The development of chaos in dynamic structure ensembles

I. S. Aranson, A. V. Gaponov-Grekhov, and M. I. Rabinovich

Institute of Applied Physics, Academy of Sciences of the USSR
(Submitted 29 January 1985)
Zh. Eksp. Teor. Fiz. **89**, 92–105 (July 1985)

We study the collective motions in an ensemble of coupled dynamic structures—autostructures described by the discrete analog of the Ginzburg-Landau equation. We show that there exists a range of parameters where all motions in the form of stationary traveling waves are unstable and where either rather complex regular regimes (in particular, quasiperiodic ones) or chaotic ones in the form of a multidimensional strange attractor arise. We obtain an upper estimate for the entropy of the chaotic motion and of the dimensionality of the strange attractor. We evaluate the way these quantities depend on the coupling parameter in the structures in one- and two-dimensional ensembles—lattices with an arbitrary though finite number of elements. We discuss the relation between the dimensionality of the strange attractor and the number of unstable solutions in the form of stationary waves. The theory we have constructed qualitatively explains the results of a numerical experiment.

1. THE MODEL

As a result of the development of spatial instabilities regular formations in the form of ensembles of identical (or similar) elementary cells or dynamical structures are often established in nonequilibrium dissipative media. Examples of such dynamical structures are rolls (ridges) observed when there is thermal convection in a plane horizontal layer,[1] Taylor vortices in Couette flow between rotating cylinders,[2] and so on. If we increase the amount by which we exceed criticality—the amount by which the medium deviates from equilibrium—the stationary state consisting of a set of fixed structures becomes unstable and is, generally speaking, destroyed. In many cases, however, before the spatial regularity is totally destroyed and turbulence or complete chaos develops in the nonequilibrium medium, several stages occur associated with gradual complication and the formation of increasingly autonomous structures. One of the most common ways this complication occurs is through the excitation of individual or collective degrees of freedom superposed on the regular background structure. As a rule, the structures become oscillatory initially—for instance, bending oscillations of convective rolls[1] or azimuthal waves on Taylor vortices[2] arise. Later, when we get further away from criticality the coupling between "neighbors" weakens and one can consider the nonequilibrium medium (in a certain range of parameters) to be a discrete ensemble of oscillating structures which interact with one another.

We emphasize that when there is a well-defined mechanism leading to the formation of nonequilibrium states in a medium it goes over to an ensemble of oscillating structures (or autostructures) directly from the rest state, skipping the stationary dynamical formation stage. This is the mechanism which gives rise to pulsating cells on the horizontal surface of a fluid in an oscillating gravitational field,[3] the rectangular and hexagonal cells on the surface of a fluid dielectric or ferroliquid placed, respectively, in uniform oscillating electrical[4] or magnetic[5] fields, etc. The increase in autonomy and stability of similar elementary oscillating structures as we pass further beyond criticality is indicated, in particular, by the formation of dislocations[6] and the subsequent occurrence of chaos[7] in an ordered ensemble.

The nature of the transition to chaos when the autostructures interact with each other is determined in the first instance by the number of degrees of freedom of the structures and the size of the ensemble. If the individual structures are not stochasticized and the number of elements in the ensemble is small, the stochastization process is connected with the complete or partial formation of structures—fusion, granulation, change of size, etc.[8] If, on the other hand, the ensemble consists of a large number of autostructures, collective effects rather than elementary processes of strong interaction between the autostructures will be the determining factor. The transition to chaos in the ensemble then takes place with no change in the composition of the ensemble, thanks to the change in the dynamics of the separate autostructures (due to their coupling to one another).

We consider here the dynamics of an ensemble of autostructures, assuming that in each of them only one degree of freedom is excited:

$$da_j/dt=a_j(1-\delta|a_j|^2), \tag{1}$$

where $a_j(t)$ is a complex variable characterizing the state of the structure, while $\delta = 1 + i\beta$ is a complex parameter determining the amplitude and frequency of its oscillations in the stationary regime. A two-dimensional ensemble of such interacting structures under the simplest assumptions about their coupling is described by a differential-difference equation of the form

$$da_{jk}/dt=a_{jk}-(1+i\beta)|a_{jk}|^2a_{jk}+\nu\hat{L}a_{jk}, \tag{2}$$

where $\hat{L}$ is a linear difference operator. If $\hat{L}$ is a second-order difference, Eq. (2) will be the discrete analog of the Ginzburg-Landau equation.[9] The magnitude of the coupling between neighboring cells in (2) is determined by the complex parameter ν. Correspondingly in the one-dimensional case when $k = \text{const}$ and $\nu\hat{L}(...) = e(1 - ic)(a_{j+1} + a_{j-1} - 2a_j)$ we shall have

$$da_j/dt=a_j-(1+i\beta)|a_j|^2a_j+e(1-ic)(a_{j+1}+a_{j-1}-2a_j), \tag{3}$$

 0038-5646/85/070052-08$04.00

where $\beta,c,e>0$. In what follows we restrict ourselves to the analysis of ensembles with periodic boundary conditions $a_j(t)=a_{j+N}(t)$, where N is the number of structures in the ensemble.

