Dynamics of integrable quantum systems

V. I. Yudson

Institute of Spectroscopy, Academy of Sciences of the USSR, Moscow
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An approach to the description of the temporal evolution of an arbitrary physical state in systems whose eigenstates are determined by means of the Bethe ansatz is proposed. The approach is based on a representation of an arbitrary state by a superposition of generalized Bethe states, which does not contain the sum over the string configurations. The one-dimensional Bose-gas with attractive forces and the Dicke model are examined. For the latter, explicit expressions are obtained for the multiphoton wave functions describing the electromagnetic field in superradiant decay of atomic excitation and in photon scattering (resonant fluorescence).

§1. INTRODUCTION

Several models in quantum field theory admit of an explicit determination of eigenstates by means of the Bethe ansatz or its algebraic version, namely, the quantum inverse problem method (see, for example, the reviews given in Refs. 1–4). In some of these models, the temporal evolution of a given initial state is of physical interest. An immediate example is the Dicke model,[5] which describes the resonant interaction between two-level atoms and an quantum electromagnetic field. The eigenstates of the model were constructed in Refs. 6 and 7 with the aid of the Bethe ansatz (the model involving the continuous resonant medium was examined in the original paper[8] by the inverse quantum problem method). Typical dynamic processes in the "atom + field" system are, for example, cooperative spontaneous emission during the decay of an excited initial state of an atomic subsystem and the scattering of photons by an atomic subsystem (resonant fluorescence).

In addition to the Dicke model, we shall consider the usual object of the theory of integrable systems, namely, the one-dimensional Bose gas with attraction [quantum nonlinear Schrödinger (NS) equation]. The dynamic problem that is of definite methodological interest in the NS model is the temporal evolution of a packet of interacting bosons.

Of course, if we know the complete set of eigenstates $\{|\lambda\rangle = |\lambda_1,...\rangle\}$ of the Hamiltonian H of a system and the corresponding energy eigenvalues $\{E(\lambda)\}$, we can, in principle, completely describe the evolution of an arbitrary state $|\Psi_0\rangle$ specified at the initial time ($t = 0$):

$$|\Psi(t)\rangle \equiv \exp(-iHt)|\Psi_0\rangle = \sum_{\lambda} \exp[-iE(\lambda)t]|\lambda\rangle\langle\lambda|\Psi_0\rangle, \tag{1}$$

i.e., we can solve the quantum Cauchy problem. For example, in the case of a simple model such as the one-dimensional Bose gas with repulsion, the N-particle Bethe states are characterized by the set of N real parameters (rapidities) $\lambda = \{\lambda_1,...,\lambda_N\}$, that independently assume values in the interval $(-\infty, \infty)$. The sum over states in (1) then reduces to the evaluation of an N-fold integral (for the N-particle state $|\Psi_0\rangle$).

A much more complex situation arises in models similar to the Dicke model and the NS model (with attraction) considered here, but which allow the formation of bound quasiparticle complexes. The Bethe N-particle state in such models is characterized by the set of N complex rapidities $\lambda_1,...,\lambda_N$, combined into "strings." A string of n rapidities is defined by

$$\lambda_j = \Lambda - i\varkappa(n+1-2j)/2, \quad j = 1, \ldots, n,$$

where $\varkappa$ is the positive interaction constant of the model (see §2) and Λ is a generalized real part ("principal rapidity") of the rapidities comprising the given string. The Bethe state is thus characterized by a set of strings (configurations). The carrier rapidities of strings assume independently arbitrary values on the real axis. Figure 1 shows possible configurations for the three-particle Bethe state. In the description of the evolution of the N-particle initial state, the summation over the Bethe states in (1) is performed by integrating over the carrier rapidities $\Lambda = \{\Lambda_1,...\}$ of a given configuration, followed by summation over all the possible configurations of the strings:

$$|\Psi(t)\rangle = \sum \int d\Lambda \exp[-iE(\lambda)t]|\lambda\rangle\langle\lambda|\Psi_0\rangle. \tag{2}$$

The number of possible configurations for given N is equal to the number of ways of representing it in the form of a sum of nonnegative integers (when the order in which the terms are arranged is unimportant), and increases rapidly with increasing N. The fact that a large number of integrals has to be taken into account (2) means that the calculation is very laborious even for small values $N \gtrsim 4$, and the general analysis of the properties of the state $|\Psi(t)\rangle$ is extremely difficult.

The well-known alternative approach to the dynamics of integrable models is related to the quantum inverse problem method and involves the solution of the quantum Gel'fand-Levitan equations.[2,9–11] In models with string solutions, these equations constitute a set of integral operator relations (cf. Refs. 10 and 11 for the NS model), the complexity of which has prevented their application.

In this paper, we propose a method of describing the temporal evolution of states, which avoids the difficulties encountered in summing over string configurations. The result of our analysis is the following representation of the state $|\Psi(t)\rangle$:

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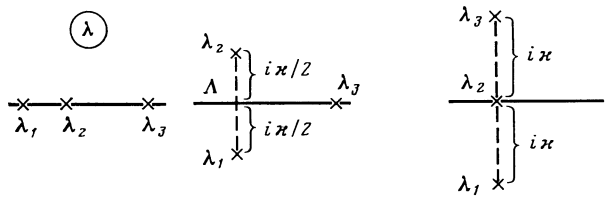

FIG. 1.

$$|\Psi(t)\rangle=\int_{\Gamma} d\lambda \exp[-iE(\lambda)t]\,|\lambda\rangle(\lambda|\Psi_0\rangle. \tag{3}$$

The symbolism used in (3) will be explained below. For the moment, it is important to note the remarkable fact that (3) does not involve summation over the string configurations, so that, as in the case of models that do not have string solutions, the state $|\Psi(t)\rangle$ is determined by a single integral over the contour Γ (complex in this case). Depending on the initial state $|\Psi_0\rangle$, this integral can be evaluated explicitly or be the basis for a systematic analysis.

