Collective modes and dissipation coefficients in He³–He⁴ quantum fluids

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Kinetic phenomena in superfluid He³–He⁴ solutions are investigated by calculating the velocities and damping of first and second sound in the hydrodynamic as well as in the kinetic regime. The results point to the presence of velocity dispersion of second sound and to its substantial absorption in the transition frequency region. The calculation procedure employed has made it possible to write down in explicit form all the dissipation coefficients of the phonon-impurity system of a superfluid He³–He⁴ solution. The times contained in the dissipation coefficients for low-density solutions differ by more than an order of magnitude from those of an earlier theory. The calculation results are compared with the available experimental data.

The fundamentals of the kinetic theory of the He³–He⁴ quantum-liquid solutions were set forth by Khalatnikov and Zharkov.[1] Using the basic ideas of this paper, Baym, Saam, and Ebner[2–4] developed a detailed theory of kinetic phenomena in the phonon-impurity system of He³–He⁴ solutions at $T < 0.6$ K. Baym's theory, while in good agreement with experiment for sufficiently concentrated solutions at relatively low temperatures, diverges substantially from it with rising solution temperature and with decreasing solution concentration. Attempting to eliminate this divergence, some workers calculated the phonon-impurity time using an unjustified averaging procedure that contradicts the theory of Refs. 1–4 and is shown in Ref. 7 to be incorrect. The use of the intermediate results of Ref. 3 gave them grounds for stating that they follow this theory. Most investigators (see, e.g., Refs. 8 and 9), after reducing the data in full accord with Baym's theory, found in a number of cases a deviation from experiment by approximately an order of magnitude. This, in particular was the size of the disparity between the observed and calculated values of the second-sound absorption coefficient.[9]

The purpose of the present paper is to calculate the velocity and absorption of second sound in the phonon-impurity system of an He³–He⁴ solution with allowance, as in Ref. 7, for the phonon-phonon small-angle scattering and for the inelasticity of the phonon-impurity scattering. A general expression is obtained in the hydrodynamic regime for the renormalization of the velocity and of first-sound absorption in superfluid He³–He⁴ solutions; this explains, in particular, the high-temperature data of Refs. 10–12. The results permitted all the dissipative coefficients of the phonon-impurity system including the second viscosities, to be written in explicit form. We have used in the calculations the results of Ref. 1 as well as of Refs. 2–4, and stress therefore that we are dealing here not with the correctness of the Baym–Saam-Ebner theory, but with the range of its validity.

VELOCITY RENORMALIZATION AND FIRST-SOUND DAMPING. DISSIPATION COEFFICIENTS

We have previously[7] obtained the renormalization and damping of first-sound in a phonon-impurity system, assuming that the condition $C_4/C_3 \ll 1$ (where C_i are the quasiparticle heat capacities per unit volume) is satisfied in the hydro-

dynamic limit. Here we consider for a phonon-roton-impurity system a hydrodynamic regime not subject to this restriction and obtain the time that determines the second viscosity in a phonon-impurity system.

Starting with the complete system of linearized equations[1–4] that describe solution of He³–He⁴ quantum liquids and yield the dispersion equation $\omega = \omega(k)$, we have in accordance with Ref. 7 for the renormalization of the velocity c of the first-sound absorption coefficient α_1, in an approximation linear in ρ_n/ρ_4 ($\rho_n = \rho_{n3} + \rho_{n4}$ is the total normal density),

$$\frac{\Delta c}{c} = \frac{\operatorname{Re}\omega - c_0 k}{c_0 k} = \frac{c - c_0}{c_0} - \frac{\delta m}{2m}\frac{\rho_3}{\rho_4} + \frac{1}{2\rho_4}\operatorname{Re} K,$$

$$\alpha_1 = -\frac{\omega}{2\rho_4 c}\operatorname{Im} K, \tag{1}$$

where

$$\frac{c - c_0}{c} = -u\left[xa + \frac{P_f}{\rho_4 c^2}\left(1 - \frac{3}{2}w\right)\right]$$
$$+ \frac{\rho_4}{2c^2}\left(\int \frac{\partial^2 E_3}{\partial \rho_4^2} f_3 d\Gamma_3 + \int \frac{\partial^2 \varepsilon_4}{\partial \rho_4^2} f_4 d\Gamma_4\right), \tag{2}$$

c_0 is the speed of sound in pure He⁴ at $T = 0$, $x = n_3/n_4$ is the concentration, P_f is the impurity-gas pressure, $\rho_i = n_i m_i$ are the densities of the solution components, ε_4 is the energy of the He⁴ quasiparticles, $E_3 = \varepsilon_0 + \varepsilon_3 = \varepsilon_0 + p_3^2/2m$ is the energy of the impurity excitation (impuriton), $\delta m = m - m_3$ is the renormalization correction to the impuriton mass, f_i is the distribution function,

$$u = \frac{\rho_4}{c}\frac{\partial c}{\partial \rho_4}, \quad a = \frac{\rho_4}{m_4 c^2}\frac{\partial \varepsilon_0}{\partial \rho_4}, \quad w = \frac{\rho_4}{m}\frac{\partial m}{\partial \rho_4}.$$

The expression for K is

$$K = -\langle V|V\rangle + \langle V|R\omega|V\rangle, \tag{3}$$

where

$$V = \begin{pmatrix} V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} \dfrac{\rho_4}{c}\dfrac{\partial E_3}{\partial \rho_4} + \dfrac{\delta m}{m}p_{3z} \\[2mm] \dfrac{\rho_4}{c}\dfrac{\partial \varepsilon_4}{\partial \rho_4} + p_{4z} \end{pmatrix},$$

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and the operator matrix is

$$R = (\omega - \mathbf{kv} - iI)^{-1}, \tag{4}$$

$$\omega - \mathbf{kv} = \begin{pmatrix} \omega - kv_3 & 0 \\ 0 & \omega - kv_4 \end{pmatrix}, \quad I = \begin{pmatrix} I_{33} + I_{34} & I_{34} \\ I_{43} & I_{44} + I_{43} \end{pmatrix}. \tag{5}$$

The z axis is chosen here along $\mathbf{k}$, $\mathbf{p}_i$ and $\mathbf{v}_i = \partial\varepsilon_i / \partial\mathbf{p}_i$ are the quasiparticle momentum and velocity, and I_{ik} are linearized collision operators. The scalar product of the bra-vector $\langle\psi|$ by the ket vector $|\varphi\rangle$ is defined as follows:

$$\langle\psi|\varphi\rangle = -\int \psi_3^* \varphi_3 f_3' \, d\Gamma_3 - \int \psi_4^* \varphi_4 f_4' \, d\Gamma_4, \tag{6}$$

where f_i' is the derivative of the distribution function with respect to energy.