We emphasize that Eq. (3) for a discrete ensemble of autostructures in many cases directly follow from the equations for the complete field in nonlinear nonequilibrium media. Here the structures may, for example, be solitons the oscillations of which are connected with the presence of an external periodic wave or a parametric wave. For instance, for the high-frequency field described by the well-known sine-Gordon equation for a weakly nonequilibrium medium[10] we can in the presence of a low-frequency wave write ($\varepsilon\ll1$)

$$\frac{\partial^2\varphi}{\partial t^2}-c_0^2\frac{\partial^2\varphi}{\partial x^2}+\sin\varphi$$
$$=\varepsilon\left[\alpha\frac{\partial\varphi}{\partial t}-\gamma\left(\frac{\partial\varphi}{\partial t}\right)^3+A_0\sin(\omega t-kx)\right]. \quad (4)$$

Here the perturbation $\alpha\varphi_t$ is responsible for the departure from equilibrium of the medium and $\gamma\varphi_t^3$ for the nonlinear damping; a slow ($v_{\rm ph}=\omega/k\ll c_0$) low-frequency wave of amplitude A_0 is assumed to be given. Because $\varepsilon\ll1$ a solution of (4) exists in the form of a chain of solitons which are weakly coupled to one another. When the amplitudes of the oscillations of these solitons in the field of the low-frequency wave are complex we obtain the set (3) (see Appendix 1).

By means of numerical experiments a rather extensive amount of information has been accumulated about the properties of the solutions of nonlinear systems of the form (2) and (3).[11–17] The present paper is devoted to an analytical study of these systems. We show, in particular, that in a well-defined range of parameters all collective motions in the form of stationary traveling waves are unstable and that in the ensemble either quasiperiodic or chaotic motions result.[1)] We give an upper bound for the Kolmogorov-Sinaĭ entropy of the chaotic set and determine the fractal dimensionality of the strange attractor. We find how the entropy and dimensionality depend on the magnitude of the coupling between structures and ensembles with an arbitrary number of elements. We establish a connection between the dimensionality of the strange attractor and the number of solutions of unstable stationary waves which occur in the medium. The theory proposed here is compared with a numerical experiment performed earlier.[16]

2. DEVELOPMENT OF CHAOS

Assuming that for some parameter range a stable stochastic regime (or, if the chaos is transitory, one with a long lifetime) develops in the ensemble (2) or (3), we determine how the characteristics of the chaos depend on the parameters of the ensemble, viz., the number of elements N and the magnitude e of the coupling between the autostructures.

To estimate how far the multidimensional chaos has evolved we shall use the following properties of stochastic sets: the normalized Kolmogorov-Sinaĭ entropy

$$H_\lambda=\sum_{j=1}^{m}\lambda_j/\lambda_1, \quad (5)$$

where the λ_j are the Lyapunov characteristic indices arranged in decreasing order: $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_m\geqslant0\geqslant\ldots\geqslant\lambda_{2N}$,[18] and the dimensionality D_λ of the attractor which is defined as[2)]

$$D_\lambda=M+d, \quad (6)$$

where M is found from the conditions

$$\sum_{j=1}^{M}\lambda_j\geqslant0,\quad \sum_{j=1}^{M+1}\lambda_j\leqslant0,$$
$$d=\sum_{j=1}^{M}\lambda_j|\lambda_{M+1}|^{-1}.$$

We emphasize that the entropy H_λ depends solely on the number of unstable directions at the attractor and the relative rate of dispersal of close trajectories along these directions. The magnitude of H_λ will clearly always be less than the dimensionality D_λ of the stochastic set, which simultaneously characterizes also the number of effective (normal) variables necessary to describe the established chaotic motion.

It is very important to obtain an approximate analytical description of the development of chaos when e decreases in one- and two-dimensional lattices for a large number of elements N in the ensemble, and to elucidate the physical nature of the change in the dimensionality of the strange attractor when the autonomy of the structures changes. For systems of arbitrary form the solution of that problem is extraordinarily difficult. However, in the present case it is possible to solve this if we take into account the form of the nonlinearity and the symmetry properties of the systems (2) or (3). One can show (see Appendix 2) that for estimates of the entropy and the dimensionality it is sufficient to evaluate averages of the eigenvalues $\sigma_j(t)$ of the auxiliary matrix $\overline{B}(t)=[B(t)+B^+(t)]/2$, where $B(t)$ is the matrix of the original system (2) or (3) written in real form and linearized near a typical solution belonging to the stochastic set; correspondingly $B^+(t)$ is the Hermitian conjugate matrix. The sum of the first characteristic indexes h_l necessary for the evaluation of the entropy and the dimensionality of the stochastic set is connected with $\sigma_j(t)$ as follows:

$$h_l=\sum_{i=1}^{l}\lambda_i\leqslant\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{j=1}^{l}\sigma_j(\tau)\,d\tau, \quad (7)$$

where $\sigma_1(t)\geqslant\sigma_2(t)\geqslant\ldots\geqslant\sigma_l(t)$.