The immediate stimulus to the present research was the following surprising fact discovered by Rupasov and the present author:[7] when (2) is used in the Dicke model to determine the final state of the radiation field emitted during the decay of an excited state of a system consisting of two or three atoms ($N=2$ or 3, respectively), the complicated expressions that appear at intermediate stages mutually cancel out and eventually reduce to the simple form of the photon wave function. The hypothesis that this type of cancellation will also occur in the case of an arbitrary number N of excited atoms was used in Ref. 7 to determine the physical characteristics of the emitted radiation. It will be shown below that the validity of this hypothesis is a special consequence of the general integral representation given by (3).

Section 2 gives the necessary information about the models under consideration, including the expressions for the eigenstates (Bethe states). Section 3 derives the generalized Bethe states (with arbitrary complex rapidities) and proves the validity, in the weak sense, of several relationships that are valid for ordinary Bethe states. The general representation (3) is obtained in Section 4 for an arbitrary state in the Dicke and NS models. Applications of these results to physical problems, such as superradiant decay and the scattering of photons by an atomic system, are discussed in Sec. 5.

§2. THE MODELS

A. The Dicke model[5] describes a set of two-level atoms confined to a small volume with linear dimensions $R \ll \omega_{12}^{-1}$ and interacting with a quantized electromagnetic field (ω_2 is the frequency of the atomic transition, and we are using the system of units in which $\hbar = c = 1$). The resonance character of the interaction was investigated in Ref. 7 in a reduction of the model to the one-dimensional case (a similar procedure was previously used for the Kondo model; cf. Ref. 3). The Hamiltonian for the effective one-dimensional system has the form

$$H=\int dx\{-i\varepsilon^+(x)\,\partial_x\varepsilon(x)-\varkappa^{1/2}[S^+(x)\,\varepsilon(x)+\varepsilon^+(x)\,S^-(x)]\}, \tag{4}$$

where the spin operator

$$\mathbf{S}(x)=\mathbf{s}\delta(x)=\sum_{a=1}^{M}\mathbf{s}_a\delta(x) \tag{5}$$

($s^{\pm}=s^x\pm is^y$) describes the localized system of M two-level atoms (the operator s_a refers to the ath atom), the operator $\varepsilon(x)$ describes the electromagnetic field, and $\varkappa$ is the positive interaction constant. The commutation relations have the form

$$[\varepsilon(x),\ \varepsilon^+(y)]=\delta(x-y), \tag{6a}$$

$$[s_a^i,\ s_b^j]=ie^{ijk}s_a^k\delta_{ab},\quad a,\ b=1,\ldots,M,\quad i,\ j,\ k=x,\ y,\ z. \tag{6b}$$

B. The model proposed by MacGillivray and Feld (MF model)[12] was devised for the classical description of Dicke superradiance in an extended resonant medium with small lateral dimensions. The quantum version of this model with discrete atoms was examined in Ref. 6, where it was referred to as the Dicke model. In the ensuing analysis, we use the more appropriate terminology "the MF model." Because of the particular geometry of the problem, the MF model is initially one-dimensional and is described by expressions that are formally identical with (4) and (6). The only difference is in the definition of the operator $S(x)$:

$$\mathbf{S}(x)=\sum_{a=1}^{M}\mathbf{s}_a\delta(x-r_a), \tag{7}$$

where r_a is the coordinate of the a-th atom and M is the number of atoms. To avoid any misunderstanding, we emphasize that the Dicke model defined by (4) and (5) is not a special case of the MF model defined by (4) and (7) in the limit as $r_a\to 0$, $a=1,\ldots,M$.

C. The MF model is described by the Hamiltonian

$$H=\int dx[\partial_x\varepsilon^+(x)\,\partial_x\varepsilon(x)-\varkappa\varepsilon^+(x)\,\varepsilon^+(x)\,\varepsilon(x)\,\varepsilon(x)] \tag{8}$$

and the commutation relations (6a). Below, we consider the case of attraction, $\varkappa>0$.

The models listed above have the following constant of motion [particle (excitation) number operator]:

$$N=\int dx\varepsilon^+(x)\,\varepsilon(x)+\sum_{a=1}^{M}\left(s_a^z+\frac{1}{2}\right)\quad(\text{Dicke + MF}),$$
$$N=\int dx\varepsilon^+(x)\,\varepsilon(x)\quad(\text{NS}), \tag{9}$$

and this can be used to classify the corresponding eigenstates. The N-particle eigenstates of the Dicke and MF models constructed with the aid of the Bethe Ansatz are characterized by the set of rapidities $\lambda=\{\lambda_1,\ldots,\lambda_N\}$ and have the form[6,7]

$$|\lambda_1,\ldots,\lambda_N\rangle=C(\lambda)\int d^Ny\prod_{l<j}\left[1+\frac{i\varkappa}{\lambda_l-\lambda_j}\operatorname{sign}(y_l-y_j)\right]$$
$$\times\prod_{j=1}^{N}f(\lambda_j,y_j)\exp(i\lambda_jy_j)\,r^+(y_j,\lambda_j)\,|0\rangle, \tag{10}$$

where

$$r^+(y,\lambda)=\varepsilon^+(y)-(\varkappa^{1/2}/\lambda)S^+(y),$$

$$f(\lambda,y)=\begin{cases}\dfrac{\lambda-(i\varkappa M/2)\operatorname{sign} y}{\lambda+i\varkappa M/2} & \text{(Dicke)}\\ \displaystyle\prod_{a=1}^{M}\frac{\lambda-(i\varkappa/2)\operatorname{sign}(y-r_a)}{\lambda+i\varkappa/2} & \text{(MF)}\end{cases} \tag{11}$$

and

$$\operatorname{sign} y=\{-1,\ y<0;\ 0,\ y=0;\ 1,\ y>0\}.$$

The factor $C(\lambda)$ is determined by the normalization conditions. In particular, it can be taken in the following form for configurations without strings:

$$C(\lambda)=(2\pi)^{-N/2}(N!)^{-1/2}\prod_{l<j}\frac{\lambda_l-\lambda_j}{\lambda_l-\lambda_j+i\varkappa}, \tag{12}$$

which corresponds to the normalization

$$\langle\lambda|\mu\rangle=(N!)^{-1}\sum\prod\delta(\lambda_j-\mu_j),$$

where the sum is evaluated over all the permutations of the parameters $\{\mu_j\}$.

