 3, 4 that the KP equation with positive dispersion has solutions in the form of two-dimensional pulses localized in all coordinates: algebraic solitons with power-law asymptotic behavior far from the peak.

Often, however, there occur situations in which expanding the dispersion equation in a Taylor series as $k\to 0$ shows the coefficient β to be anomalously small. For instance, for gravitational-capillary waves on shallow water[1]

$$\beta=(c_0/6)\left[(3\sigma/\rho g)-H^2\right],$$

where $c_0=(gH)^{1/2}$; g is the gravitational acceleration; H is the depth of the fluid; σ is the surface tension coefficient; and ρ is the density of the fluid. It is clear from this expression that when $H=(3\sigma/\rho g)^{1/2}$ the dispersion parameter β vanishes (for pure water the corresponding value of the depth is 0.48 cm). A similar situation occurs for magnetosonic waves in a cold plasma:[1] $\beta\sim\cot^2\theta - m_e/m_i$, where m_e, m_i are, respectively, the electron and ion masses; θ is the angle between the direction of wave propagation and that of the external magnetic field. When $\theta=\arctan(m_i/m_e)^{1/2}$ we have $\beta=0$. The vanishing of the dispersion parameter β does not mean that the dispersion in the medium has completely disappeared, but simply that in that case it is necessary to retain the next term in the Taylor series expansion in k of the complete dispersion equation. As a rule, the next term turns out to be proportional to k^5. Plane, weakly nonlinear perturbations for anomalously small values of β are described by the generalized KdV equation[5,6]

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}-\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^5 u}{\partial x^5}=0. \tag{1}$$

For multi dimensional perturbations in the KP approximation (i.e., assuming the characteristic scale of changes in the field along y to be much larger than the corresponding scale along x) one can obtain a two-dimensionally generalized analog of Eq. (1). In the frame of reference moving along the x-axis with a velocity c_0 the corresponding equation has the form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}-\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^5 u}{\partial x^5}\right)=-\frac{c_0}{2}\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The problem of the existence of two-dimensional stationary solutions of this equation and their structure is of interest.

2. It is impossible to find an analytical solution of Eq. (2) and we therefore used the numerical method for finding stationary solutions proposed by Petviashvili.[3] First of all, we rewrite Eq. (2) in dimensionless form:

$$\frac{\partial^2 v}{\partial\xi^2}\mp\frac{\partial^2 v}{\partial\eta^2}=\frac{1}{2}\frac{\partial^2 v^2}{\partial\xi^2}-\varepsilon\frac{\partial^4 v}{\partial\xi^4}-\frac{\partial^6 v}{\partial\xi^6}, \tag{3}$$

where

$$\xi=(-V/\gamma)^{1/4}(x-Vt),\quad \eta=(\pm 2V/c_0)^{1/2}(-V/\gamma)^{1/4}y,$$
$$v=\frac{\alpha}{V}u,\quad \varepsilon=\frac{\beta}{V}\left(-\frac{V}{\gamma}\right)^{1/2}.$$

In those formulae the upper sign occurs when $V>0$ and the lower sign when $V<0$; moreover, γ and V must have differ-

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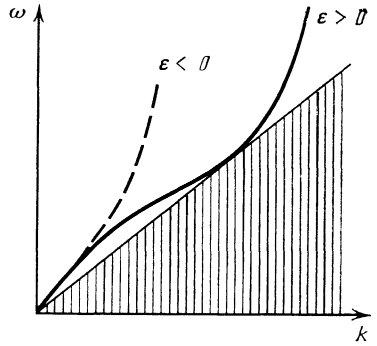

FIG. 1. Qualitative form of the dispersion curve when the dispersion changes its character. When $\varepsilon > 0$ there appears a point of inflection.

ent signs. We then Fourier transform Eq. (3) in the variables ξ and η and write down the basic formula for the iteration process for the numerical search for stationary solutions (see Ref. 3):

$$\tilde{v}_{n+1} = \frac{1}{2} M^p \frac{k_\xi^2 \tilde{v}_n^2}{k_\xi^2 \mp k_\eta^2 - \varepsilon k_\xi^4 + k_\xi^6}. \tag{4}$$