We use Eq. (7) to estimate the dimensionality of the stochastic set in the phase space of the system (3). Bearing in mind the translational symmetry of the chain we restrict our discussion to spatially uniform regimes which are stochastic on the average:

$$\langle|a_j^0|^2\rangle_t=|a^0|^2,\quad \langle a_j^{02}\rangle_t=a^{02}. \quad (8)$$

Moreover, we assume that the average pulsation intensities are small:[3)]

$$|a_j^0|^2-|a^0|^2=z_j, \quad [\langle z_j^2\rangle]^{1/2}\ll|a^0|^2. \tag{9}$$

Numerical experiments validate these assumptions.[16] Under the assumptions made one can write the system $\dot{\zeta}=\overline{B}\zeta$, where

$$\zeta=(x_1, y_1, x_2, y_2, \dots, x_N, y_N),$$
$$x_j=\mathrm{Re}(a_j-a_j^0), \quad y_j=\mathrm{Im}(a_j-a_j^0)$$

in the form

$$\begin{aligned}\dot{x}_j=&(1-2|a^0|^2)x_j+e(x_{j+1}+x_{j-1}-2x_j)-\alpha_1x_j\\&-\alpha_2y_j-2z_jx_j-\bar{z}_jx_j-\bar{\bar{z}}_jy_j,\\ \dot{y}_j=&(1-2|a^0|^2)y_j+e(y_{j+1}+y_{j-1}-2y_j)-\alpha_2x_j\\&+\alpha_1y_j-2z_jy_j-\bar{\bar{z}}_jx_j+\bar{z}_jy_j,\end{aligned} \tag{10}$$

where

$$\alpha_1=\mathrm{Re}(1+i\beta)a^{02}, \quad \alpha_2=\mathrm{Im}(1+i\beta)a^{02},$$
$$\bar{z}_j=\mathrm{Re}(1+i\beta)(a_j^2-a^{02}), \quad \bar{\bar{z}}_j=\mathrm{Im}(1+i\beta)(a_j^2-a^{02}).$$

To first order we can neglect the pulsation terms $z_j, \bar{z}_j, \bar{\bar{z}}_j$; One then easily finds all eigenvalues of the matrix $\overline{B}(t)$ in which we are interested:

$$\begin{aligned}\sigma_{n+1}&=1-4e\sin^2\frac{\theta_n}{2}+[(1+\beta^2)^{1/2}-2]|a^0|^2,\\ \sigma_{N+n+1}&=1-4e\sin^2\frac{\theta_n}{2}-[(1+\beta^2)^{1/2}+2]|a^0|^2,\end{aligned} \tag{11}$$

where $\theta_n = \pm 2\pi n/N$ $(n=0,1, \dots, N/2)$. There are here $2N$ eigenvalues, all degenerate, except those corresponding to $\theta_n = 0,\pi$.[4)]

In accordance with the definitions (5) and (6), to estimate the entropy H_λ and the dimensionality D_λ of the stochastic set it is necessary to find the sum of all positive σ_j and the number of the first few σ_j, the sum of which is close to zero from the left. In order that the result of the estimate depends as little as possible on the choice of the trajectories we perform the calculation for $\beta^2=3$. It is clear from (11) that then the first N eigenvalues $\sigma_j=\sigma_{n+1}$ are completely independent of $a^0(t)$, while errors due to the contribution of σ_{N+n+1} to the estimates of H_λ and D_λ change little when β is changed.

In the present case $\sigma_1=1$, so that the upper estimate for H_λ will simply be a sum of positive σ_j:

$$H_\lambda\leqslant\sum_{j=1}^{m}\sigma_j=H_m \tag{12}$$

(clearly $m\leqslant N$). Bearing in mind that

$$\sum_{n=1}^{k}\sin^2\frac{\theta_n}{2}=\frac{(2k+1)}{4}-\frac{\sin[(2k+1)\theta_0/2]}{4\sin(\theta_0/2)} \quad \left(\theta_0=\frac{2\pi}{N}\right), \tag{13}$$

we get from (11), (12)

$$H_m=m(1-2e)+2e\frac{\sin(m\pi/N)}{\sin(\pi/N)}, \tag{14}$$

where m is determined from the condition $\sigma_{m+1}\geqslant0$, $\sigma_{m+2}<0$. Thus $H_m(e)$ is a piecewise linear function, undergoing a jump in derivative at the points $e_n=[4\sin^2(\theta_n/2)]^{-1}$. We give the function $H_m(e)$ in Fig. 1a for arbitrary N. It is clear that the entropy increases fairly rapidly with increasing autonomy of the structures.