In the calculation of the second term in (3) it is necessary to separate in R the part corresponding to the subspace of the collision invariants of the operator I, and then use the τ-approximation for the operators I_{ii}. We choose as the basis vectors of the subspace of the collision invariants

$$|\tilde{c}_1\rangle = \left|\begin{matrix} \tilde{c}_{31} \\ \tilde{c}_{41} \end{matrix}\right\rangle = \left[\frac{1}{n_3}\left(\frac{\partial P_f}{\partial n_3}\right)_{Tn_4}\right]^{-1/2} \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle,$$

$$|\tilde{c}_2\rangle = (C_V T)^{-1/2} \left|\begin{matrix} \Delta\varepsilon_3 \\ \varepsilon_4 \end{matrix}\right\rangle, \quad |\tilde{c}_3\rangle = \rho_n^{-1/2} \left|\begin{matrix} p_{3z} \\ p_{4z} \end{matrix}\right\rangle, \tag{7}$$

where $C_V = C_3 + C_4$ is the heat capacity, $\Delta\varepsilon_3 = \varepsilon_3 - \bar{\varepsilon}_3$, and $\bar{\varepsilon}_3 = \frac{3}{2}(\partial P_f / \partial n_3)_{Tn_4}$. The vectors $|\tilde{c}_4\rangle$ and $|\tilde{c}_5\rangle$, which contain x and y components of the momenta, drop out of the calculations, since the z axis is directed along $\mathbf{k}$.

We introduce, for the mixture of the phonon and impuriton gases, operators for projection on the subspace of the collision invariants

$$\mathscr{P}_c = \sum_j |\tilde{c}_j\rangle\langle\tilde{c}_j| \tag{8}$$

and on the subspace $\mathscr{P}_n = 1 - \mathscr{P}_c$ orthogonal to it. Further transformations are carried out to go from the operator I to $\mathscr{P}_n I \mathscr{P}_n$, where the τ approximation is used. We start from the equation

$$(\omega - \mathbf{kv} - iI)R = 1.$$

We multiply it from the left and from the right by $\mathscr{P}_c$, and then from the left by $\mathscr{P}_n$ and from the right by $\mathscr{P}_c$. Solving the system of two equations we obtain

$$\mathscr{P}_c R \mathscr{P}_c = \mathscr{P}_c [\omega - \mathscr{P}_c \mathbf{kv}\mathscr{P}_c - \mathscr{P}_c \mathbf{kv}\mathscr{P}_n$$

$$\times (\omega - \mathscr{P}_n \mathbf{kv}\mathscr{P}_n - i\mathscr{P}_n I\mathscr{P}_n)^{-1} \mathscr{P}_n \mathbf{kv}\mathscr{P}_c]^{-1}\mathscr{P}_c,$$

$$\mathscr{P}_n R\mathscr{P}_c = \mathscr{P}_n(\omega - \mathscr{P}_n \mathbf{kv}\mathscr{P}_n - i\mathscr{P}_n I\mathscr{P}_n)^{-1}\mathscr{P}_n \mathbf{kv}\mathscr{P}_c R\mathscr{P}_c. \tag{9}$$

In the hydrodynamic limit, which will be considered for first sound, the terms $\omega - \mathscr{P}_n \mathbf{kv}\mathscr{P}_n$ can be left out of the expressions in (9). We calculate similarly in this limit

$$\mathscr{P}_n R\mathscr{P}_n = i\mathscr{P}_n(\mathscr{P}_n I\mathscr{P}_n)^{-1}\mathscr{P}_n, \quad \mathscr{P}_c R\mathscr{P}_n = \mathscr{P}_c R\mathscr{P}_c \mathbf{kv}\mathscr{P}_n R\mathscr{P}_n. \tag{10}$$

From (3), (9), and (10) we obtain

$$K = -\langle V|V\rangle + \omega \langle V|\mathscr{P}_c(\omega - \mathscr{P}_c \mathbf{kv}\mathscr{P}_c)^{-1}\mathscr{P}_c|V\rangle$$

$$+ \frac{i\omega}{c^2}\langle \tilde{V}|(\mathscr{P}_n I\mathscr{P}_n)^{-1}|\tilde{V}\rangle, \tag{11}$$

$$|\tilde{V}\rangle = c\mathscr{P}_n[1 + \mathbf{kv}\mathscr{P}_c(\omega - \mathscr{P}_c \mathbf{kv}\mathscr{P}_c)^{-1}\mathscr{P}_c]|V\rangle. \tag{12}$$

The matrix elements of the operator $\mathscr{P}_c \mathbf{k}\cdot\mathbf{v}\mathscr{P}_c$ are calculated by starting from the definitions (6) and (7):

$$\langle\tilde{c}_1|kv_z|\tilde{c}_3\rangle = ku_{2N}, \quad \langle\tilde{c}_2|kv_z|\tilde{c}_3\rangle = ku_{2\delta}, \tag{13}$$

where

$$u_{2N}^2 = \frac{n_3}{\rho_n}\left(\frac{\partial P_f}{\partial n_3}\right)_{Tn_4}, \quad u_{2\varepsilon}^2 = \frac{T\bar{S}^2}{\rho_n C_V}, \quad \bar{S} = S_4 + \frac{2}{3}C_3, \tag{14}$$

with S_4 the entropy per unit volume of pure He^4.

Calculating the matrix elements of the operator $(\omega - \mathscr{P}_c \mathbf{k}\cdot\mathbf{v}\mathscr{P}_c)^{-1}$, we obtain for (12) after rather laborious algebra

$$|\tilde{V}\rangle = \frac{2}{\sqrt{3}}A_\eta|\varphi_\eta\rangle + A_\zeta|\varphi_\zeta\rangle + A_\varkappa\left(\frac{\rho_n u_{2\varepsilon}^2}{C_V c^2}\right)^{1/2}|\varphi_\varkappa\rangle$$

$$+ \left(A_D + A_\varkappa k_T \frac{\rho_4\bar{S}}{\rho_3 C_V}\right)\left(\frac{\rho_n u_{2N}^2}{c^2}\right)^{1/2}|\varphi_D\rangle, \tag{15}$$

where

$$|\varphi_\eta\rangle = \left|\begin{matrix} \varphi_{3\eta} \\ \varphi_{4\eta} \end{matrix}\right\rangle = \frac{1}{2\sqrt{3}}\left|\begin{matrix} 3p_{3z}v_{3z} - p_3 v_3 \\ 3p_{4z}v_{4z} - p_4 v_4 \end{matrix}\right\rangle,$$

$$|\varphi_\zeta\rangle = \frac{1}{3C_V}\left|\begin{matrix} C_4\Delta\varepsilon_3 \\ -C_3\varepsilon_4 + 3C_V\varphi_{0\zeta} \end{matrix}\right\rangle \tag{16}$$

are vectors that determine the first and second viscosities of the solutions, while

$$|\varphi_\varkappa\rangle = T^{-1/2}\left|\begin{matrix} \Delta\varepsilon_3 v_{3z} - \dfrac{2TC_3}{3\rho_{n3}}p_{3z} \\ \varepsilon_4 v_{4z} - \dfrac{TS_4}{\rho_{n4}}p_{4z} \end{matrix}\right\rangle$$

$$|\varphi_D\rangle = \frac{1}{m\rho_n}\left[\frac{1}{n_3}\left(\frac{\partial P_f}{\partial n_3}\right)_{Tn_4}\right]^{1/2}\left|\begin{matrix} \rho_{n4}p_{3z} \\ -\rho_{n3}p_{4z} \end{matrix}\right\rangle \tag{17}$$

are vectors that determine the heat conduction and the diffusion

$$A_\zeta\varphi_{0\zeta} = \rho_4\frac{\partial\varepsilon_4}{\partial\rho_4} - u_4\varepsilon_4 + A_\eta\left(\frac{1}{3}p_4 v_4 - \frac{S_4}{C_4}\varepsilon_4\right),$$

$$u_4 = \frac{\rho_4}{\langle\varepsilon_4|\varepsilon_4\rangle}\left\langle\frac{\partial\varepsilon_4}{\partial\rho_4}\middle|\varepsilon_4\right\rangle = -\frac{\rho_4}{C_4}\left(\frac{\partial S_4}{\partial\rho_4}\right)_T.$$