The tilde indicates here the Fourier transform of the corrsponding function; p is an arbitrary real number selected empirically to optimize the rate of convergence of the iteration process (in our calculations $p = 2$);

$$M = \frac{\int_{-\infty}^{\infty} (k_\xi^2 \mp k_\eta^2 - \varepsilon k_\xi^4 + k_\xi^6)(\tilde{v}_n)^2 dk_\xi\, dk_\eta}{{}^1/_2 \int_{-\infty}^{\infty} k_\xi^2 \tilde{v}_n^2 \tilde{v}_n\, dk_\xi\, dk_\eta}.$$

As a start one can use here an average smooth two-dimensional function with a single maximum. One sees easily that the iteration processes is not well defined if the denominator in Eq. (4) vanishes for finite values of k_ξ, k_η. This situation will occur when we choose the upper sign in Eqs. (3), (4), i.e., when $V > 0$. We shall therefore in what follows assume that $V < 0$, and, hence that $\gamma > 0$. The positive value of γ corresponds to a positive dispersion in the short-wavelength region. It is necessary to impose yet another restriction on the magnitude of the parameter ε in order that the polynomial in k_ξ in the denominator of Eq. (4) not have real roots for $k_\xi > 0$. If $\varepsilon < 0$ ($\beta > 0$) this condition is certainly satisfied. One easily checks that it is also satisfied when $\varepsilon < 2$. This imposes a restriction on the soliton velocity V: the condition $V < V_{\mathrm{ph,min}} \equiv -\beta^2/4\gamma$ must be satisfied, i.e., the soliton velocity must be less than the minimum phase velocity of the

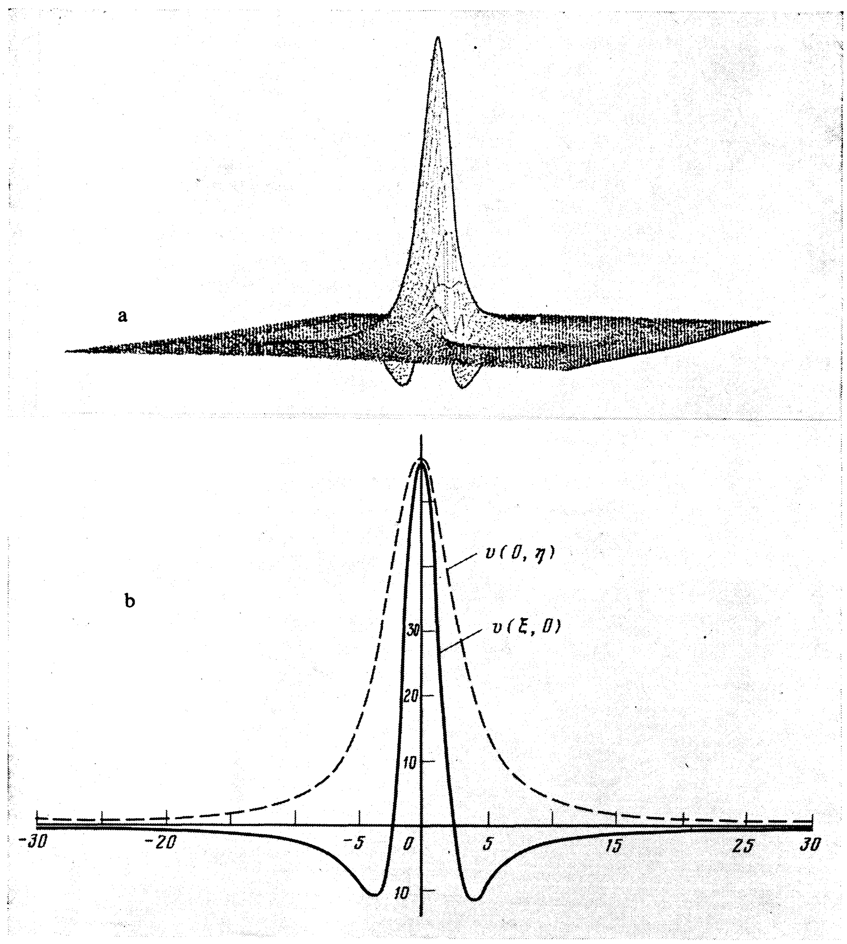

FIG. 2. Two-dimensional soliton for $\varepsilon = 0$: a: general form; b: main cross sections along the motion (solid curve) and at right angles to the motion (dashed curve).

