Weakly bound electron with arbitrary orbital angular momentum / in a Coulomb field

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The energy spectrum is found for an electron with arbitrary orbital angular momentum l weakly bound by a short-range potential $U(|\mathbf{r}|)$ and a Coulomb potential which is distinct from it. It is shown that for $l \neq 0$ as the result of the centrifugal barrier the spectrum differs radically from the case $l = 0$. In the classically allowed region $|E| \lesssim 1/R$ ($\hbar = m = e = 1$, R is the distance between centers), depending on the ratio between the width of the ionic energy level (21) and the distance between the Coulomb levels, there is either a quasi-intersection of terms with a nonexponential splitting or a rearrangement of the spectrum.

1. INTRODUCTION

The behavior of an electron in the field of a short-range potential $U(|\mathbf{r}|)$ and a long-range field $V_f(\mathbf{r})$ has been the subject of considerable attention recently. In Refs. 1 and 2 is reviewed the work on this problem with respectively an alternating electric field and a constant magnetic field for V_f. This interest is due first of all to the development of experimental methods and, second, to the fact that the electron-impurity interaction itself for $l \neq 0$ leads to a number of new physical effects.

The two-center problem of an electron in the field of a short-range potential well and a Coulomb potential belongs to the same class. Finding the energy levels of such a system is important also for a number of applications: the theory of complexes of the type H^-–H^+ in semiconductors,[3] the theory of collisions of atomic particles,[4] and so forth. The spectrum of such a system was calculated for the first time[1)] in Ref. 5 and the resulting equations have been analyzed in a number of studies (see the reviews in Ref. 6 and Ref. 4). However, the treatment in Refs. 4–6 employed the method of a zero-range potential,[6] which takes into account only the s wave in the electron-impurity interaction.

In interaction of an electron with $l \neq 0$ with a short-range potential $U(|\mathbf{r}|) < 0$ of radius r_c there is a centrifugal barrier, as a result of which quasistationary states can exist with energy $E_l > 0$ of width[2)] $\Gamma_l \sim E^{l+1/2} r_c^{2l-1}$.[8,9] On turning on an attractive Coulomb potential, the energy reference level is reduced by R^{-1} and the quasistationary level goes over into a stationary level. Then (in contrast to the case $l = 0$) in the classically allowed region of energies $R^{-1} \lesssim E < 0$ there is an ionic level. Here, depending on the ratio between the width of the ionic term $\tilde{\Gamma}_l \sim (R^{-1} - |E|)^{l+1/2} r_c^{2l-1}$ (see Section 4) and the distance between the Coulomb levels $\Delta n \sim n_0^{-3}$ (n_0 is the number of the Coulomb term corresponding to quasi-intersection with the ionic term), the following situations can be realized. a) For $\tilde{\Gamma}_l \ll n_0^{-3}$ there are a Coulomb term (the electron is localized near the Coulomb potential) and an ionic term (the electron is localized near the short-range potential), which in the case when they do not coincide undergo a slight shift, and if they coincide there is a quasi-intersection with a splitting which depends nonexponentially on the distance between the centers (here the electron is found near the two centers with approximately equal probability). b) For $\tilde{\Gamma}_l \gg n_0^{-3}$ the levels undergo a rearrangment—for a large number of levels it is impossible to introduce the concept of Coulomb and ionic terms individually—the electron is found near the two centers with equal probability. Here the number of highly perturbed terms is close to the number of Coulomb levels encompassed by the width of the initial well level and, depending on the ratio between $\tilde{\Gamma}_l$ and n_0^{-2}, the number can be both finite and infinite.

2. EQUATIONS OF THE SPECTRUM

Let us consider an electron interacting with Coulomb and short-range potentials (see Fig. 1). It is assumed that at the short-range center in the absence of an external field there is a shallow stationary or quasistationary state $|E| \ll r_c^{-2}$ with angular momentum l. The short-range center we shall place at the origin, and the Coulomb center at the point R on the z axis. The projection of the angular momentum m on the z axis will be conserved in view of the axial symmetry of the problem.