The vectors (16) and (17) are mutually orthogonal. They contain expressions that enter in the hydrodynamic definitions of the corresponding dissipative coefficients:

$$A_\eta = \left[\frac{\rho_{n4}}{\rho_n} + \left(\lambda - \frac{3}{2}w\beta\right)\frac{\rho_{n3}}{\rho_n} + \frac{T\bar{S}}{\rho_n c^2}\frac{u_4 C_4 - wC_3}{C_V}\right]$$

$$\times\left(1 - \frac{u_2^2}{c^2}\right)^{-1},$$

$$A_\zeta = \left[A_\eta\left(2 - 3\frac{S_4}{C_4}\right) - 3(u_4 + w)\right],$$

$$A_D = \left(\lambda - \frac{3}{2}w\beta - 1\right)\beta^{-1} + A_\eta, \tag{18}$$

$$A_\varkappa = (u_4 C_4 - w C_3) \bar{S}^{-1} + A_\eta, \quad \lambda = \frac{m_4}{m}\left(a + \frac{\delta m}{m_4}\right),$$

$$u_2^2 = u_{2N}^2 + u_{2e}^2. \tag{19}$$

Here u_2^2 is the squared second-sound velocity in the solution, and

$$\beta = \frac{1}{mc^2}\left(\frac{\partial P_f}{\partial n_3}\right)_{T_{n_4}} \tag{20}$$

From the relations (11) and (15)–(18) and from the definition (1) we have for the first-sound absorption coefficient

$$\alpha_1 = \frac{\omega^2}{2\rho_4 c^3}\left[\frac{4}{3}A_\eta^2 \eta + \zeta_1 + \frac{\rho_n \varkappa}{C_V}\frac{u_{2e}^2}{c^2}A_\varkappa^2\right.$$
$$\left. + \rho_n D \frac{u_{2N}^2}{c^2}\left(A_D + k_T \frac{\rho_4 \bar{S}}{\rho_3 C_V}A_\varkappa\right)^2\right], \tag{21}$$

where

$$\eta = -\langle\varphi_\eta| (\mathscr{P}_n I \mathscr{P}_n)^{-1}|\varphi_\eta\rangle, \quad \varkappa = -\langle\varphi_\varkappa| (\mathscr{P}_n I \mathscr{P}_n)^{-1}\varphi_\varkappa\rangle \tag{22}$$

are the velocity and heat-conduction coefficients:

$$D = -\langle\varphi_D| (\mathscr{P}_n I \mathscr{P}_n)^{-1}|\varphi_D\rangle,$$

$$k_T = \left(\frac{2}{3}\frac{C_3 T}{\rho_{n3}c^2} - \frac{S_4 T}{\rho_{n4}c^2}\right)\frac{\rho_3}{\rho_4 \beta} \tag{23}$$

are the diffusion coefficient and the thermal-diffusion ratio;

$$\zeta_1 = -A_\zeta^2\langle\varphi_\zeta| (\mathscr{P}_n I \mathscr{P}_n)^{-1}|\varphi_\zeta\rangle$$

$$= \zeta_2 A_\eta^2 + 2\rho_4 \zeta_1 A_\eta(1-A_\eta) + \rho_4^2 \zeta_3(1-A_\eta)^2 \tag{24}$$

is a combination of the second-viscosity coefficients. Relation (21) was recast in a form that allows us to compare it with the corresponding result obtained from the hydrodynamics equations.[13] For nondegenerate solutions, the thermal-diffusion ratio defined by (23) coincides in this case with the expression given in Ref. 1. Calculation in accordance with (21) yields for the phonon system the results of Ref. 14.

We calculate now the dissipation coefficients contained in (21) for the phonon-impurity system ($T < 0.6$ K). We rewrite the matrix element for the first (shear) viscosity η in (22) in the form

$$\langle\varphi_\eta| (\mathscr{P}_n I \mathscr{P}_n)^{-1}|\varphi_\eta\rangle = \langle\varphi_\eta|\chi_\eta\rangle, \tag{25}$$

where $|\chi_\eta\rangle$ is defined as

$$\mathscr{P}_n I \mathscr{P}_n|\chi_\eta\rangle = |\varphi_\eta\rangle. \tag{26}$$

Allowing for the orthogonality of $|\chi_{3\eta}\rangle$ to the subspace of the collision invariants of the operator I_{33}, we have in expanded form

$$(-\tau_{33}^{-1} + I_{34})|\chi_{3\eta}\rangle + I_{34}|\chi_{4\eta}\rangle = |\varphi_{3\eta}\rangle,$$
$$[-\tau_{\parallel}^{-1}(1-\mathscr{P}_\parallel) + I_{43}]|\chi_{4\eta}\rangle + I_{43}|\chi_{3\eta}\rangle = |\varphi_{4\eta}\rangle. \tag{27}$$

To derive (27) from (26) we used the τ-approximation for the impurity-impurity interaction

$$I_{33} = -\tau_{33}^{-1}(1 - \mathscr{P}_{3c}) \tag{28}$$

and for the phonon-phonon interaction

$$I_{44} = -\tau_{\parallel}^{-1}(1 - \mathscr{P}_\parallel), \tag{29}$$

where

$$\mathscr{P}_{3c} = \sum_j |c_{3j}\rangle\langle c_{3j}|, \quad \mathscr{P}_\parallel = \frac{|p_4\rangle_{e\ e}\langle p_4|}{{}_e\langle p_4|p_4\rangle_e}, \tag{30}$$

$|c_{3j}\rangle$, in contrast to $|\tilde{c}_{3j}\rangle$, is normalized to quantities pertaining only to the impurity subsystem. The operators I_{44} and I_{43} exert different actions in the angle and energy subspaces. We have therefore introduced vectors $|b\rangle_e$ and $|a\rangle_y$ defined respectively in energy and angle subspace, and the corresponding scalar products.[7] In I_{44}, account is taken only of the fast longitudinal relaxation that has a characteristic time $\tau_\parallel$ and is due to three-phonon processes. This approximation is valid if $x > 10^{-4}$, since the establishment of total equilibrium in the phonon system of these solutions is governed by phonon-impurity collisions rather than by transverse phonon relaxation.