To estimate the dimensionality D_λ for strong coupling $(e>1/2)$, just as for the estimate of H_λ, we need know only the values σ_{n+1}. If $e<1/2$, however, it is necessary also to take into account σ_{N+n+1} [see (11)]. The fact is that when $e=1/2$ and $M=N$ [see (14)] the sum $H_M(e)=0$, i.e., for the given value of the coupling parameter, $D_\lambda\leqslant N$. When $e<1/2$ we have accordingly $D_\lambda>N$ and when $e<1/2$ we have $D_\lambda<N$. For strong coupling $(e>1/2)$ an upper estimate on D_λ is obtained directly from the equation

$$H_{D_\lambda}(e)=D_\lambda(1-2e)+2e\frac{\sin(D_\lambda\pi/N)}{\sin(\pi/N)}=0. \tag{15}$$

In the case of weak coupling $(e<1/2)$ we must add to the sum of the N eigenvalues σ_{n+1}, which according to (14) for $m=N$ equals $H_N(e)=N(1-2e)$, another r values σ_{N+n+1}, the sum of which

$$\sum_{j=0}^{r}\sigma_{j+1+N}$$

can also be written in the form (13). Bearing in mind that almost all σ satisfy the inequality $|\sigma_{N+n+1}|\gg\sigma_{n+1}$ we may assume for large r that $r\ll N$. In that case we get, expanding the sine (13) in a series for small arguments,

$$\sum_{n=0}^{r}\sigma_{N+1+n}\approx r(1-4|a^0|^2),$$

and from the condition

$$\sum_{n=0}^{N}\sigma_{n+1}+\sum_{n=0}^{r}\sigma_{N+n+1}=N(1-2e)+n(1-4|a^0|^2)=0$$

we finally find $(|a^0|^2\approx1)$

$$D_\lambda\leqslant N+n=N\left(1+\frac{1-2e}{4|a^0|^2-1}\right)\approx N\left(1+\frac{1-2e}{3}\right). \tag{16}$$

We show in Fig. 1b how D_λ depends on the magnitude of the coupling between structures, using Eqs. (15) (for $e>1/2$) and (16) (for $e<1/2$).

For two-dimensional lattices the dynamics of which according to (2) is described by the equation

$$\begin{aligned}da_{jk}/dt=&a_{jk}-(1+i\beta)|a_{jk}|^2a_{jk}+e(1-ic)\\&(a_{j+1,k}+a_{j-1,k}+a_{j,k+1}+a_{j,k-1}-4a_{jk}),\\ a_{jk}=&a_{(j+J)k}, \quad a_{jk}=a_{j(k+K)},\end{aligned} \tag{17}$$

one can construct in a similar way estimates for the entropy and the dimensionality of the stochastic set. Instead of the eigenvalues (11) of the one-dimensional chain we shall in this case have

$$\begin{aligned}\sigma_{(n+1)(l+1)}&=1-4e+2e(\cos\theta_n+\cos\theta_l)+[(1+\beta^2)^{1/2}-2]|a^0|^2,\\ \sigma_{(K+n+1)(J+l+1)}&=1-4e+2e(\cos\theta_n+\cos\theta_l)-[(1+\beta^2)^{1/2}+2]|a^0|^2,\end{aligned} \tag{18}$$

where $\theta_n=\pm2\pi n/K$ $(n=0,1,\dots,K/2)$, $\theta_l=\pm2\pi/J$ $(l=0,1,\dots,J/2)$. Repeating the considerations and calculations given above we find, for instance, for $e<1/4$

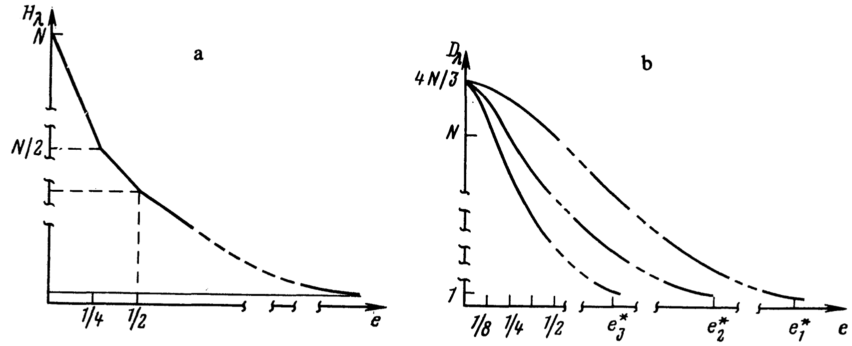

FIG. 1. (a) The entropy H_λ as function of the coupling constant e in a one-dimensional chain of N structures; (b) The dimensionality D_λ of the stochastic set in an ensemble of N autostructures as function of the coupling constant e; the upper curve corresponds to a one-dimensional "lattice" ($e_1^* = N^2/4\pi^2$), the middle one to a square lattice ($e_2^* = e_1^*/2N$), the lower one to a cubic lattice ($e_3^* = e_1^*/3N^{4/3}$).