A model-independent method of calculation of the spectra of weakly bound states of a particle in external fields for the condition of isolation of one partial wave was devel-

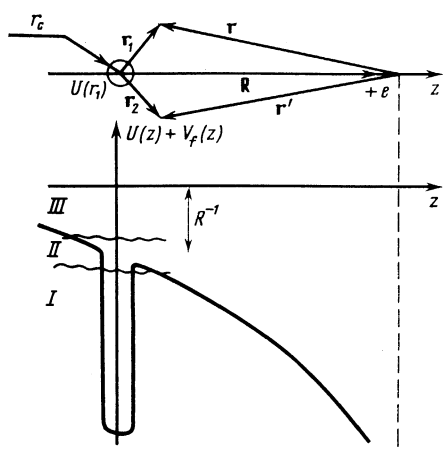

FIG. 1. Diagram of energy regions: I—classically forbidden region ($\varepsilon - 1/R \gtrsim 0$); II—intermediate region ($\varepsilon - 1/R \approx 0$); III—classically allowed region ($\varepsilon - 1/R \lesssim 0$).

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oped by Andreev *et al.*[10,11] According to this method, the equations of the spectrum

$$-\frac{1}{a_l}+\frac{r_l k^2}{2}=A_{lm}{}^l(E)\,(2l-1)!!\,(2l+1)!!,\quad k=(2E)^{1/2} \quad (1)$$

are determined by the parameters of the low-energy scattering by the short-range potential—the scattering length a_l and the effective range r_l (Ref. 12) (according to Ref. 9 the parameter r_l is proportional to $-r_c^{-2l+1}$) and by the coefficients $A^{\,l}_{lm}$ which contain all of the information on the external field.[3)] The latter are uniquely determined by the solutions of the Schrödinger equation which are regular at infinity in the external field G^f_{lm},[10,11] which for $r_1\to 0$ contains singular terms of the form $r_1^{-l'-1}Y_{l'm}(\mathbf{n}_1)$ only with $l'=l$:

$$G_{lm}{}^f=r_1{}^{-l-1}Y_{lm}(\mathbf{n}_1)+\ldots+\sum_{l'}A_{lm}{}^{l'}(r_1{}^{l'}Y_{l'm}(\mathbf{n}_1)+\ldots). \quad (2)$$

We obtain a spectrum for the case where V_f is a Coulomb potential in the most important case $l=|m|$ (Ref. 10).[4)] The solution with a specified singularity of the type (2) can be found by proceeding from the known Green function $G(\mathbf{r},\mathbf{r}',E)$ in a Coulomb field[13]:

$$G_{ll}{}^f\infty\lim_{r_2\to 0} r_2{}^{-l}\int Y_{ll}^*(\mathbf{n}_2)\,G(\mathbf{r},\mathbf{r}',E)\,d\mathbf{n}_2. \quad (5)$$

The integral representation[13] is used here for $G(\mathbf{r},\mathbf{r}',E)$. The integral over φ_2 is obtained from the handbook.[14] We then utilize the explicit form of the functions Y_u and the relation between θ and θ_1 and between θ' and θ_2 (see Fig. 1). After taking the limit $r_2\to 0$ and expansion in power series in r_1 Eq. (5) reduces to the form (2). In addition it is evident that Eq. (5) is a solution of the Schrödinger equation with a Coulomb potential.

The resulting equation for the spectrum has the form

$$(-1)^l\frac{\Gamma(1-n)}{n^{2l+1}}\sum_{k=0}^{2l+2}(-1)^k C^k_{2l+2}M^{(k)}_{n,1/2}\left(\frac{2R}{n}\right)W^{(2l+2-k)}_{n,1/2}\left(\frac{2R}{n}\right)$$

$$=\frac{1}{a_l}+r_l\left(\varepsilon-\frac{1}{R}\right). \quad (6)$$

Here $\varepsilon=-E$, $n=(2\varepsilon)^{-1/2}$, C^k_n are binomial coefficients, and M and W are the Whittaker functions.[15] In the right-hand side of Eq. (6) we have taken into account the shift of the energy reference level under the action of the Coulomb potential.