The action of the operator I_{34} and I_{43} on a vector is defined in standard fashion.[7] We shall find it useful to express I_{43} in terms of projection operators on a basis of Legendre polynomials P_l:

$$\mathscr{P}_l = \frac{|P_l\rangle_{y\ y}\langle P_l|}{{}_y\langle P_l|P_l\rangle_y}. \tag{31}$$

Starting with (31), we can represent the operator I_{43} that acts in phonon-vector space in the form[7]

$$I_{43} = -\sum_{l=1}^{\infty} t_l^{-1}\mathscr{P}_l + \frac{{}_y\langle P_0|I_{43}|P_0\rangle_y}{{}_y\langle P_0|P_0\rangle_y}\mathscr{P}_0, \tag{32}$$

where, using the corrected numerical values of the impurity spectrum, we have according to Refs. 2 and 3

$$t_1 = \frac{7.7n_4}{xcp_4^4}, \quad t_2 = \frac{11.3n_4}{xcp_4^4}, \quad t_{l\geqslant 3} \equiv t_{phi} = \frac{10.7n_4}{xcp_4^4}. \tag{33}$$

At the temperatures and concentrations we have considered, the inequalities $\rho_{n4}/\rho_{n3} \ll 1$ and $\beta \ll 1$ hold. This allows us to leave out of (27) the terms $I_{34}|\chi_{3\eta}\rangle$, $I_{34}|\chi_{4\eta}\rangle$, $I_{43}|\chi_{3\eta}\rangle$. The system (27) breaks up then into two independent equation, and the shear viscosity becomes a sum of an impurity viscosity

$$\eta_3 = \langle\varphi_{3\eta}|\varphi_{3\eta}\rangle\tau_{33} = P_f\tau_{33} \tag{34}$$

and a phonon viscosity

$$\eta_4 = \langle\varphi_{4\bar{\eta}}|[\tau_\parallel^{-1}(1-\mathscr{P}_\parallel) + t_2^{-1}]^{-1}|\varphi_{4\eta}\rangle. \tag{35}$$

In the derivation of (35) it was recognized that, in accordance with (32),

$$I_{43}|\chi_{4\eta}\rangle = -t_2^{-1}|\chi_{4\eta}\rangle.$$

Relation (34) coincides with the expression obtained in Ref. 2, and goes over for the nondegenerate case into the result of elementary gas kinetic theory. We note that the equality $\eta = \eta_3 + \eta_4$ agrees with the results of Ref. 1 and, as follows from the calculations presented here, is valid only in the zeroth approximation in ρ_{n4}/ρ_{n3} and in β. The matrix element (35) is calculated in the same manner as in Ref. 7:

$$\eta_4 = {}^1/_5\rho_{n4}c^2\tau_2, \tag{36}$$

where τ_2 is determined from the expression

$$\tau_l = \left[\int_0^\infty t_l (t_l + \tau_\parallel)^{-1} p_4{}^4 f_4{}' \, dp_4\right]\left[\int_0^\infty (t_l + \tau_\parallel)^{-1} p_4{}^4 f_4{}' \, dp_4\right]^{-1}.$$

$$(37)$$

Equation (37) represents various mechanisms, discussed in Refs. 7 and 15, that establish equilibrium in the phonon system when impurities are present. In low-concentration solutions, for which $\tau_l \gg \tau_\parallel$, we must put $\tau_\parallel = 0$ in (37), and then

$$\tau_l{}^0 = -6\pi^2 \rho_{n4} \left(\int_0^\infty t_l{}^{-1} p_4{}^4 f_4{}' \, dp_4\right)^{-1}.$$

$$(38)$$

Equation (38) differs substantially from the theoretical results[2-4] in which no account was taken of the phonon times; this corresponds to the limit $\tau_\parallel \to \infty$, for which (37) yields

$$\tau_l{}^\infty = -(6\pi^2 \rho_{n4})^{-1}\int_0^\infty t_l p_4{}^4 f_4{}' \, dp_4.$$

$$(39)$$

Substitution of (33) in (39) produces at the lower limit a divergence that was eliminated in Refs. 2–4 by taking into account the absorption of long-wave phonons by the impurities, with a characteristic time t_a. Equation (37) is therefore valid at $\tau_\parallel \ll t_a$. This inequality does not hold for concentrated solutions at low temperatures and at high pressures, when the phonon spectrum becomes nondecaying. In this case Baym's theory is applicable, as confirmed by experiment. In the general case it is necessary to replace $t_l{}^{-1}$ in (37) by $t_l{}^{-1} + t_a{}^{-1}$. Since the averaged quantities are the frequency $t_l{}^{-1}$ with weight $p_4{}^4 f_4'$ in (38) and the time t_l appears in (39), it follows that $\tau_l{}^0$ differs from $\tau_l{}^\infty$ by more than an order of magnitude, as shown in Fig. 1. It can be seen from this figure that in the general case τ_l has a rather complicated dependence on temperature and concentration. At pressures 15–20 atm, when the phonon spectrum is non-decaying, (38) should go over into (39), as was indeed observed in Refs. 16 and 17 for $x \approx 10^{-3}$. The large measurement error, however, prevents an accurate quantitative comparison.

The thermal conductivity $\varkappa = \varkappa_3 + \varkappa_4$ is calculated in the same manner as the viscosity. In accordance with Ref. 1 we have here $\varkappa_4 = 0$, and

$$\varkappa_3 = \langle \varphi_{3\varkappa} | \varphi_{3\varkappa} \rangle \tau_{33}.$$

$$(40)$$

For the nondegenerate case we obtain

$$\varkappa_3 = {}^5\!/_2 n_3 \tau_{33} T/m.$$

The matrix element in (23), which determines the diffusion, is

$$\langle \varphi_D | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_D \rangle = \langle \varphi_D | \chi_D \rangle,$$

$$(41)$$

where $|\chi_D\rangle$ is given by an equation similar to (35).

In Eq. (41) the $|\chi_{iD}\rangle$ are determined in terms of $|\varphi_{iD}\rangle$, which contain vectors that pertain the subspace of the collision invariants of the operators I_{ii}. To use the τ-approximation it is therefore necessary to separate in $|\chi_{iD}\rangle$ the terms pertaining to the subspace of the collision invariants of these operators:

$$|\chi_{iD}\rangle = |\chi_{iD}{}^n\rangle + a_{iD}|\varphi_{iD}\rangle$$

$$(42)$$

with the orthogonality condition

$$\langle \chi_{iD}{}^n | \varphi_{iD} \rangle = 0.$$

$$(43)$$

We express the desired matrix element (41) in terms of a_{iD} in the form

$$\langle \varphi_D | \chi_D \rangle = \left(\frac{\rho_{n4}}{\rho_{n3}} a_{3D} + a_{4D}\right)\frac{\rho_{n4}}{\rho_n} u_{2N}{}^2.$$

$$(44)$$

Substituting (42) in the equation of $|\chi_D\rangle$ in the τ-approximation, we get

$$(-\tau_{33}{}^{-1} + I_{34})|\chi_{3D}{}^n\rangle + I_{34}|\chi_{4D}{}^n\rangle = |\varphi_{3D}\rangle - I_{34}|a_{3D}\varphi_{3D} + a_{4D}\varphi_{4D}\rangle,$$

$$(-\tau_\parallel{}^{-1} + I_{43})|\chi_{4D}{}^n\rangle + I_{43}|\chi_{3D}{}^n\rangle = |\varphi_{4D}\rangle - I_{43}|a_{3D}\varphi_{3D} + a_{4D}\varphi_{4D}\rangle.$$

$$(45)$$

Solving the system (45) for $|\chi_{iD}{}^n\rangle$ with the condition (43), subject to the constraints above, we obtain the diffusion coefficient

$$D = (\rho_{n4}/\rho_n) u_{2N}{}^2 \tau_1.$$

$$(46)$$

Substituting D and k_T in the relation for the effective heat-conduction coefficient,[1] we obtain

$$\varkappa_{eff} = {}^1\!/_3 C_4 c^2 \tau_1 + \varkappa_3.$$

$$(47)$$

Relation (47) allows us to determine the time τ_1 from heat-conduction experiments (Fig. 1). It can be seen from this figure that the times obtained in Refs. 6, 18, 19, and 5, 16, 17 by the same procedure differ by approximately a factor of two. The reason for this difference is apparently that in the first set of references the width of the heat pulse was a factor of ten larger than in the second. Therefore in the first case the regime was closer to hydrodynamic, determined by the time τ_1, while in the second it was closer to kinetic with a time $\tau_{phi} > \tau_1$. The difference between τ_1 and τ_{phi} agrees approximately with that observed in experiment. We note that the time obtained in Ref. 20 from thermal-conductivity data[8] agrees with τ_1 of Fig. 1.