$$D_\lambda = JK(1+(1-4e))\{[(1+\beta^2)^{1/2}+2]|a^0|^2-1\}^{-1}.$$

It is natural to compare the dimensionality of the stochastic sets in the one- and two-dimensional ensemble of structures with identical parameters and identical number of elements. The corresponding results are for a square lattice ($J = K = N^{1/2}$) shown in Fig. 1. It is clear that the dimensionality of the chaos in the two-dimensional ensemble for a given magnitude of the coupling between the structures is always smaller than in the one-dimensional case for the same number of elements. One can explain this simply. The fact is that the degree of autonomy of the structures (where the dimensionality of the chaos increases when this decreases) is determined not only by the magnitude of the coupling between neighbors but also by their number—as the number of bonds increases the autonomy of the elementary structure is effectively decreased and, hence, the order of the collective motions increases. This in turn corresponds to a lower value of the dimensionality.

3. CONNECTION BETWEEN THE DIMENSIONALITY OF THE CHAOS AND THE UNSTABLE STATIONARY WAVES

In order to explain the physical nature of the increase in the dimensionality of stochastic motions with increasing autonomy of the oscillating structures, i.e., when e decreases, we turn to an analysis of the properties of the regular collective motions in the form of stationary traveling waves

$$a_j(t) = A_n \exp[i(\omega_n t + j\theta_n)]. \tag{19}$$

Substituting (19) into (3) yields the dependence of the intensity of the stationary waves on their propagation constant θ_n in the form

$$|A_n|^2 = 1-4e\sin^2(\theta_n/2) \tag{20}$$

and the dispersion law for these waves

$$\omega_n = -\beta+4e(\beta+c)\sin^2(\theta_n/2), \tag{21}$$

where $\theta_n = \pm 2\pi n/N$, $n = 0,1,\ldots,N/2$. The collective excitations (19) are for $n=0$ spatially uniform oscillations and for $n = N/2$ π-oscillations, while for $n \neq 0, N/2$ they are waves traveling to the right or to the left. It is possible to study the stability of such waves in detail thanks to the appreciable symmetry of Eq. (3) which, in particular, allows a continuous transformation of the form

$$a_{j\,\text{new}} = a_{j\,\text{old}}\, e^{i\varphi}, \tag{22}$$

where φ is real. We use the invariance of (3) under the substitution (22) to study the stability of the solutions of (19), in particular, to evaluate the characteristic Lyapunov indices corresponding to them.

For waves with a propagation constant θ_n we make the substitution

$$b_j = a_j \exp[-i(\omega_n t + j\theta_n)]. \tag{23}$$

In the new variables the traveling wave stops propagating there corresponds in the new variables to a limiting cycle in the phase space of the system (3) a circle, each point of which is an equilibrium state. As each of these equilibrium states is changed into another through a simple rotation, it is sufficient to determine the index of only one of them. Substitution of (23) reduces (3) to the form

$$\dot{b}_j = (1-i\omega_n)b_j - (1+i\beta)|b_j|^2 b_j + e(1-ic)(b_{j+1}e^{i\theta_n} + b_{j-1}e^{-i\theta_n} - 2b_j). \tag{24}$$

The equilibrium state of interest to us is $|b_j^0|^2 = A_n^2$. We may assume here that A_n is real because (3) is invariant under the substitution (22). We make the substitution $\xi_j = b_j - b_j^0 = b_j - A_n$. We then get for the variable ξ_j the equation

$$\dot{\xi}_j = -(1+i\beta)A_n^2(\xi_j+\xi_j^*) + e(1-ic)(\xi_{j+1}e^{i\theta_n} + \xi_{j-1}e^{-i\theta_n} - 2\cos\theta_n\xi_j) - (1+i\beta)(\xi_j|\xi_j|^2 + 2A_n|\xi_j|^2 + A_n\xi_j^2). \tag{25}$$

Separating the real and the imaginary parts $\xi_j = x_j + iy_j$ and linearizing (25) we shall look for a solution of the system obtained in the form

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \exp(\lambda_l t + ij\theta_l),$$

where $\theta_l = \pm 2\pi l/N$, $l = 0,1,\ldots,N/2$. From the condition that the system for X, Y be soluble we get the characteristic equation

$$\Delta(\lambda_{ln}) = (\lambda_{ln}-\gamma_{ln})(2A_n^2-\gamma_{ln}+\lambda_{ln}) - \rho_{ln}(\rho_{ln}-2\beta A_n^2) = 0,$$

$$\gamma_{ln} = -4e\sin^2\frac{\theta_l}{2}\cos\theta_n + 2eci\sin\theta_l\sin\theta_n, \tag{26}$$

$$\rho_{ln} = 4ec\sin^2\frac{\theta_l}{2}\cos\theta_n + 2ei\sin\theta_l\sin\theta_n,$$

whence follows an expression for the characteristic indexes

$$\lambda_{ln} = \gamma_{ln} - A_n^2 \pm (A_n^4 - \rho_{ln}^2 + 2\beta\rho_{ln}A_n^2)^{1/2}. \tag{27}$$

Altogether there are here $2N$ indexes—the number of independent directions in the phase space of a system of N degrees of freedom. The analysis of (27) enables us completely to study the properties of the elementary excitations of the form (19).