By means of the Whittaker differential equation[15] it is possible to express the sum in the left-hand side of Eq. (6) in terms of Whittaker functions and their first derivatives. In particular, for $l=0$, $m=0$ Eq. (6) coincides with the known equation of the spectrum,[4–6] differing in the term with the effective range, which is unimportant for $l=0$. For the most important case with nonzero angular momentum $l=1$, $|m|=1$ in which it is possible to discuss the rearrangement of the spectrum without taking into account renormalization of the scattering length by the external field,[5)] Eq. (6) takes the form

$$\frac{n\Gamma(1-n)}{2}\left[2\left(\varepsilon-\frac{1}{R}\right)\left[\frac{d}{dR}M_{n,1/2}\left(\frac{2R}{n}\right)\frac{d}{dR}W_{n,1/2}\left(\frac{2R}{n}\right)\right.\right.$$
$$\left.-2\left(\varepsilon-\frac{1}{R}\right)M_{n,1/2}\left(\frac{2R}{n}\right)W_{n,1/2}\left(\frac{2R}{n}\right)\right]+\frac{1}{R^2}\left[\frac{1}{R}M_{n,1/2}\left(\frac{2R}{n}\right)\right.$$
$$\left.\left.\times W_{n,1/2}\left(\frac{2R}{n}\right)+\frac{1}{2}\frac{d}{dR}\left(M_{n,1/2}\left(\frac{2R}{n}\right)W_{n,1/2}\left(\frac{2R}{n}\right)\right)\right]\right]$$
$$=\frac{1}{a_1}+r_1\left(\varepsilon-\frac{1}{R}\right). \quad (7)$$

The condition of applicability of Eqs. (6) and (7) is fulfillment of the inequalities

$$r_c\ll n,\quad r_c\ll R. \quad (8)$$

In particular, Eq. (7) gives the proper corrections of first order to the energy of the ionic and Coulomb levels respectively in the charge e^2 and the short-range potential.

3. SPECTRUM IN THE CLASSICALLY FORBIDDEN REGION

To find the asymptotic behavior of the spectrum equation (6) in the classically forbidden region $2R^{-1}\lesssim n^{-2}$ we note that the sum over k in the left-hand side of (6) can be reduced to the form

$$\frac{d^{2l+2}}{d\Delta^{2l+2}}M_{n,1/2}\left(\frac{2R}{n}-\Delta\right)W_{n,1/2}\left(\frac{2R}{n}+\Delta\right)\Big|_{\Delta=0}. \quad (9)$$

Substituting into Eqs. (6) and (9) the quasiclassical asymptotic behavior of the Whittaker functions,[4] we have

$$(-1)^l p^{2l+1}+\frac{\operatorname{ctg}\pi n\,(2l+1)!!\,e^{-2S}}{2^{l+2}p^{l+2}R^{2l+2}}=\frac{1}{a_l}+\frac{r_l}{2}p^2, \quad (10)$$

where

$$p=\left(2\varepsilon-\frac{2}{R}\right)^{1/2},\quad S=pR-2n\ln\left\{\left(\frac{R}{2n^2}\right)^{1/2}+\left(\frac{R}{2n^2}-1\right)^{1/2}\right\}$$

are the quasiclassical momentum and action; $n\gg 1$, $S\gg 1$, $R\gtrsim \frac{1}{2}n^2$, $n\gg l$.