The matrix element

$$\langle \varphi_\zeta | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_\zeta \rangle = \langle \varphi_\zeta | \chi_\zeta \rangle,$$

$$(48)$$

which defines in (24) a combination of second viscosities, is calculated in analogy with the diffusion. The difference is that the relaxation time τ_ζ is substantially longer than not
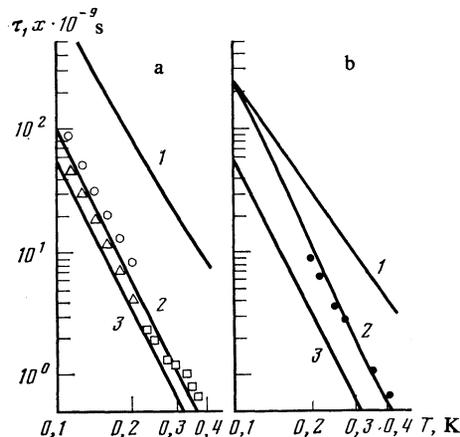

FIG. 1. Temperature dependence of the phonon-impurity relaxation times for $x = 10^{-3}$ (a) and $x = 1.3 \cdot 10^{-2}$ (b): 1—theoretical calculation,[2-4] 2—calculated from (37), 3—calculated from (38); $\triangle$ data of Refs. 5, 18, and 19, $\bigcirc$—of Refs. 6, 16, and 17, $\square$—of Ref. 20, $\bullet$—of Ref. 21.

only τ_{33} but also $\tau_\parallel$. In this approximation we obtain

$$\zeta_{\rm I} = \left(u+w-\frac{\lambda}{3}\right)^2 \left(\frac{C_3}{C_V}\right)^2 C_4 T \tau_\zeta, \qquad (49)$$

where

$$\tau_\zeta = -\langle \varepsilon_4/\varepsilon_4\rangle / \langle \varepsilon_4 | I_{43} | \varepsilon_4 \rangle \qquad (50)$$

is the time that determines the second viscosity. According to (32) and to the definition of the action of the operator I_{43} on an arbitrary vector we have

$$\langle \varepsilon_4 | I_{43} | \varepsilon_4 \rangle = -\frac{1}{2} \int w_{34} f_3' f_4' (\varepsilon_4 - \varepsilon_4')^2 \, d\Gamma_3 \, d\Gamma_3' \, d\Gamma_4 \, d\Gamma_4', \quad (51)$$

where w_{34} is the transition-probability density. We note that in perfectly elastic scattering ($\varepsilon_4 = \varepsilon_4'$) expression (51) vanishes. It is therefore necessary to retain in the difference $\varepsilon_4 - \varepsilon_4'$ the terms that contain the small parameter v_3/c; this is equivalent to allowance for the inelasticity of the phonon-impurity scattering. According to the conservation laws we have

$$\varepsilon_4 - \varepsilon_4' = \mathbf{v}_3 (\mathbf{p}_4 - \mathbf{p}_4') + (\mathbf{p}_4 - \mathbf{p}_4')^2/2m.$$

We ultimately obtain, accurate to within terms of order $(v_3/c)^2$,

$$\tau_\zeta = \beta^{-1} \tau_1^0. \qquad (52)$$

The time τ_ζ describes the slow establishment of energy equilibrium between the phonons and the impurities. The situation is similar here to that considered in Ref. 14 for a phonon-roton system. The difference is that in the latter the parameter β is not as small, so the first-viscosity time is of order τ_ζ.

Leaving out of (18) the terms containing the small parameters β and $(u_2/c)^2$, we obtain according to (21) for the coefficient of first-sound absorption in the phonon-impurity system

$$\alpha_1 = \frac{\omega^2}{2\rho_4 c^3} \left[\frac{4}{3} \lambda^2 \eta + \zeta_{\rm I} + \frac{\varkappa_3 \rho_n}{C_V} - \frac{u_{2e}^2}{c^2} \left(\lambda - \frac{uC_4 - wC_3}{C_V} \right)^2 \right.$$
$$\left. + \frac{\rho_{n3} D}{\beta} \left(1 - \lambda + \frac{\lambda S + uC_4 - wC_3}{C_V} \right)^2 \right]. \qquad (53)$$

The dissipation coefficients $\eta = \eta_3 + \eta_4, \varkappa_3, D$, and $\zeta_{\rm I}$ in (53) are given by expressions (34), (36), (40), (46), and (49).

When the inequality $C_4/C_3 \ll 1$ is satisfied, Eq. (53) goes over into the result of the Ref. 7, where the third term, proportional in this case to the small quantity β, was left out. The contribution of the second viscosity to the first-sound absorption coefficient was calculated here in explicit form. If the phonon heat capacity and the impurity-gas thermal conductivity are neglected, Eq. (53) agrees with the result of Ref. 4 to an accuracy of order the times τ_1, τ_2, and τ_ζ. The differences between the times, as noted above, is due to neglect, in Refs. 2–4, of the three-phonon processes and of the inelasticity of the phonon-impurity scattering. It was assumed on this basis that the energy equilibrium between the phonon and impurity systems is ensured by processes in which the phonons are absorbed by impuritons, with a characteristic time t_a that determines in that case the second viscosity.

It follows from our present result that the establishment

of the indicated equilibrium and the second viscosity are determined by two processes that proceed in parallel: inelastic scattering and phonon absorption by the impurity. Numerical estimates show that absorption must be taken into account at low temperatures and high densities, and at increased pressures. In this case τ_ζ^{-1} must be replaced by $\tau_\zeta^{-1} + t_a^{-1}$.

The main contribution to first-sound absorption is made by second viscosity. From this viewpoint it would be of interest to measure sound absorption in the hydrodynamic regime at $T < 0.6$ K, for this would permit comparison of the theoretical value of τ_ζ with the experimental one. To our knowledge, only kinetic data on sound absorption in this temperature region are available at present,[22] and agree with the calculations of Ref. 7.