For the NS model, the $|\lambda\rangle$ is described by (10) with $r^+(y,\lambda)\to\varepsilon^+(y), f(\lambda,y)\to 1$:

$$|\lambda\rangle=C(\lambda)\int d^Ny\prod_{l<j}\left[1+\frac{i\varkappa}{\lambda_l-\lambda_j}\operatorname{sign}(y_l-y_j)\right]\times\prod_{j=1}^{N}\exp(i\lambda_jy_j)\,\varepsilon^+(y_j)\,|0\rangle. \tag{13}$$

The admissible values of the parameters $\lambda_1,\ldots,\lambda_N$ are determined by the requirement that the wave functions must be bonded, and are described in the Introduction. The vacuum state $|0\rangle$ in (10) and (13) satisfies the conditions $\varepsilon(x)|0\rangle=0$ and (in the Dicke and MF models) $s_a^-|0\rangle=0$, $a=1,\ldots,M$. We conclude our description of models by listing the energy eigenvalues in the state $|\lambda_1,\ldots,\lambda_N\rangle$:

$$E(\lambda)=\sum_{j=1}^{N}\begin{cases}\lambda_j & \text{(Dicke + MF)}\\ \lambda_j^2 & \text{(NS)}\end{cases}. \tag{14}$$

§3. GENERALIZED BETHE STATES

The proposed approach is based on the representation of the state $|\Psi(t)\rangle$ in the form of the integral (3) of the Bethe state $|\lambda_1,\ldots,\lambda_N\rangle$ over a contour Γ in the complex space. We emphasize in this connection that expressions such as (3) are only formal because, for a complex contour Γ, they involve Bethe states $|\lambda_1,\ldots,\lambda_N\rangle$ with rapidities that do not satisfy the boundedness condition for the wave functions. Such states ("generalized Bethe states") can be given a meaning because it is possible to form matrix elements $\langle F|\lambda\rangle$ with states $|F\rangle$ whose boson wave functions vanish outside a finite range of coordinates or decrease sufficiently rapidly (or oscillate—see below) with increasing coordinates. For brevity, we shall use the phase "F-state." The simplest example of an F-state is $\varepsilon^+(x_1)\ldots\varepsilon^+(x_N)|0\rangle$; a trivial example is the state (31), which does not contain bosons at all. Any physical state is, of course, an F-state.

It is obvious that the effect of the Hamiltonian, which is a local operator, on an F-state is another F-state, which enables us to define the matrix element $\langle F|H|\lambda\rangle$. Next, we note that verification of the relation $H|\lambda\rangle=E(\lambda)|\lambda\rangle$ reduces to a chain of algebraic transformations, in which the values of the parameters $\{\lambda_j\}$ are unimportant, and to operations involving integration by parts. Owing to the presence of the F-state in the matrix element $\langle F|H|\lambda\rangle$, which effectively cuts off the region of integration, it is possible to integrate by parts even for an arbitrary set $\{\lambda_j\}$, so that we have the equation $\langle F|H|\lambda\rangle=E(\lambda)\langle F|\lambda\rangle$ and its immediate generalization

$$\langle F|H^n|\lambda\rangle=E^n(\lambda)\langle F|\lambda\rangle,\quad n=1,\ 2,\ldots. \tag{15}$$

Finally, consider the matrix element of the evolution operator $\langle F|\exp(-iHt)|\lambda\rangle$. In the Dicke and MF models, in which the excitation propagates from the source with velocity less than or equal to the velocity of light $c=1$, the application of the evolution operator to an F-state, which gives a meaning to the expression $\langle F|\exp(-iHt)|\lambda\rangle$ for arbitrary $\{\lambda_j\}$. The situation is somewhat more complicated in the NS model in the sense that the excitation "diffuses" and, for any nonzero instant of time, there is a definite probability of finding the boson arbitrarily far from its point of creation (at $t=0$). For example:

$$\exp(-iHt)\,\varepsilon^+(x)\,|0\rangle=(2\pi t)^{-1/2}e^{-i\pi/4}\int dy\exp\left[i\frac{(y-x)^2}{2t}\right]\varepsilon^+(y)\,|0\rangle. \tag{16}$$

However, owing to the rapid oscillation of the amplitude with increasing distance from the source, integrals of expressions of the form $\exp[iy^2/2t+i\lambda y]$ converge for any complex λ, which enables us to determine the matrix element $\langle F|\exp(-iHt)|\lambda\rangle$ for arbitrary $\{\lambda_j\}$.

The relation given by (15) and the fact that we can define the matrix element $\langle F|\exp(-iHt)|\lambda\rangle$ lead to the following important consequence:

$$\langle F|\exp(-iHt)|\lambda\rangle=\exp[-iE(\lambda)t]\langle F|\lambda\rangle, \tag{17}$$

which is valid for the Dicke, MF, and NS models with arbitrary parameters $\{\lambda_j\}$.

In the ensuing presentation, we shall, as a rule, write out not the matrix elements but the states $|\lambda\rangle$ themselves or their superpositions. If the rapidities $\{\lambda_j\}$ do not then belong to the admissible Bethe set, we shall always imply the subsequent formation of matrix elements of the form (17) or $\langle F|O|\lambda\rangle$ with arbitrary local operator O.

§4. EVOLUTION OF AN ARBITRARY INITIAL STATE

The proposed approach involves the representation of an arbitrary initial state $|\Psi_0\rangle$ by the superposition of generalized Bethe states of a particular string configuration (without summation over configurations!). By virtue of (17), the description of the temporal evolution of the state $|\Psi_0\rangle$ will be reduced to the multiplication of the states $|\lambda\rangle$ in this superposition by the factor $\exp[-iE(\lambda)t]$.

The assumed generality of the required representation restricts its possible form: for example, to extract the state

$$|\Psi_0\rangle=\varepsilon^+(x_1)\dots\varepsilon^+(x_N)|0\rangle,$$

from the superposition of Bethe states (10) and (13), we must construct N δ-functions $\delta(y_j-x_j)$ in the integrands in (10) and (13). These functions may arise if we take integrals of $\exp[i\lambda_j(y_j-x_j)]$ over contours extending to infinity in the complex plane of λ. This establishes the necessity for the N-fold integration with respect to $\{\lambda_1,\dots,\lambda_N\}$, and hence the selection from all the configurations of the single configuration without strings (N trivial strings).