For the case $l=0$ Eq. (10) coincides with the corresponding result of Ref. 4 with the exception of the term with the effective range, which is small in comparison with $p^{1/2}$: $r_0p^2\sim r_cp^2\ll p$. However, for $l\geqslant 1$ the term with r_l is large in comparison with p^{2l+1}: $r_lp^2\sim r_c^{-2l+1}p^2\gg p^{2l+1}$ and, as will be shown below, it is just this term which determines the spectrum. Indeed, for analysis of the equation (10) we shall write it in a form similar to that of Ref. 4:

$$\operatorname{tg}\pi n\left[(-1)^{l+1}p^{2l+1}+\frac{1}{a_l}+r_l\left(\varepsilon-\frac{1}{R}\right)\right]$$
$$=\frac{(2l+1)!!\,e^{-2S}}{2^{l+2}p^{l+2}R^{2l+2}}\equiv T_l. \quad (11)$$

(Here T_l has exponential smallness.)

As in the case $l=0$ (see Refs. 4 and 6), the energy levels determined by Eq. (11) are broken up into two groups (see Figs. 2 and 3). The first group is made up of the ionic terms (the electron is localized near the short-range potential), which are shifted by R^{-1} as the result of the long-range interaction with the positive charge (cf. Eq. (3.21) in Ref. 16)[6)]:

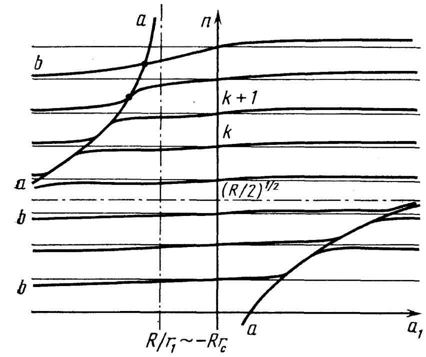

FIG. 2. Energy $\varepsilon = \frac{1}{2}n^2$ as a function of scattering length for $l = 1$, $|m| = 1$. Curve a is the shifted ionic level, and the curves b are the true level.

$$(-1)^{l+1}p^{2l+1} + \frac{1}{a_l} + \frac{r_l p^2}{2} = 0,$$

$$n_1 \approx \left(-\frac{2}{r_l a_l} + \frac{2}{R}\right)^{-1/2}, \quad l \neq 0; \quad n_1 \approx \left(\frac{1}{a_0^2} + \frac{2}{R}\right)^{-1/2}, \quad l = 0. \tag{12}$$

The second group consists of the shifted Coulomb levels (the particle is near the positive charge), which undergo an exponentially small shift as the result of penetration of the electron through the Coulomb potential barrier (see Fig. 1):

$$n = n_0 + \frac{1}{\pi} \operatorname{arctg}\left[T_l \left((-1)^{l+1}p^{2l+1} + \frac{1}{a_l} + \frac{r_l}{2}p^2 \right)^{-1} \right]. \tag{13}$$

At the point of quasi-intersection of the terms, where the particle is found near the two centers with approximately equal probability, we have

$$n = \frac{n_1 + n_0}{2} \pm \left[\left(\frac{n_1 - n_0}{2} \right)^2 + (\Delta n)^2 \right]^{1/2}; \tag{14}$$

the splitting of the terms is determined by the expression

$$(\Delta n)^2 = p n^3 T_0 / \pi, \qquad l = 0,$$

$$(\Delta n)^2 = -n^3 T_l / \pi r_l, \qquad l \neq 0. \tag{15}$$

The splitting of the levels (15) is exponentially small. In addition, $(\Delta n)^2$ for the case $l \neq 0$ is proportional to r_c^{2l-1}.

In conclusion of this section we shall give the spectrum

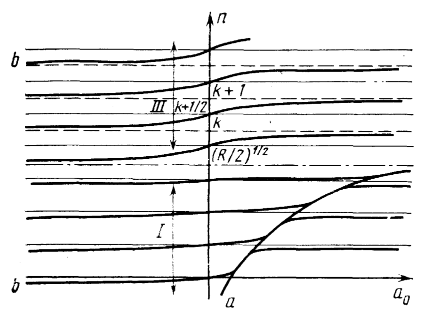

FIG. 3. The energy $\varepsilon = \frac{1}{2}n^2$ as a function of the scattering length for $l = 0$, $m = 0$. The curves a are the shifted ionic level, and the curves b are the true level.