Let us calculate the renormalization of first-sound velocity in the hydrodynamic limit. According to (1) and (11) we have

$$\frac{\Delta c}{c} = \frac{c - c_0}{c_0} + \frac{1}{2\rho_4} \left[\omega \langle V | \mathscr{P}_c (\omega - \mathscr{P}_c k v \mathscr{P}_c)^{-1} \mathscr{P}_c | V \rangle \right.$$
$$\left. - \langle V | V \rangle - \frac{\delta m}{m} \rho_3 \right]. \qquad (54)$$

The calculation takes the rotons into account, so that this result can be compared with the available experimental data. Calculating the matrix element contained in (54), we get

$$\frac{\Delta c}{c} = \frac{c - c_0}{c_0} + \frac{\rho_n}{2\rho_4} \left[A_\eta^2 \left(1 - \frac{u_2^2}{c^2} \right) \right.$$
$$\left. - (u_4 + w)^2 \frac{C_3 C_4 T}{C_V \rho_n c^2} - \frac{\rho_{n4} + n_3 \delta m}{\rho_n} \right] + \left(\frac{\Delta c}{c} \right)_4, \qquad (55)$$

where

$$\left(\frac{\Delta c}{c} \right)_4 = \frac{C_4 T}{2\rho_4 c^2} \left(u_4^2 - \frac{\rho_4^2}{C_4 T} \left\langle \frac{\partial \varepsilon_4}{\partial \rho_4} \Big| \frac{\partial \varepsilon_4}{\partial \rho_4} \right\rangle \right). \qquad (56)$$

The hydrodynamic expression (55) is exact in the approximation linear in ρ_n/ρ_4. If the impurity-gas and roton densities are small compared with the phonon density ρ_{ph}, then Eq. (55) leads to the result of Ref. 14 in the zeroth approximation in the small parameters ρ_{n3}/ρ_{ph} and ρ_r/ρ_{ph}. For the roton system, Eq. (55) with the numerical values of the parameters taken into account goes over into the result of Ref. 23, in which the second derivatives of the roton-spectrum parameters with respect to density were disregarded. Numerical estimates show that the contribution of the latter is insignificant at saturated vapor pressure. Neglecting the roton contribution, we obtain from (55) at $C_4/C_3 \ll 1$ the results of Refs. 7 and 24.

Equation (55) with account taken of the rotons and of the results of Ref. 7 explains the experimentally observed[10] decrease of the temperature-dependent part of $\Delta c/c$ at $T > 0.6$ K. A quantitative comparison, however, is hardly useful here, since the hydrodynamic conditions do not obtain at all times in Ref. 10. A numerical comparison of (55) with experiment in the high-temperature region bounded by the inequality $\rho_n/\rho_4 \ll 1$ is shown in Fig. 2, where calculated and experimental data are given for a solution with

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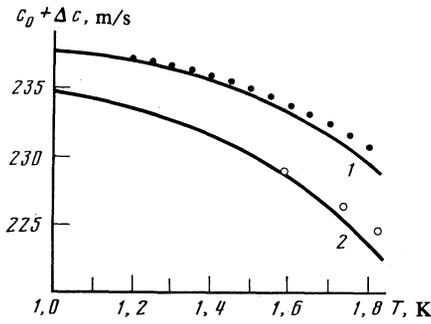
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FIG. 2. Second-sound velocity of $c_0 + \Delta c$ at $c_0 = 238.3$ m/s: 1—calculated from (55) for $x = 0$; 2—calculated from (55) for $x = 6.3 \cdot 10^{-2}$; O—data of Ref. 11, ●—data of Ref. 12.

$x = 6.3 \cdot 10^{-2}$ (Ref. 11) and for pure He[4] (Ref. 12). It follows from (56) that the phonon contribution to the renormalization of the first-sound velocity by the phonon-roton cross term is significant at $T > 1$ K.

SECOND-SOUND DAMPING AND VELOCITY DISPERSION. SECOND-VISCOSITY COEFFICIENTS

We calculate now the velocity and absorption of second sound in a phonon-impurity system when $\omega \tau_{33} \ll 1$ holds in the zeroth approximation in the small parameter ρ_n / ρ_4 the problem reduces to solution of the kinetic equations

$$(\omega - \mathbf{k} \mathbf{v}_3 - iI_{33} - iI_{34}) g_3 - iI_{34} g_4 = 0,$$
$$(\omega - \mathbf{k} \mathbf{v}_4 - iI_{44} - iI_{43}) g_4 - iI_{43} g_3 = 0, \tag{57}$$

where g_i determines the deviation of f_i from equilibrium.[7]
The system (57) yields

$$(\omega - \mathbf{k} \mathbf{v}_3 - iI_{33} - iI_{34} + I_{34} R_4 I_{43}) g_3 = 0, \tag{58}$$
$$R_4 = (\omega - \mathbf{k} \mathbf{v}_4 - iI_{44} - iI_{43})^{-1}. \tag{59}$$

The problem of finding $\omega = \omega(k)$ that satisfy Eq. (58) is equivalent to determining the poles of the resolvent of this equation

$$R_3 = (\omega - \mathbf{k} \mathbf{v}_3 - iI_{33} - iI_{34} + I_{34} R_4 I_{43})^{-1}. \tag{60}$$

For an impurity system in the hydrodynamic approximation, we must project Eq. (60) on the subspace of the invariants I_{33} and find the poles of the expression

$$\mathscr{P}_{3c} R_3 \mathscr{P}_{3c} = \mathscr{P}_{3c} (\omega - \mathscr{P}_{3c} \Omega \mathscr{P}_{3c})^{-1} \mathscr{P}_{3c}, \tag{61}$$

$$\Omega = \mathbf{k} \mathbf{v}_3 + iI_{34} - I_{34} R_4 I_{43} - i \mathbf{k} \mathbf{v}_3 \mathscr{P}_{3n} \mathbf{k} \mathbf{v}_3 \tau_{33}. \tag{62}$$

The derivation of (61) and (62) is similar to the derivation of (9). It was recognized in the derivation of (62) that the parameters ρ_{n4} / ρ_n, β and $\omega \tau_{33}$ are small.

It is necessary next to find the matrix of the operator $\omega - \mathscr{P}_{3c} \Omega \mathscr{P}_{3c}$ in the basis $|c_{3j}\rangle$. By setting the determinant of this matrix equal to zero we get a dispersion equation for $\omega(k)$ in the form

$$\left(\frac{\omega}{k} \right)^2 = \frac{u_{2N}^2}{1 + b_{11}} + \frac{u_{2e}^2 (1 + b_{01})^2}{(1 + b_{00})(1 + b_{11})}, \tag{63}$$

where

$$b_{00} = i \frac{k^2 \varkappa_3}{\omega C_V} - \frac{1}{\omega} \langle \tilde{c}_{42} | (\omega - \mathbf{k} \mathbf{v}_4) R_4 (\omega - \mathbf{k} \mathbf{v}_4) | \tilde{c}_{42} \rangle,$$

$$b_{11} = i \frac{4}{3} \frac{k^2 \eta_3}{\omega \rho_n} - \frac{1}{\omega} \langle \tilde{c}_{43} | (\omega - \mathbf{k} \mathbf{v}_4) R_4 (\omega - \mathbf{k} \mathbf{v}_4) | \tilde{c}_{43} \rangle, \tag{64}$$

$$b_{01} = \frac{1}{k u_{2e}} \langle \tilde{c}_{42} | (\omega - \mathbf{k} \mathbf{v}_4) R_4 (\omega - \mathbf{k} \mathbf{v}_4) | \tilde{c}_{43} \rangle.$$

The energy and momentum conservation laws were used in the derivation of (63).