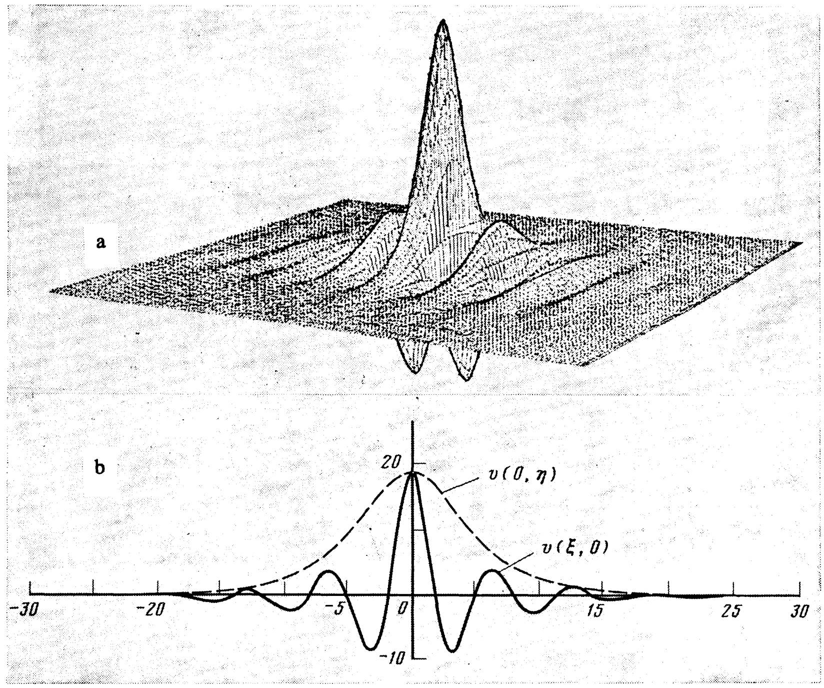

FIG. 3. Two-dimensional soliton for $\varepsilon = 1.9$: a: general form; b: main cross-sections along the motion (solid curve) and at right angles to the motion (dashed curve).

linear perturbations. We depict in Fig. 1 the qualitative shape of the dispersion curve in the rest frame for the case considered. The soliton velocity must lie in the hatched region under the dispersion curve. Such a situation occurs for surface waves on water for depths $H \gtrsim 0.5$ cm. In that case, if $u(x,y,t)$ describes the change in the mean level of the liquid we have in Eq. (2)

$$\gamma = \frac{c_0}{6}\left[H^2\left(\frac{2}{5}H^2 - \frac{\sigma}{\rho g}\right) - \frac{1}{12}\left(\frac{3\sigma}{\rho g} - H^2\right)^2\right], \quad \alpha = \frac{3}{2}\frac{c_0}{H};$$

the expressions for c_0 and β were given earlier.

3. When $\varepsilon = 0$ the structure of the two-dimensional solitons, found numerically, qualitatively does not differ from the algebraic KP solitons[3,4] (Fig. 2). The two-dimensional solitons, depending on the sign of α, have both positive and negative polarity in the original variables $u(x,y,t)$. For instance, on the surface of a liquid they are depressions. When $\varepsilon > 0$, for instance, when the depth of the liquid is increased from the value $H = (3\sigma/\rho g)^{1/2}$, there appears on the dispersion curve a point of inflection (see Fig. 1). As a result of this the structure of the solitons also changes: they decrease monotonically as before from a maximum to zero in the transverse direction, but along the direction of the motion they alternate in sign (Fig. 3). We note that the one-dimensional solitons described by Eq. (1) have a similar structure.[5,6] When ε increase the number of oscillations on their tails increases so that gradually the solitons become more and more like high-frequency wavetrains, i.e., envelope two-dimensional solitons. There thus occurs a smooth transition from the "video-pulse" KP solitons to the "radio-pulse" envelope solitons. Thanks to the fact that the velocity of these solitons must be less than the minimum phase velocity of the linear perturbations, their amplitude which is porportional to their velocity must be larger than some threshold value. For surface waves this threshold value of the amplitude of the solitons is determined by the depth of the liquid.