equation in the far classically forbidden region ($2R/n^2 \to \infty$, n arbitrary):

$$(-1)^l n^{-2l-1} + \pi e^{-z} \operatorname{ctg} \pi n \frac{z^{2(n-l-1)} 2^{l+1} (2l+1)!!}{n^{2l+1}(n-l-1)!(n-1)!}$$

$$= \frac{1}{a^l} + r_l \left(\frac{1}{2n^2} - \frac{1}{R} \right), \tag{16}$$

where $z = 2R/n$, $z/n \gg 1$; this equation is obtained by means of the corresponding asymptotic behaviors of the Whittaker functions.[15] The term splitting Δn which is given by Eq. (16) for $l = 0$ coincides with the result of Refs. 5 and 17.

4. THE SPECTRUM IN THE CLASSICALLY ALLOWED REGION

In the classically allowed region (III in Fig. 1) $2R^{-1} \gtrsim n^{-2}$ the spectrum equation (6) with accuracy to quasiclassically small corrections (which oscillate as $\cos 2S$) has the form

$$\operatorname{ctg} \pi n \left(\frac{2}{R} - 2\varepsilon \right)^{l+1/2} = \frac{1}{a_l} + r_l \left(\varepsilon - \frac{1}{R} \right),$$

$$S = -R \left(\frac{2}{R} - 2\varepsilon \right)^{1/2} - 2n \left[\arcsin \left(\left(\frac{R}{2n^2} \right)^{1/2} \right) - \frac{\pi}{2} \right] \tag{17}$$

$$+ \frac{\pi}{4} \gg 1, \quad n \gg 1, \quad R \gg 1, \quad l \ll R \left(\frac{2}{R} - 2\varepsilon \right)^{1/2}.$$

For the case $l = 0$, Eq. (17) coincides with the corresponding result of Ref. 4.

For $l = 0$ the spectrum of Ref. 4 is given by Eq. (18) see Fig. 3). Depending on the value of the quantity $a_0(2/R - 2\varepsilon)^{1/2}$, the Coulomb spectrum is either weakly perturbed ($\Delta n \ll 1$) if $a_0(2/R - 2\varepsilon)^{1/2} \ll 1$ or undergoes rearrangement (is strongly shifted, $\Delta n \sim 1$) if $a_0(2/R - 2\varepsilon)^{1/2} \gg 1$ (see Fig. 3). The ionic term is not present. There is no quasi-intersection.[4]

The spectrum for $l \neq 0$ differs radically from the case of zero angular momentum in the presence of two clearly expressed level systems—ionic and Coulomb—with a quasi-intersection between them which depends nonexponentially on R. As in Eq. (9), Eq. (16) for $l \neq 0$ describes two groups of levels—shifted Coulomb levels:

$$n = n_0 + \frac{1}{\pi} \operatorname{arctg} \left[\left(\frac{2}{R} - 2\varepsilon \right)^{l+1/2} \left(\frac{1}{a_l} + r_l \left(\varepsilon - \frac{1}{R} \right) \right)^{-1} \right] \tag{18}$$

and shifted levels in the field of the short-range center:

$$\frac{1}{a_l} + r_l \left(\varepsilon - \frac{1}{R} \right) = 0, \quad n_1 = \left(\frac{2}{-r_l a_l} + \frac{2}{R} \right)^{-1/2}, \quad l \neq 0. \tag{19}$$