As we noted in the first section, immediately above the threshold for the occurrence of structures, i.e., just above criticality, the coupling between the cells is rather strong. The ensemble of the structures then demonstrates only a trivial regular behavior. We show this by considering $e \sim e^* = 1/4 \sin^2(\pi/N)$. We emphasize that it is clear from (20) that when $e > e^*$ no solutions of the form (19) exist in the system except a spatially uniform one, $n = 0$. According to (27) the stability of the spatially uniform regime is determined by the indexes

$$\lambda_{l0} = -1 - 4e\sin^2\frac{\theta_l}{2} \pm \left[1 - \left(4ec\sin^2\frac{\theta_l}{2}\right)^2 + 8\beta ce\sin^2\frac{\theta_l}{2}\right]^{1/2}, \quad (28)$$

whence it follows that when

$$e > e_0 = (\beta c - 1)\,[2(1 + c^2)\sin^2(\pi/N)]^{-1} \quad (29)$$

where $\lambda_{l0} \leqslant 0$ and the regime of spatially uniform oscillations in an ensemble of N cells is stable. We note that as $N \to \infty$ from (29) we obtain the well known[19] condition of stability for the continuum model, $\beta c \leqslant 1$.

If $\beta c > 1$, when e decreases from the value e_0 associated with the trivial equilibrium state $a_j = 0$ there originate in the system consecutively (for $e = e_n$) new collective motions (19) with, respectively, $n = 1,2, \ldots, N/2$. In the phase space of the system (3) this corresponds for $e = e_n$ to the weak creation from the equilibrium state of a pair of limiting cycles (corresponding to direct and counter waves) and all periodic motions thus created are unstable. One can verify this by considering the characteristic indexes of the trivial equilibrium $a_j = 0$:

$$\lambda_l = 1 - 4e\sin^2\frac{\theta_l}{2} + 4ice\sin^2\frac{\theta_l}{2}. \quad (30)$$

Comparing (30) with (20) one notes easily that the number of unstable directions, i.e., the number of positive $\mathrm{Re}\lambda_l$ for a given e^0 is the same as the number of stationary waves produced when e is decreased from e_0 to e^0, while the growth rates along the unstable directions are determined by the intensities of these waves, $\mathrm{Re}\lambda_l = |A_l|^2$. We emphasize that as long as the periodic motions appear from the trivial equilibrium in a weak fashion they inherit (at the time when a new unstable direction is produced) the index of the equilibrium state along the remaining independent directions. Thus, at the time when the periodic motion with $l = 1$ is generated it is characterized by a single unstable direction (it corresponds to the growth of spatially uniform perturbations), the motion with $l = 2$ by three, and so on. If we now assume that again the newly unstable motions pertain to a strange attractor one can easily interpret the above result that the dimensionality of the chaos increases monotonically with the increase in the autonomy of the structures—as e decreases trajectories appear in the attractor, and the number of unstable directions among them is of the order of the number of the generated cycle (periodic motion with $\theta = \theta_l$). The number of unstable periodic solutions of the form (19) existing for a given e gives thus a lower bound for the dimensionality of the attractor.

4. COMPARISON WITH THE RESULTS OF A NUMERICAL EXPERIMENT

Numerical experiments with two-dimensional[12,13] and one-dimensional[11,14,16] chains described by equations such as (2) have demonstrated the possibility of establishing as $t \to \infty$ in such systems irregular stochastic motions. A strange attractor describes their form in phase space. We use here the results of Ref. 16 in which the system (3) was studied with periodic boundary conditions and with $N = 9,10,50$. One observed, in particular, that the regime of spatially uniform oscillations which arises for strong coupling changes when e is decreased to the beat regime (to which a two-dimensional torus corresponds in phase space). When $e = e^0$ (when $N = 9$ and $\beta = c = 1.71$, the value $e^0 = 1.033$) the quasiperiodic regime with two incommensurate frequencies is destroyed and a regime is established, characterized by a positive Kolmogorov-Sinaĭ entropy, to which a strange attractor corresponds.