We calculate now the matrix elements in (64). The vectors $(\omega - \mathbf{k} \cdot \mathbf{v}_4) | \tilde{c}_{4j} \rangle$ that they contain are proportional to $|p_4\rangle$ and belong to a subspace projected out by the operator $\mathscr{P}_\parallel$. Equation (64) therefore usually contains matrix elements of the operator $\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel$, that can be reduced to the form

$$\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel = (1 - i \tau_\parallel^{-1} \mathscr{P}_\parallel R_\parallel \mathscr{P}_\parallel)^{-1} \mathscr{P}_\parallel R_\parallel \mathscr{P}_\parallel,$$
$$R_\parallel = (\omega - \mathbf{k} \mathbf{v}_4 + i \tau_\parallel^{-1} - iI_{43})^{-1}. \tag{65}$$

Relation (65) is obtained from the equation

$$R_4 = R_\parallel + i \tau_\parallel^{-1} R_\parallel \mathscr{P}_\parallel R_4, \tag{66}$$

for R_4. Multiplying (66) from the left and from the right by $\mathscr{P}_\parallel$ we obtain Eq. (65).

Since the times (33) differ little, we shall simplify the derivations and the final results by regarding them as identical and equal to t_{phi}. This assumption is inessential, since in the kinetic limit, when $\omega \tau_{phi} > 1$, all equations contain only τ_{phi}, as follows from the calculations, and the generalization of the hydrodynamic equal-time approximation to include the case $t_1 \neq t_2 \neq t_{phi}$ is trivial; this is done by replacing τ_{phi} by τ_2 in the term corresponding to the shear viscosity, and by replacing τ_{phi} by τ_1 in the diffusion and thermal-diffusion terms.

Consider the frequency region

$$\omega (\tau_\parallel^{-1} + \tau_{phi}^{-1})^{-1} \ll u_2/c. \tag{67}$$

In this case

$$R_\parallel = -i (\tau_\parallel^{-1} - I_{43})^{-1}. \tag{68}$$

Substitution of (68) in (65) yields

$$\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel = \frac{1}{3 \rho_{n4}} |p_4\rangle_\varepsilon (\omega - \mathbf{k} \mathbf{v}_4 - i \langle I_{43} \rangle_R)^{-1} {}_\varepsilon \langle p_4|, \tag{69}$$

where

$$\langle I_{43} \rangle_R = {}_\varepsilon \langle p_4 | (\tau_\parallel^{-1} - I_{43})^{-1} | p_4 \rangle_\varepsilon^{-1} {}_\varepsilon \langle p_4 | (\tau_\parallel^{-1} - I_{43})^{-1} I_{43} | p_4 \rangle_\varepsilon \tag{70}$$

is an operator in angle space. From the definitions (32), (37), (52), and (70) we have in the equal-time approximation

$$\langle I_{43} \rangle = - \tau_{phi}^{-1} + (1 - \tilde{\beta}) \tau_{phi}^{-1} \mathscr{P}_0, \tag{71}$$

where $\tilde{\beta} = \tau_{phi} / \tau_\zeta$. To obtain (71) we left out the terms containing $\beta (\tau_\parallel / \tau_{phi}) \ll 1$. By an iteration procedure we reduce (69) to the form

$$\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel$$
$$= \frac{1}{3 \rho_{n4}} |p_4\rangle_\varepsilon \left[R_{phi} - R_{phi} \frac{(1-\tilde{\beta}) ck \mathscr{P}_0}{ick \tau_{phi} + (1 - \tilde{\beta}) Q_0} R_{phi} \right]_\varepsilon \langle p_4|, \tag{72}$$

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where

$$R_{phi} = (\omega - kv_4 + i\tau_{phi}^{-1})^{-1},$$

$$Q_0 = \frac{1}{2}\ln\frac{q_0+1}{q_0-1}, \quad q_0 = \frac{1}{ck}(\omega + i\tau_{phi}).$$

Calculating in (64) the matrix elements of the operator (72), we obtain after straightforward but rather laborious calculations

$$\frac{C_V}{C_3}(1+b_{00}) = 1 + i\frac{k^2 \varkappa_3}{\omega C_3} + \frac{i}{\omega\tau_\zeta}\frac{C_4}{C_3}\frac{ick\tau_{phi}+Q_0}{ick\tau_{phi}+(1-\beta)Q_0},$$

$$\frac{\rho_n}{\rho_{n3}}(1+b_{11}) = 1 + i\frac{4}{3}\frac{k^2\eta_3}{\omega\rho_{n3}} + \frac{i}{\omega\tau_{phi}}\frac{\rho_{n4}}{\rho_{n3}}\Big\{1$$

$$-\frac{3i(q_0 Q_0-1)}{ck\tau_{phi}}\Big[q_0 + \frac{(1-\beta)(q_0 Q_0-1)}{ick\tau_{phi}+(1-\beta)Q_0}\Big]\Big\} \quad (73)$$

$$b_{01} = \Big(3\frac{C_4\rho_{n4}}{C_V\rho_n}\Big)^{1/2}\frac{i}{ku_{2c}\tau_{phi}}\frac{q_0 Q_0-1}{ick\tau_{phi}+(1-\beta)Q_0}.$$

Relations (63) and (73) solve the problem of finding the second-sound dispersion law at frequencies that satisfy the inequality (67).

The result is greatly simplified in the kinetic limit $\omega\tau_{phi}\gg u_2/c$ (but $\omega\tau_{33}\ll 1$ and $\omega\tau_\parallel \ll u_2/c$). In this case we have from (63) and (73) for the second-sound velocity and absorption coefficient

$$u_3 = \Big(\frac{5}{3}\frac{P_f}{\rho_{n3}}\Big)^{1/2}, \quad (74)$$

$$\alpha_2 = \frac{\omega^2}{2\rho_{n3}u_3^2}\Big(\frac{4}{3}\eta_3 + \frac{4T}{9u_3^2}\varkappa_3\Big) + \frac{\rho_{n4}}{2\rho_{n3}u_3}\Big(\frac{1}{\tau_{phi}} + \frac{4c^2}{3u_3^2}\frac{1}{\tau_\zeta}\Big). \quad (75)$$

The physical meaning of (74) is obvious and is similar to that described in Ref. 14 for a phonon-roton system. When the phonon mean free path $c\tau_{phi}$ substantially exceeds the sound wavelength k^{-1}, the second sound propagates through the impurity gas with velocity u_3 and damping α_2, the latter determined by the impuriton hydrodynamics and by the phonon kinetics. For the impurity system we have $\rho_{n4} = 0$ and $C_4 = 0$, and (75) becomes equal to the result of Ref. 7.

In the opposite limiting case $\omega\tau_{phi}\ll u_2/c$, according to (63) and (73) the dispersion equation takes the form

$$\Big(\frac{\omega}{k}\Big)^2 = u_{2N}^2 + u_{2\varepsilon}^2\Big[1 + \Big(\frac{\rho_{n4}C_4}{3\rho_n C_V}\Big)^{1/2}\frac{c}{u_{2\varepsilon}}\frac{i\omega\tau_\zeta}{1-i\omega\tau_\zeta}\Big]^2$$

$$\times\Big[1 + i\frac{k^2\varkappa_3}{\omega C_V} + i\frac{C_4}{C_V}\frac{c^2k^2\tau_{phi}}{\omega} + \frac{C_4}{C_V}\frac{i\omega\tau_\zeta}{1-i\omega\tau_\zeta}\Big]^{-1}$$

$$-\frac{\rho_{n4}c^2}{3\rho_n}\frac{i\omega\tau_\zeta}{1-i\omega\tau_\zeta} - i\frac{4}{3}\frac{\eta_3}{\rho_n}\omega \quad (76)$$

to simplify the result. We have left out of (76) small terms due to the phonon shear viscosity.