1. Let us begin by considering the Dicke model (after necessary reduction, the results will give us the required representation for the NS model). The basis state of the model is

$$|\Psi_0\rangle=(S^+)^m\prod_j\varepsilon^+(x_j)|0\rangle, \tag{18}$$

where $x_1>\dots>x_k>0=x_{k+1}=\dots=x_{k+m}>x_{k+m+1}>\dots>x_N$ and we are dealing with the product of boson operators $\varepsilon^+(x_j)$ with $j\in\{1,\dots,k;\,k+m+1,\dots,N\}$. Equation (18) describes the state with $N-k-m$ incident and k departing photons under m-fold excitation of the atomic subsystem ($m\leqslant M$; the ratio of N and M is arbitrary). We shall seek the representation for $|\Psi_0\rangle$ in the form

$$|\Psi_0\rangle=\int_\Gamma d^N\lambda\,|\lambda_1,\dots,\lambda_N\rangle A(\{\lambda_j\},\{x_j\}). \tag{19}$$

The coefficient $A(\{\lambda_j\},\{x_j\})$ is chosen by the analogy with the expression $\langle\lambda|\Psi_0\rangle$ in the form

$$A(\{\lambda_j\},\{x_j\})$$

$$=\frac{(-\varkappa^{1/2})^m(N!)^{1/2}M!}{(2\pi)^{N/2}(M-m)!}\prod_{j=k+1}^{k+m}\lambda_j^{-1}\prod_{j=1}^{N}\overline{f(\bar\lambda_j,x_j)}\exp(-i\lambda_jx_j). \tag{20}$$

Let us establish the validity of (19) and (20):

$$|\Psi_0\rangle=\frac{(-\varkappa^{1/2})^mM!}{(M-m)!}\int d^Ny\int_\Gamma\frac{d^N\lambda}{(2\pi)^N}$$

$$\times\prod_{l<j}\frac{\lambda_l-\lambda_j+i\varkappa\,\mathrm{sign}\,(y_l-y_j)}{\lambda_l-\lambda_j+i\varkappa}\prod_{j=k+1}^{k+m}\lambda_j^{-1}$$

$$\times\prod_{j=1}^{N}\frac{\lambda_j-(i\varkappa M/2)\,\mathrm{sign}\,y_j}{\lambda_j+i\varkappa M/2}\,\frac{\lambda_j+(i\varkappa M/2)\,\mathrm{sign}\,x_j}{\lambda_j-i\varkappa M/2}$$

$$\times\exp[i\lambda_j(y_j-x_j)]$$

$$\times\left[\varepsilon^+(y_j)-\frac{\varkappa^{1/2}}{\lambda_j}S^+\delta(y_j)\right]|0\rangle \tag{21}$$

for the required contour $\Gamma=\gamma_1\otimes\dots\otimes\gamma_N$. The component contours γ_j will be assumed to be parallel to the real axes in the planes of the complex variables λ_j: $\mathrm{Im}\gamma_j=\mathrm{const}$. We shall also require that

$$\mathrm{Im}\,\gamma_{j+1}-\mathrm{Im}\,\gamma_j>\varkappa, \tag{22a}$$

$$\mathrm{Im}\,\gamma_j<\varkappa M/2,\quad j=1,\dots,k, \tag{22b}$$

$$-\varkappa M/2<\mathrm{Im}\,\gamma_j<\varkappa M/2,\quad j=k+1,\dots,k+m, \tag{22c}$$

$$\mathrm{Im}\,\gamma_j>-\varkappa M/2,\quad j=k+m+1,\dots,N. \tag{22d}$$

It is readily verified that this set of equations is consistent for $m\leqslant M$. The improper integrals of the form

$$I=\int\frac{d\lambda}{2\pi}\varphi(\lambda)\exp[i\lambda(y-x)],$$

that appear in (21), where $\varphi(\lambda)\to1$ and $|\lambda|\to\infty$, will be looked upon as integrals over the region $\mathrm{Re}\lambda\in(-\Lambda,\Lambda)$, $\Lambda\to\infty$, and we shall thus locate the singular contribution:

$$I=\delta(y-x)+\int\frac{d\lambda}{2\pi}[\varphi(\lambda)-1]\exp[i\lambda(y-x)].$$

We shall first prove that the integrand in (21) is different from zero only for $\{y_j\leqslant x_j;j=1,\dots,N\}$. In fact, for $y_N>x_N$, the singular part of the integral with respect to λ_N provides no contribution and, if we bend the contour λ_N into the upper half-plane, we see that, by virtue of (22a,d), the region it bounds does not contain the poles $\lambda_N=\lambda_j+i\varkappa(j=1,\dots,N-1)$ and $\lambda_N=-i\varkappa M/2$, and that the factor $(\lambda_N-i\varkappa M/2)^{-1}$ cancels with the numerator of the fraction (since $x_N<0$). Hence, it follows that the integrand in (21) is nonzero only for $y_N\leqslant x_N$; the inequality symbol here and henceforth reminds us of the possible singularity at $y=x$. If now $y_{N-1}>x_{N-1}$, we deform the contour γ_{N-1} for integration with respect to λ_{N-1} in the upward direction. The new feature, as compared with the last case, is the appearance of the pole $\lambda_{N-1}=\lambda_N-i\varkappa$ inside the contour γ_{n-1}. However, the inequality $y_{N-1}-y_N>x_{N-1}-x_N>0$ leads to the appearance of the factor $\lambda_{N-1}-\lambda_N+i\varkappa\,\mathrm{sign}(+)$ in (21), which cancels this pole. This results in the condition $y_{N-1}\leqslant x_{N-1}$, and, in precisely the same way, in the analogous conditions for all $j>k+m$. The argument has to be modified somewhat for $j\leqslant k+m$. The cancellation of the singularity $(\lambda_j-i\varkappa M/2)^{-1}$ for $y_j>x_j\geqslant0$ then occurs because of the presence of the factor $\lambda_j-(i\varkappa M/2)\,\mathrm{sign}\,y_j$ [moreover, for $j\leqslant k(x_j>0)$, the poles $\lambda_j=-i\varkappa M/2$ are absent]. This leads us to the conclusion that the integrand in (21) is different from zero only in the region $y_j\leqslant x_j;j=1,\dots,N$.