With respect to the ionic terms determined by Eq. (19) we must note the following. In a purely short-range potential there is a shallow l-level with energy $E = -\varepsilon = 1/a_l r_l \sim -r_c^{2l-1}/a_l$. For $a_l > 0$ we have $E < 0$: the level is stationary. For $a_l < 0$ we have $E > 0$: as a result of the centrifugal barrier the level becomes quasistationary with a width[9]

$$\Gamma_l = 2(2E)^{l+1/2} / |r_l| \sim E^{l+1/2} r_c^{2l-1}. \tag{20}$$

On turning on the Coulomb potential, the energy reference level at the point of location of the short-range center $U(r_1)$ is shifted down by R^{-1} and therefore part of the quasistationary levels due to the well (corresponding to $a_l < R/r_l \sim -Rr_c^{2l-1}$) become strictly stationary. Here, however, it is just the width of the shifted level

$$\tilde{\Gamma}_l = 2\left(\frac{2}{R} - 2\varepsilon\right)^{l+1/2} \Big/ |r_l| \sim r_c^{2l-1}\left(\frac{2}{R} - 2\varepsilon\right)^{l+1/2}, \tag{21}$$

which determines the nature of the interaction of the ionic and Coulomb terms.

If $\tilde{\Gamma}_l \ll n_0^{-3}$ (where n_0 is the number of the Coulomb level corresponding to the quasi-intersection), then the spectrum is described by Eqs. (18) and (19) with a nonexponential splitting (14) at the quasi-intersection:

$$(\Delta n)^2 = \tilde{\Gamma}_l n_0^3 / 2\pi \ll 1; \tag{22}$$

the ionic level at the quasi-intersection interacts with one Coulomb term.

In the other limiting case $\tilde{\Gamma}_l \gg n_0^{-3}$ there is a rearrangement of the spectrum: near the quasi-intersection point not one, but many Coulomb terms are distorted (see Fig. 2). Here it is impossible to speak individually of Coulomb and ionic terms. Analyzing Eq. (17), we note the agreement of the exact solution (17) with the ionic term at $n = n_0 + 1/2$, where n_0 is an integer representing the number of the Coulomb term. Writing n in the form $n = n_0 + 1/2 + \Delta n$ and using Eq. (17), we find

$$n = n_0 + \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg}\left[\left(\frac{1}{a_l} + \frac{r_l}{2}\left(\frac{1}{n^2} - \frac{2}{R}\right)\right) \times \left(\frac{2}{R} - \frac{1}{n^2}\right)^{-l-1/2}\right]. \tag{23}$$

It follows from Eq. (23) that the Coulomb levels which are encompassed by the width of the former level in the short-range potential are highly distorted ($\Delta n \sim 1$). Depending on the ratio between $\tilde{\Gamma}_l$ and n_0^{-2} a finite or infinite number of Coulomb levels n_2 are distorted:

$$n_2 \sim n_0^3 \tilde{\Gamma}_l(n_0) \quad \text{for} \quad n_0^{-3} \ll \tilde{\Gamma}_l(n_0) \ll n_0^{-2}, \tag{24}$$

$$\infty < n_2 \leqslant (\tilde{\Gamma}_l(n_0))^{-1/2} \quad \text{for} \quad \tilde{\Gamma}_l(n_0) \gg n_0^{-2}. \tag{25}$$

We note that, by replacing the summation over angular momenta in Eq. (30) of Ref. 18 by integration, after nontrivial transformations it is possible to show that for $\Delta n \ll 1$ in the quasiclassical limit in the classically allowed region the series[18] which determines the energy coincides with the expression obtained for the energy from the simple formula of Eq. (17). In addition, for $l = |m|$, $l = n - 1$ the series of Ref. 18 coincides with Eq. (6) in the limit $\Delta n \ll 1$.

We note also that the spectrum equation for $l > |m|$ in the classically allowed region also has the form of Eq. (17) with accuracy to quasiclassically small corrections.

The spectrum equation of the type (17) in the classically allowed region was obtained by Ivanov,[19] who considered an electron in a Coulomb field and the field of two zero-range potentials. In the corresponding expression from Ref. 19 there is a width of the two-center resonance, renormalized as the result of the Coulomb interaction.