In the limit $\omega\tau_\zeta\ll 1$ it follows from (63) and (73) that

$$u_2 = \Big[\frac{T\bar{S}^2}{\rho_n C_V} + \frac{n_3}{\rho_n}\Big(\frac{\partial P_f}{\partial n_3}\Big)_{Tn_4}\Big]^{1/2}, \quad (77)$$

$$\alpha_2 = \frac{\omega^2}{2\rho_n u_2^3}\Big[\frac{4}{3}\eta + \zeta_{II} + \frac{\varkappa_3\rho_n}{C_V}\frac{u_{2\varepsilon}^2}{u_2^2}$$

$$+ \rho_n D\frac{u_{2N}^2}{u_2^2}\Big(1 + k_T\frac{\rho_4\bar{S}}{\rho_3 C_V}\Big)^2\Big], \quad (78)$$

where

$$\zeta_{II} = \zeta_2 - 2\rho_4\zeta_1 + \rho_4^2\zeta_3 = \frac{1}{9}\Big(\frac{C_3}{C_V}\Big)^2 C_4 T\tau_\zeta \quad (79)$$

is a combination of the second-viscosity coefficients. Taking an earlier remark into account we have written out here the result with the various τ_l that are contained in the corresponding dissipation coefficients.

Expression (78) was reduced to the same form as the result of Ref. 13, which follows from the complete system of hydrodynamic equations. Analysis of the result (78) shows that the main contribution to second-sound absorption is made by diffusion, thermal diffusion, and second viscosity. Figure 3 shows the temperature dependence, calculated from (78), of the second-sound absorption coefficient in a solution with $x = 1.3\cdot10^{-2}$, as well as experimental data from Ref. 9 and a curve plotted in accordance with the theory of Baym and Saam. So large a deviation of the latter from the experimental values and from those calculated from (78) is due to the need for taking small-angle phonon-phonon scattering into account; this scattering, as follows from Fig. 1, is important for the establishment of total equilibrium in the phonon system. In addition, the equation used for second-sound absorption in Ref. 9 differs, in particular, also by a numerical coefficient from the result of (78). It is impossible at present to identify the cause of the difference, since Ref. 9 cites only the results of an unpublished paper by Saam. We note that the discrepancy seen in Fig. 3 between our theoretical values and the experimental data at $T < 0.2$ K is due to the fact that the hydrodynamic approximation no longer holds at low temperatures.

The discrepancy between the calculation by the Baym theory and experiment increases with decreasing concentration. Measurements of second-sound absorption in low-con-
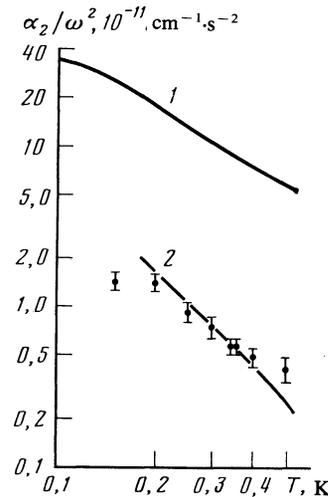

FIG. 3. Temperature dependence of the second-sound absorption coefficient in a solution with $x = 1.3\cdot10^{-2}$: 1—values calculated by the Baym-Saam-Ebner theory, ●—experimental data of Ref. 9; 2—calculations in accordance with (78) with the time shown in Fig. 1.

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centration solutions with $x \approx 10^{-3}$ are contained in Refs. 17 and 19. For these solutions, just as for $x = 1.3 \cdot 10^{-2}$, calculations by means of (78) with the time shown in Fig. 1 yield values close to the experimental ones, and the latter differ by approximately an order of magnitude from those calculated with the times of the Baym-Saam theory.[2–4]

Relations (74), (76), and (77) allow us to track the dispersion of the second sound whose velocity ranges from u_3 in the high-frequency limit to u_2 in the hydrodynamic case. Analysis of the dispersion equations (63), (73), and (76) shows that there are two second-sound dispersion domains: a space domain, where $ck\tau_{phi} \sim 1$ and the second-sound wavelength is of the order of the mean free path, and a frequency domain where $\omega\tau_\zeta \sim 1$ and the dispersion is due to the slow establishment of energy balance between the phonon and the impurity systems.

The second-viscosity coefficients ζ_1, ζ_2, ζ_3, and ζ_4 can be determined by comparing (53) and (78) with their hydrodynamic expressions. We take into account here that according to Refs. 14 and 25 the second-viscosity coefficients are connected by the relations

$$\zeta_4 = \zeta_1, \qquad \zeta_1^2 = \zeta_2\zeta_3.$$

We finally obtain

$$\rho_4\zeta_1 = (u + w - {}^1\!/_3)(u + w)(C_3/C_V)^2 C_4 T \tau_\zeta,$$

$$\zeta_2 = (u + w - {}^1\!/_3)^2 (C_3/C_V)^2 C_4 T \tau_\zeta, \qquad (80)$$

$$\rho_4^2\zeta_3 = (u + w)^2 (C_3/C_V)^2 C_4 T \tau_\zeta.$$

The time τ_ζ is given by (52), according to which the second viscosity is determined by the slow establishment of energy balance between the phonon and impurity systems.

CONCLUSION

Collective modes in the quantum-liquid He³–He⁴ solutions have been investigated in a wide frequency range. The calculation procedure employed made it possible to express in explicit form all the dissipation coefficients of a phonon-impurity system, including the second-viscosity coefficients (80). The times (37) and (52) that determine these coefficients differ substantially from those obtained earlier in Refs. 2–4. The projection-operator method employed can be used to solve in practice similar problems for a mixture of arbitrary gases.

A general expression (21) was obtained in the hydrodynamic limit for the coefficient of first-sound absorption by a system of phonons, rotons, and impuritons. Calculation by means of (21) yields for a phonon-impuriton system an expression that differs substantially, in the pertinent limiting case, from that given in Ref. 4. The difference is due to two circumstances: allowances for the small-angle phonon-phonon scattering that causes rapid two-stage relaxation in the phonon gas, and the phonon-impurity scattering inelasticity that determines, according to (52) and (80) the second-viscosity coefficients.

The relaxation time obtained agrees with the experimental data (Fig. 1) and differs substantially from that calculated earlier.[2–4]

Renormalization of the first-sound velocity in a phonon-roton-impurity system explains qualitatively the experimentally observed abrupt decrease of this velocity at $T > 0.6$ K, and agrees qualitatively with experiments in the high-temperature region (Fig. 2).

The second-sound velocity and damping in a phonon-impurity system have been determined in the frequency, temperature, and concentration ranges defined by the inequality (67). Relations (74), (76), and (77) permit the second-sound velocity dispersion to be tracked from the value u_3 (74) in the high-frequency limit down to u_2 (19) in the hydrodynamic limit.

The calculated second-sound absorption coefficient (79) agrees with the experimental data (Fig. 3) and differs substantially from the result obtained in the theory of Baym et al.

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