We show in Fig. 2 how the entropy and the dimensionality, obtained in the numerical experiment of Ref. 16 depend on the parameter e. It is clear that as the coupling e decreases in the stochastic regions both the entropy and the dimensionality increase—the motion at the attractor remains ever more unstable as the autonomy of the structures increases and chaos becomes more and more developed. We draw attention to the ranges $0.75 \leqslant e \leqslant 0.9$ and $0 \leqslant e \leqslant 0.125$ in which $H_\lambda = 0$ and $D_\lambda = 1$ (periodic motion) or $D_\lambda = 2$ (quasiperiodic motion with two incommensurate frequencies). In these ranges a regime of regular pulsations is established as $t \to \infty$; however, also for those values of the coupling parameter transitional (nonstationary) oscillations may be chaotic (Fig. 3).[16]

We show in Fig. 2 by the dashed line the dimensionality as function of the magnitude of the coupling e [for the one-dimensional chain (3)] calculated using Eqs. (15) and (16) for the same values of the parameters as in the numerical experiment. It is clear that in this case we have excellent agreement if we exclude relatively narrow ranges of e (e in the range 0.5–0.65 or 0.75–0.9), where as a result of the

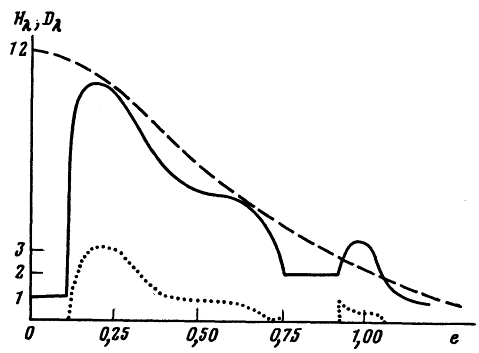

FIG. 2. Comparison of the estimate for D_λ with a numerical experiment for a one-dimensional lattice with $N = 9$ ($\beta = c = 1.71$): the solid curve corresponds to the numerical experiment of Ref. 16, the dashed curve to the analytical estimate (the dotted curve is the experimental function $H_\lambda(e)$).

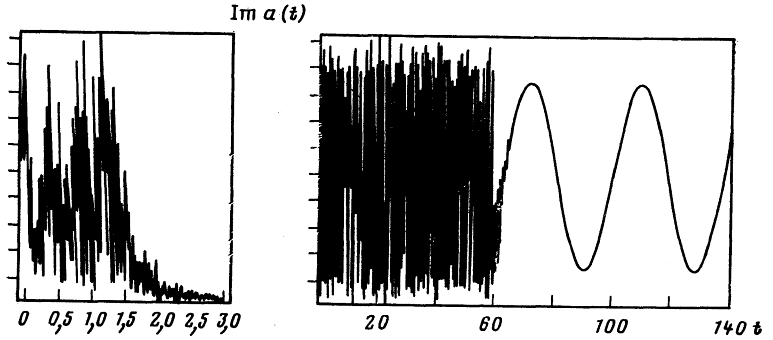

FIG. 3. Transitional one-dimensional chaos observed in a one-dimensional chain.[16] The departure from the periodic regime for large times shows the existence of a stable cycle inside the stochastic set (on the left we give the power spectrum of the transitional chaos).

synchronization of separate modes only transitional chaos is observed.

5. CONCLUSION

The estimates obtained for the entropy and the dimensionality of the chaos arising in nonequilibrium media of the form (2) or (3) enables one not only to predict the way several properties of the turbulent regime depend on the parameters but also to understand the physical mechanisms of the development of the chaos. The connection of the chaos dimensionality with stationary waves means that when the parameter changes at the moment when the next stationary wave is generated at the strange attractor (to be more precise at the trajectories belonging to it), a new unstable direction appears. The parameters of the stationary waves, for instance, their amplitudes and phases, can thus be considered as new normal variables, the number of which (less than or equal to $2N$) determines roughly the chaos dimensionality in the ensemble of structures. This estimate is suitable also for a continuous ring-shaped system (with a finite number of elements in the ring). We note that the chaos dimensionality for such a system can directly be expressed in terms of the length of the ring and one can easily find the connection between the dimensionality and the effective Reynolds number.

In a two-dimensional system the chaos dimensionality depends strongly for given parameters (number N of elements and coupling strength e) on the geometry—the ratio of the number of elements along and at right angles to the lattice. It turns out that the smallest dimensionality occurs for a square lattice, which is natural as its spectrum of stationary waves is very sparse.

We add in conclusion that the results given here can rather easily be generalized also to the case of three-dimensional ensembles of autostructures. We give in Fig. 1 the results of this generalization.