In Fig. 4 we have depicted the dimensionless soliton amplitude as a function of the parameter ε obtained by means of numerical calculations. For water waves we give here the minimum soliton amplitude as function of the depth of the liquid (H and u in mm):

H:	5	6	7	8	9	10
u_{min}:	0,013	0,1	0,19	0,28	0,36	0,44

For a depth $H = 1$ cm the soliton velocity does not exceed 28.5 cm/s. For an amplitude $u = 0.125$ mm the characteristic size of the soliton in the longitudinal direction is 25 cm and in the transverse direction 55 cm. The wavelength of the coverage equals 5.3 cm.

The presence of local field minima in the structure of the solitons enables us to conjecture that Eq. (2) may have complicated multisoliton solutions which form either excited or stationary bound states. The search for stationary

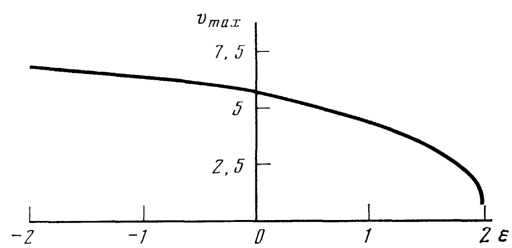

FIG. 4. The dimensionless amplitude of the soliton as function of the parameter ε.

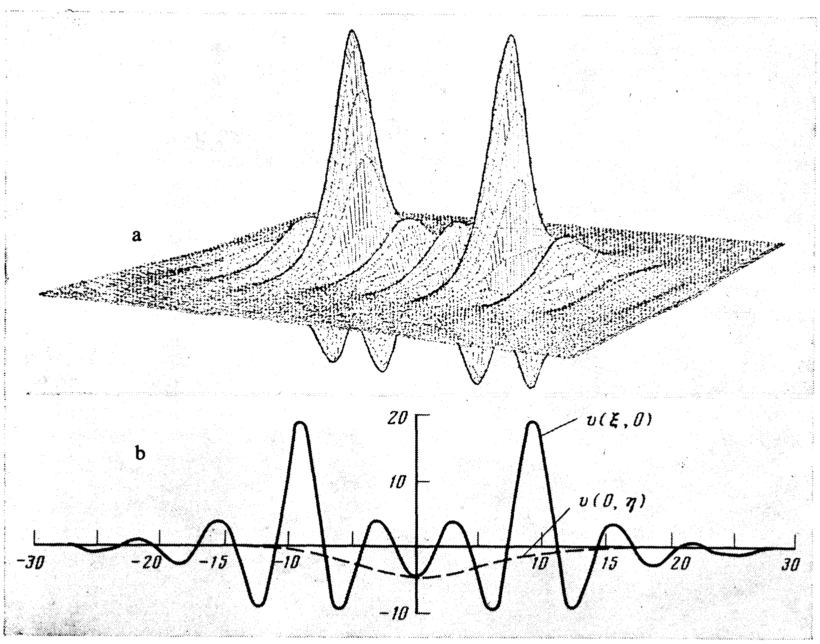

FIG. 5. Two-dimensional bisoliton for $\varepsilon = 1.9$: a: general form, b: main cross sections along the motion (solid curve) and at right angles to the motion (dashed curve). In figures 2, 3, and 5 change in the figure B to b.

multisolitons can be carried out using the same numerical procedure as was used for the calculations of individual solitons. As initial function for the iteration process (4) we used a superposition of smooth two-dimensional functions with peaks separated from one another along the ξ-axis by a distance ξ_0. For small ξ_0 the iteration process converged to the already known individual solitons. Starting from some values of ξ_0 we obtained bisolitons (Fig. 5). The structure of the field of the latter far from their peak is qualitatively similar to the structure of the field of an individual soliton. It is thus natural to expect that in the framework of Eq. (3) there may also exist more complicated stationary formations, namely, multisolitons. However, a numerical search for them involves great difficulties, because for this one needs to expand appreciably the volume of the operational memory, increase the time of the calculation, and also "guess" more accurately the initial function since otherwise the iteration process (4) may lead to some simpler, already known solution.

In conclusion we note that an experimental verification of the existence of the two-dimensional structures described here is of interest. In our opinion, the simplest would be to observe them in laboratory troughs on liquid surfaces.

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