5. DISCUSSION OF RESULTS

For the model considered of a short-range potential with a hard core, the method used has the following limitations. The radius of the center r_c must be small in comparison with the electron wavelength in the external long-range field V_f at the point of occurrence of $U(|\mathbf{r}_1|)$:

$$r_c \ll \min\{|1/n^2 - 2/R|^{-1/2},\ R\}. \tag{26}$$

It is also required that the probabilities of transitions to states with other angular momenta be small:

$$r_c^3/R^2 \ll 1. \tag{27}$$

In addition, taking account of the field V_f in the region $r \lesssim r_c$ must give a small correction to the level shift $\Delta\varepsilon$ found from the spectrum equation:

$$\Delta\varepsilon \gg \max\{r_c^2/R^3,\ r_c^4/R^4\}. \tag{28}$$

Finally, the states investigated must remain in the discrete spectrum:

$$1/n^2 \gg \max\{r_c^2/R^3,\ r_c^4/R^4\}. \tag{29}$$

It can be shown that if (26) and (27) are satisfied, the inequality (28) is satisfied for $l = 1$ (for $R > 1$). For $l \geqslant 2$ the latter inequality, depending on the additional relations between the parameters, can in general be violated. However, this leads only to renormalization of the scattering length: the spectrum equations remain valid if (26) and (27) are satisfied, but the scattering length in them will depend not only on U, but also on the external field V_f.

Another limitation on the applicability of the results is due to the multielectron structure of the short-range potential and the necessity of taking into account the long-range polarization interaction. For a polarization potential $U_p \sim -\alpha/r^4$ for $l = 0$ it is possible to determine the scattering length.[16] For $l \neq 0$ the concept of scattering length for U_p cannot be introduced.[16] However, the condition of neglect of the polarization potential in scattering is that the amplitude for scattering by the short-range potential be much greater than the amplitude for scattering by U_p:

$$\frac{k^{2l}}{|-1/a_l + k^2 r_l/2 - ik^{2l+1}|} \gg \frac{k}{|8(l+3/2)(l^2-1/4)/\alpha\pi - ik^2|}. \tag{30}$$

If the condition $1/\alpha \gg k^2\pi/8(l+3/2)(l^2 - 1/4)$ is satisfied, Eq. (30) is valid for any l near resonance,

$$\Delta\varepsilon \ll \frac{1}{\pi} 8(l+3/2)(l^2-1/4)\frac{k^{2l-1}}{|r_l|\alpha} \sim \frac{(r_c k)^{2l-1}}{\alpha} \tag{31}$$

in a region substantially broader than the level width Γ_l.

In conclusion the author thanks S. P. Andreev for numerous discussions of this work and also B. M. Karnakov and V. D. Mur for valuable remarks.

1)In 1959 Zel'dovich[7] considered an electron in the field of a Coulomb potential distorted at the origin by a well of small radius.

2)We use units $\hbar = m = e = 1$.

3)We give the value of the effective range for a rectangular potential well of radius r_c which has a shallow (stationary or quasistationary) l level[9]:

$$r_l = -r_c^{-2l+1}(2l+1)!!(2l-1)!!/(2l-1). \tag{3}$$

4)One partial wave can be isolated if

$$|f_l| \gg |f_{l'\neq l}|, \tag{4}$$

where f_l is the partial amplitude for scattering by U. For low-energy scattering ($kr_c \ll 1$) in the absence of accidental degeneracy and for $l = |m|$ Eq. (2) is satisfied for arbitrary values of a_l, and for $l > |m|$ it is satisfied only in a narrow energy region $|E - E_l| \ll (kr_c)^{2(l-|m|)} r_c^{-2}$ for the condition of existence of a quasistationary level with energy $E_l > 0$.

5)A renormalization rises as the result of the difference of the field V_f from a constant R^{-1} at $r \lesssim r_c$ (see Section 5).

6)We note that in Eqs. (10) and (12) we have taken into account only the most obvious long-range correction to the energy R^{-1} corresponding to the charge-charge interaction. The next corrections for $l = 0$ are of the order R^{-4},[4,6] and for $l \neq 0$ they are of the order R^{-3}, which corresponds to interaction between the charge and the induced dipole and between the charge and the quadrupole.

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