Let us now suppose that $y_1<x_1$. If we bend the contour γ_1 so that it enters the lower half-plane of the complex variables λ_1, we note that, by virtue of (22a,b) and the presence of the factor $\lambda_1+(i\varkappa M/2)\,\mathrm{sign}\,x_1$, the interior of the contour γ_1 does not contain any singularities. Thus, the nonzero contribution to the integral with respect to y_1 is entirely due to the singularity at $y_1=x_1$. Next, if $y_2<x_2$, the singularities inside the contour γ_2, closed in the lower half-plane, are absent by virtue of (27a,b) and the factors $\lambda_2+(i\varkappa M/2)\,\mathrm{sign}\,x_2$ and $\lambda_1-\lambda_2-i\varkappa\,\mathrm{sign}(+)$, where the form of the latter follows from the inequality $y_1-y_2>x_1-x_2>0$. Analogous discussions for arbitrary j will show that the integral (21) is exclusively determined by the contribution due to singularities at the points $y_j=x_j$, $j=1,\dots,N$. The singularity for $j\leqslant k$ and $j>k+m$ is due to the divergence of the integrals with respect to λ_j. For $k<j\leqslant k+m$, the integrals with respect to λ_j

converge and the singular contribution is due to the δ-functions next to the spin operators. The integral (21) is thus finally given by

$$|\Psi_0\rangle = \frac{\varkappa^m M!}{(M-m)!}\int_{\tilde{\Gamma}}\prod_{j=k+1}^{k+m}\frac{d\lambda_j}{2\pi}\frac{1}{\lambda_j^2+(\varkappa M/2)^2} \times \prod_{k<l<j}^{k+m}\frac{\lambda_l-\lambda_j}{\lambda_l-\lambda_j+i\varkappa}(S^+)^m\prod_{j\leqslant k,\ j>k+m}\varepsilon^+(x_j)|0\rangle, \tag{23}$$

where $\tilde{\Gamma}=\gamma_{k+1}\otimes\ldots\otimes\gamma_{k+m}$ and we have explicitly written out the operators that create the state $|\Psi_0\rangle$ (18). The integral in (23) is relatively simple to evaluate and the result cancels the factor in front of the integral which, of course, is assured by the correct choice of the coefficient $A(\{\lambda_j\},\{x_j\})$ (20). We have thus proved the validity of (19) and (20).

The representation given by (19) and (20) can be written in compact form by introducing the auxiliary state

$$|\lambda_1,\ldots,\lambda_N\rangle = \frac{(N!)^{1/2}}{(2\pi)^{N/2}}\int d^N y\,\theta(y_1\geqslant\ldots\geqslant y_N) \times\prod_{j=1}^{N}e^{i\lambda_j y_j}f(\lambda_j,y_j)r^+(y_j,\lambda_j)|0\rangle, \tag{24}$$

where the nonrigorous inequalities in the argument of the θ-function serve to remind us of the δ-functions next to the spin operators, and it is assumed, by definition, that

$$\theta(\ldots>y_k>y_{k+1}=\ldots=y_{k+m}=0>y_{k+m+1}>\ldots) =(1/m!)\theta(\ldots>y_k>0>y_{k+m+1}>\ldots).$$

The parentheses in (24) emphasize the difference between this (non-Bethe) generalized state and the corresponding Bethe state (10).

It is readily verified that the coefficient $A(\{\lambda_j\},\{x_j\})$ (20) is given by $A(\{\lambda_j\},\{x_j\})=\langle\bar{\lambda}_1,\ldots,\bar{\lambda}_N|\Psi_0\rangle$. As a consequence, we obtain the compact version of the superposition (19):

$$|\Psi_0\rangle=\int_\Gamma d^N\lambda\,|\lambda_1,\ldots,\lambda_N\rangle\langle\bar{\lambda}_1,\ldots,\bar{\lambda}_N|\Psi_0\rangle. \tag{25}$$

The state $|\Psi(t)\rangle=\exp(-iHt)|\Psi_0\rangle$ is determined with the aid of (17):

$$|\Psi(t)\rangle=\int_\Gamma d^N\lambda\exp[-iE(\lambda)t]\,|\lambda_1,\ldots,\lambda_N\rangle\langle\bar{\lambda}_1,\ldots,\bar{\lambda}_N|\Psi_0\rangle, \tag{26}$$

which is the basic result of the present paper. We note that, in some sectors of the space of Dicke states, the contour Γ can be taken in a form that does not depend on the type of state (18), so that (26) can be written in the form of a representation of the evolution operator in the given sector:

$$\exp(-iHt)=\int_\Gamma d^N\lambda\exp[-iE(\lambda)t]\,|\lambda_1,\ldots,\lambda_N\rangle\langle\bar{\lambda}_1,\ldots,\bar{\lambda}_N| \tag{27}$$

[when $t=0$, this formula gives the representation of a unit operator in the given sector]. For example, in the sector of N-particle states with $N\leqslant M$, the contour $\Gamma=\gamma_1\otimes\ldots\otimes\gamma_N$ can be taken in the form

$$\mathrm{Im}\,\gamma_j=-{}^1/_2\varkappa(N+1-2j-j\delta),\quad \delta\to+0. \tag{28}$$

In general, the contour Γ depends on the type of state $|\Psi_0\rangle$ (18) [cf. inequalities (22)].

2. As noted above, we shall now apply the above approach, after the necessary simplifications, to the NS model (13). The simplifications are as follows:

a) the basis vector of the model is

$$|\Psi_0\rangle=\prod_{j=1}^{N}\varepsilon^+(x_j)|0\rangle,\quad x_1>\ldots>x_N; \tag{29}$$

b) the trivial nature of the function $f(\lambda,y)\equiv1$ in the definition of the Bethe state removes from the Dicke model the necessity for bothering about the correct way to integrate around the poles $\lambda_j=\pm i\varkappa M/2$; correspondingly, we need not satisfy (22b–d), which restrict the choice of the contour Γ

c) the expression for the auxiliary generalized state $|\lambda_1,\ldots,\lambda_N\rangle$ in the NS model is obtained from the state (24) by the reduction $f\to1$, $r^+(y,\lambda)\to\varepsilon^+(y)$.

Thus, the evolution of the state $|\Psi_0\rangle$ given by (29) is described in the NS model by (26), and the choice of the contour is restricted by the single condition (22a). This condition, which is independent of the state $|\Psi_0\rangle$ (29), is satisfied, for example, by the contour defined by (28). Thus, in contrast to the Dicke model, the representation (27) is always valid for the evolution operator in the NS model.

3. The approach described in Section 1 will have to be modified to enable us to describe evolution in the MF model. The basis vector in the MF model takes the form

$$|\Psi_0\rangle=\prod_{j=1}^{N}p^+(x_j)|0\rangle\quad(x_1>\ldots>x_N), \tag{30}$$

where $p^+(x_j)=s_a^+$, if $x_j=r_a$ $(a=1,\ldots,M)$ and $p^+(x_j)=\varepsilon^+(x_j)$ if $x_j\notin\{r_a\}$. The auxiliary state $|\lambda\rangle$ for the MF model is given by (24). The representations (25) and (26) are valid for the states $|\Psi_0\rangle$ and $|\Psi(t)\rangle$, where the contour $\Gamma=\Gamma_1\otimes\ldots\otimes\Gamma_N$ is defined by the following conditions:

$\Gamma_j=\gamma_j+\gamma_j^+-\gamma_j^-$; $\mathrm{Im}\,\gamma_{j+1}-\mathrm{Im}\,\gamma_j>\varkappa$, $\mathrm{Im}\,\gamma_1>\varkappa/2$, $\mathrm{Im}\,\gamma_j^\pm=\varkappa/2\mp\delta$ $(\delta\to+0)$.

The proof of this statement is similar to that given in Section 1. We note that, in contrast to the Dicke model, the choice of the contour in the MF model does not depend on the type of states $|\Psi_0\rangle$ (30), so that (27) is valid for the evolution operator in the MF model.

Examples of the application of the results obtained in this section to the solution of physical problems are given below.

§5. PHYSICAL APPLICATIONS

1. Let us first demonstrate the efficiency of the general approach developed in the previous section by considering the example of superradiant decay of N-fold excited localized systems consisting of M atoms ($N\leqslant M$). The problem is to determine $t\to+\infty$ the photon component of the state which, at $t=0$, has the form (Dicke state[5])

$$|\Psi_0\rangle=\left[\frac{(M-N)!}{M!N!}\right]^{1/2}(S^+)^N|0\rangle\quad(\langle\Psi_0|\Psi_0\rangle=1). \tag{31}$$

The evolution of the state $|\Psi_0\rangle$ is described by[1)] (26) where the contour Γ is given by (28). The photon component

of the state (26) has the following form for $t\to+\infty$ (out-state):

$$|\text{out}\rangle=\int d^N y\varphi_N(y_1,\ldots,y_N)\prod_{j=1}^{N}\varepsilon^+(y_j)|0\rangle, \tag{32}$$

where the wave function (unsymmetrized) is given by

$$\varphi_N(y_1,\ldots,y_N)=(-1)^N\varkappa^{N/2}\left[\frac{M!}{(M-N)!N!}\right]^{1/2}\int_\Gamma\frac{d^N\lambda}{(2\pi)^N}$$

$$\times\prod_{l<j}\frac{\lambda_l-\lambda_j+i\varkappa\,\mathrm{sign}(y_l-y_j)}{\lambda_l-\lambda_j+i\varkappa}\prod_{j=1}^{N}\frac{\exp[i\lambda_j(y_j-t)]}{\lambda_j+i\varkappa M/2}. \tag{33}$$

It is readily verified (by successively deforming the contours $\gamma_N,\ldots,\gamma_1$ in the upward direction) that the wave function (33) is different from zero only in the causal region $\{y_j\leqslant t\}$. If we successively deform the contours $\gamma_1,\ldots,\gamma_N$ in the downward direction, we find that the value of the function (33) is determined exclusively by a set of poles that do not coincide for different variables λ_j; if we take into account the presence of, say, two identical poles, $\lambda_1=-i\varkappa M/2=\lambda_2$, we find that the factor $\mathrm{sign}(y_1-y_2)$ appears in the integrand, which is symmetric under the interchange $y_1\leftrightarrow y_2$. We now take all this into account and perform the successive integrations with respect to $\lambda_1,\ldots,\lambda_N$, bearing in mind the poles $\lambda_1=-i\varkappa M/2$, $\lambda_2=-i\varkappa M/2+i$ (for $y_1<y_2$), $\lambda_3=-i\varkappa M/2+2i$ ($y_2<y_3$), and so on. The final result is

$$\varphi_N(y_1,\ldots,y_N)=\theta(y_1<\ldots<y_N<t)\,i^N\varkappa^{N/2}\left[\frac{M!N!}{(M-N)!}\right]^{1/2}$$

$$\times\exp\left[\frac{\varkappa}{2}\sum_{j=1}^{N}(M-2j+2)(y_j-t)\right]. \tag{34}$$

This is identical with equation (36) of Ref. 7, which proves the validity of the hypothesis used by Rupasov and the present author[7] on the summation over configurations in the case of an arbitrary number of excited atoms.

We now reproduce the explicit expression for the state $|\Psi(t)\rangle$ at an arbitrary time, deduced with the aid of the representation (26):

$$|\Psi(t)\rangle=\left[\frac{N!}{M!(M-N)!}\right]^{1/2}\sum_{m=0}^{N}\frac{(i\varkappa^{1/2})^{N-m}(M-m)!}{m!}$$

$$\times\exp\left[-\frac{\varkappa m(M-m+1)t}{2}\right]$$

$$\times\int dy_{m+1}\ldots dy_N\theta(0<y_{m+1}<\ldots<y_N<t)$$

$$\times\exp\left[\frac{\varkappa}{2}\sum_{j=m+1}^{N}(M+2-2j)(y_j-t)\right](S^+)^m\prod_{j=m+1}^{N}\varepsilon^+(y_j)|0\rangle, \tag{35}$$

This enables us to calculate all the physical observables. It is readily seen that, in the limit as $t\to+\infty$, the state given by (35) is entirely determined by the photon ($m=0$) component of (34), which describes the radiation field produced as a result of the complete decay of the excited atomic subsystem.

2. Let us now consider, in the Dicke model, the scattering of N photons by an unexcited atomic subsystem. There is particular physical interest in the scattering of light by an individual atom ($M=1$), which corresponds to typical experiments on resonant fluorescence. The description of this phenomenon involves the examination of the evolution of the initial state $|\Psi_0\rangle$ (18) with $k=m=0$. The photon wave function in the final ($t\to+\infty$) state (32) can be determined with the aid of the representation (26) in the form (unsymmetrized):

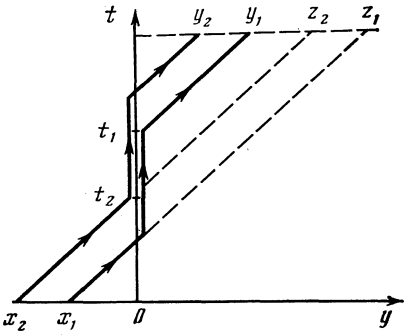

FIG. 2.

$$\varphi_N(y_1,\ldots,y_N)$$

$$=\int_\Gamma\frac{d^N\lambda}{(2\pi)^N}\prod_{l<j}\frac{\lambda_l-\lambda_j+i\varkappa\,\mathrm{sign}(y_l-y_j)}{\lambda_l-\lambda_j+i\varkappa}\prod_{j=1}^{N}\frac{\lambda_j-i\varkappa/2}{\lambda_j+i\varkappa/2}$$

$$\times\exp[i\lambda_j(y_j-z_j)], \tag{36}$$

where $z_j=x_j+t$ and the corresponding contours γ_j satisfy inequalities (22a,d). In contrast to the preceding example, the region $\{y_j\leqslant z_j;j=1,\ldots,N\}$, in which the function (36) is nonzero, is now asymmetric under the permutations $\{y_j\}$, so that we can take the residues with coincident (in different variables) poles. Moreover, it may be shown that the only important poles are $\lambda_j=-i\varkappa/2$, $j=1,\ldots,N$. For this, we note that the causal nature of the propagation of excitation in the Dicke model leads to the fact that the symmetrized wave function

$$\varphi_N^s(y_1,\ldots,y_N)=\frac{1}{N!}\sum_\sigma\varphi_N(y_{\sigma_1},\ldots,y_{\sigma_N}) \tag{37}$$

(summation over all permutations σ), considered in the sector $y_1>\ldots>y_N$, is different from zero only for

$$y_N\leqslant z_N\leqslant y_{N-1}\leqslant\ldots\leqslant z_2\leqslant y_1\leqslant z_1. \tag{38}$$

The graph (Fig. 2) of the "forbidden" two-photon scattering process ($N=2$) will explain the origin of the limitations defined by (38). The appearance of two photons in the region $y_2<y_1<z_2$ is forbidden because this process requires that the system contains no photons in the final time interval (t_2,t_1). This is, of course, impossible in the Dicke model (two-level atom and resonant excitation).

Since the function $\varphi_N(y_{\sigma1},\ldots,y_{\sigma N})$ is zero outside the region $\{y_{\sigma j}\leqslant z_j;j=1,\ldots,N\}$, we find that the contribution to (37) in the region defined by (38) is provided only by permutations $\sigma=\{\sigma_j\}$ for which $\sigma_j\geqslant j-1$, $j=2,\ldots,N$. Nonidentical permutations then ensure that the function (37) differs from zero only on a set of measure zero, the contribution of which can be due to only the singular part of the integrals with respect to the corresponding rapidities. In the case

where $\sigma_N = N-1$, this enables us to replace the factor $[\lambda_l - \lambda_N + i\varkappa \operatorname{sign}(y_{\sigma l} - y_{\sigma N})]/[\lambda_l - \lambda_N + i\varkappa]$ in (36) with unity; when $\sigma_N = N$, the factor is equal to unity by virtue of the inequality (38). Repeating the discussion for the remaining part of the integrand in (36), we finally obtain

$$\varphi_{N^s}(y_1, \ldots, y_N)$$

$$= \frac{1}{N!} \sum_{\sigma}{}' \int_{\Gamma} \frac{d^N\lambda}{(2\pi)^N} \prod_{j=1}^{N} \frac{\lambda_j - i\varkappa/2}{\lambda_j + i\varkappa/2} \exp[i\lambda_j(y_{\sigma_j} - z_j)]. \qquad (39)$$

If we now evaluate the integral, we obtain the explicit expression for the photon wave function of the final state of scattered photons in the sector $y_1 > \ldots > y_N$:

$$\varphi_{N^s}(y_1, \ldots, y_N)$$

$$= \theta(y_N \leqslant z_N \leqslant \ldots \leqslant y_1 \leqslant z_1) \frac{1}{N!} \sum_{\sigma}{}' \prod_{j=1}^{N} \left\{ \delta(y_{\sigma_j} - z_j) \right.$$

$$\left. - \varkappa\theta(y_{\sigma_j} < z_j) \exp\left[\frac{\varkappa}{2}(y_{\sigma_j} - z_j)\right] \right\}; \qquad (40)$$

where we have restored the θ-function that is implied in (37) and (39); the sum is evaluated only over the permutation $\sigma = \{\sigma_j \geqslant j-1; j = 2, \ldots, N\}$.

3. The initial state in the problem of light emission by excited atoms in an extended medium (MF model) has the form

$$|\Psi_0\rangle = \prod_{j=1}^{N} s_{a_j}^{+} |0\rangle, \quad \{a_j; j = 1, \ldots, N\} \subset \{1, \ldots, M\}, \qquad (41)$$

where the subscript a_j labels the excited atoms. The evolution of the state $|\Psi_0\rangle$ according to (4) is described by the representation (26); cf. Section 4.3. The final ($t \to +\infty$) state of the electromagnetic field is given by

$$|\text{out}\rangle = (-\varkappa^{1/2})^N \int d^N y \int_{\Gamma} \frac{d^N\lambda}{(2\pi)^N} \prod_{l<j} \frac{\lambda_l - \lambda_j + i\varkappa \operatorname{sign}(y_l - y_j)}{\lambda_l - \lambda_j + i\varkappa}$$

$$\times \prod_{j=1}^{N} \exp[i\lambda_j(y_j - z_j)] \frac{(\lambda_j - i\varkappa/2)^{a_j - 1}}{(\lambda_j + i\varkappa/2)^{a_j}} \varepsilon^{+}(y_j) |0\rangle. \qquad (42)$$

It is readily seen that the wave function differs from zero only in the causal region $\{0 < y_j < z_j \equiv x_j + t\}$, where integration over the contour Γ in this region reduces to negative circuits around the (multiple) poles $\lambda_j = -i\varkappa/2$. In contrast to the above emphasis, here, unfortunately, we cannot evaluate the N-fold integral over the variables $\{\lambda_j\}$ for arbitrary N and M. Expression (42) can also be written in the form

$$|\text{out}\rangle = \frac{i^N \varkappa^{N/2}}{N!} \int d^N y \prod_{l<j} \int_0^{\infty} d\alpha_{lj} [\delta(\alpha_{lj}) - 2\varkappa\theta(y_l - y_j)]$$

$$\times \exp(-\varkappa\alpha_{lj})$$

$$\times \sum_{\sigma} \prod_{j=1}^{N} \theta(y_j - z_{\sigma_j}) \frac{\exp[-\varkappa(y_j - z_j)/2]}{(a_j - 1)!} L_{a_j - 1}$$

$$\times \left[\varkappa\left(z_j - y_j + \sum_{l=j+1}^{N} \alpha_{jl} - \sum_{l=1}^{j-1} \alpha_{lj}\right)\right] \varepsilon^{+}(y_j) |0\rangle, \qquad (43)$$

where, by definition, the integral of $\delta_+(\alpha)g(\alpha)$ over $(0, +\infty)$ is equal to $g(0)$; $L_n(y)$ is the Laguerre polynomial. The complexity of (42) and (43) reflects the variety of processes in the MF model, which is more complex than the Dicke model.

To conclude this section, we also reproduce the expression for the out-state of the electromagnetic field when a packet of N photons (29) passes through an extended resonant (unexcited) medium:

$$|\text{out}\rangle = \int d^N y \int_{\Gamma} \frac{d^N\lambda}{(2\pi)^N} \prod_{l<j} \frac{\lambda_l - \lambda_j + i\varkappa \operatorname{sign}(y_l - y_j)}{\lambda_l - \lambda_j + i\varkappa}$$

$$\times \prod_{j=1}^{N} \left[\frac{\lambda_j - i\varkappa/2}{\lambda_j + i\varkappa/2}\right]^M \exp[i\lambda_j(y_j - z_j)] \varepsilon^{+}(y_j) |0\rangle. \qquad (44)$$

The contour Γ is described in Section 4.3; here, it reduces to the contour $\gamma_1 \otimes \ldots \otimes \gamma_N$. For $N = 2$, which is of particular independent interest (paramagnetic mixing of light), the wave function in the state (44) has the form

$$\varphi_2(y_1, y_2) = \exp\left[\frac{\varkappa}{2}(y_1 + y_2 - z_1 - z_2)\right]$$

$$\times \int_0^{\infty} d\alpha [\delta_+(\alpha) - 2\varkappa\theta(y_1 < y_2) \exp(-\varkappa\alpha)]$$

$$\times \prod_{j=1}^{2} \left\{ \delta(y_j - z_j + \alpha_j) - \frac{\theta(y_j - z_j + \alpha_j < 0)}{(M-1)!} \left(\frac{\partial}{\partial y_j} + \varkappa\right) \right.$$

$$\left. \times L_{M-1}[\varkappa(z_j - y_j - \alpha_j)] \right\}_{\alpha_1 = \alpha = -\alpha_2}. \qquad (45)$$

The expressions obtained in this Section for the photon wave functions of the electromagnetic field in a number of processes in resonant quantum optics enable us to calculate all the physical quantities and hence provide a basis for a rigorous quantum-mechanical description of these processes.

§6. CONCLUSION

The above approach is based on the representation of an arbitrary physical state of the systems that we have considered by the superposition of generalized Bethe states of a single (stringless) configuration. The absence of summation over an enormous number of string configurations is an important advantage of this representation as compared with the usual expansion over the Bethe basis. The simplicity of the representation enables us to perform an explicit description of the temporal evolution of an arbitrary physical state, and thus approach the rigorous theory of dynamic processes in such systems.

The efficacy of the approach has been demonstrated for processes that are of particular physical interest, namely, superradiant decay of atomic excitation and scattering of photons by atoms (resonant fluorescence). Explicit expressions, given by (34) and (40), have been obtained for multiphoton wave functions describing the state of the radiated (scattered) electromagnetic field. Their structure shows that there are photon-photon correlations due to the nonlinearity (restricted excitation spectrum) of the atomic subsystem.

The representation that we have obtained can be uniquely extended to models whose Hamiltonians are higher-order constants of motion of the systems described above (and, consequently, have a more general Bethe basis). The form of the representation and, in particular, the choice of the contour of integration are determined, above all, by the structure of the S-matrix (which is the same for all the models that we have considered). It would be interesting to extend the approach to models with a different algebraic structure.

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[1)] We note a special property of the Dicke state $|\Psi_0\rangle$ (31): the corresponding state $|\Psi(t)\rangle$ admits of the general representation (26) as well as the representation

$$|\Psi(t)\rangle = P\exp(-iHt)\oint d\Lambda |\lambda_1,\ldots,\lambda_N\rangle\langle\bar{\lambda}_1,\ldots,\bar{\lambda}_N|\Psi_0\rangle,$$

where $|\lambda\rangle$ is the Bethe vector (10) corresponding to the N-string with carrier rapidity Λ; the contour of integration describes a negative circuit around the point $\Lambda = -ix(M-N+1)/2$; the operator P executes a projection into the causal region with photon coordinates $\{0<y_j<t\}$.

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