APPENDIX 1

When $\varepsilon = 0$ (4) has a solution in the form of a soliton

$$\varphi^0(\xi)=4\,\mathrm{arctg}\left(\pm\exp\frac{\xi}{(1-v^2)^{1/2}}\right),\quad \xi=x-vt. \qquad \text{(A1.1)}$$

In the case $\varepsilon \ll 1$ we shall look for a solution of (4) as a sum of a forced solution

$$\varphi_b\approx A_0(1+k^2-\omega^2)^{-1}\sin(\omega t-kx)$$

and a perturbed soliton solution, for which we get from (4) ($c_0^2=1$)

$$\frac{\partial^2\varphi_s}{\partial t^2}-\frac{\partial^2\varphi_s}{\partial x^2}+\sin\varphi_s =\varepsilon\left[\alpha\frac{\partial\varphi_s}{\partial t}-\gamma\left(\frac{\partial\varphi_s}{\partial t}\right)^3+(1-\cos\varphi_s)\varphi_b\right]. \qquad \text{(A1.2)}$$

Looking for the solution of (5) in the form of a series ($\tau = t$)

$$\varphi_s(x,t)=\varphi^0(\xi,\tau)+\sum_{n=1}\varepsilon^n\varphi^{(n)}(\xi,\tau,x) \qquad \text{(A1.3)}$$

and applying an asymptotic method[20] we get for the perturbation of the velocity v of the soliton in the presence on a nonresonant low-frequency wave

$$\frac{d}{dt}\frac{v}{(1-v^2)^{1/2}}=\frac{\alpha v}{(1-v^2)^{1/2}}-\frac{8}{3}\gamma\frac{v^3}{(1-v^2)^{3/2}}+A\sin(\omega t-kx),$$
$$A=\frac{\pi}{4}\frac{A_0}{1+k^2-\omega^2}. \qquad \text{(A1.4)}$$

Hence we have for the relative coordinate $y = x - v_{\mathrm{ph}}t$ of the soliton center of mass when $v^2, v_{\mathrm{ph}}^2 \ll 1$

$$\frac{d^2y}{dt^2}=\left[\alpha-3\left(\frac{8}{3}+\alpha\right)v_{\mathrm{ph}}^2\right]\frac{dy}{dt}-\left(\frac{8}{3}\gamma+\alpha\right)\left(\frac{dy}{dt}\right)^3 -A\sin ky+\frac{3}{2}A\left(\frac{dy}{dt}\right)^2\sin ky +3Av_{\mathrm{ph}}\frac{dy}{dt}\sin ky-\left(\frac{8}{3}\gamma+\alpha\right)v_{\mathrm{ph}}^3 -3\left(\frac{8}{3}\gamma+\alpha\right)v_{\mathrm{ph}}\left(\frac{dy}{dt}\right)^2+\alpha v_{\mathrm{ph}} \qquad \text{(A1.5)}$$

According to (A1.5) solitons perform in the field of a wave oscillations near the equilibrium state $A\sin ky_0\approx\alpha v_{\mathrm{ph}}+O(v_{\mathrm{ph}}^2)$. For not too large an amplitude they are described by an equation for $S = y - y_0$ of the form

$$\frac{d^2S}{dt^2}-\left(\tilde{\alpha}-\beta\left(\frac{dS}{dt}\right)^2\right)\frac{dS}{dt} +\frac{3}{2}kAS\left(\frac{dS}{dt}\right)^2-\frac{Ak^3}{6}S^3+AkS\approx 0, \qquad \text{(A1.6)}$$

$$\tilde{\alpha}=\left(\frac{8}{3}\gamma+\alpha\right)v_{\mathrm{ph}}^2-\alpha,\quad \beta=\frac{8}{3}\gamma+\alpha>0.$$

In the case of small deviations from equilibrium, $\alpha < \alpha_{cr} = 8\gamma(3/v_{ph}^2 - 3)^{-1}$, these oscillations are damped and a stationary structure (a soliton) develops. If, however, the deviation from equilibrium is somewhat larger, i.e., $\alpha > \alpha_{cr}$, the equilibrium state $S = 0$ becomes unstable and the soliton becomes oscillatory—it performs undamped self-oscillations in a "well" [when $\alpha = \alpha_{cr}$ a stable limiting cycle is formed from the equilibrium state in the system (A1.6)]. Considering further the interaction between neighboring solitons which are at a distance apart $y_{j+1} - y_j \sim L = (2\pi/k)l$ $(l = 1,2,\ldots)$ and bearing in mind that $|S_{j+1} - S_j| = |(y_{j+1} - y_j) - L| \ll L$ and $|S_j| \ll 1$ we obtain for the coordinate of the jth soliton a differential-difference equation[5)]

$$\ddot{S}_j = \tilde{\alpha} \dot{S}_j - \beta \dot{S}_j^3 - {}^3/_2 \Omega^2 S_j \dot{S}_j^2 + {}^1/_6 k^2 \Omega^2 \dot{S}_j^3 - \Omega^2 S_j + 2e^{-L}(S_{j+1} + S_{j-1} - 2S_j), \quad (A1.7)$$

where $\Omega^2 = Ak$. Going over to new variables

$$\begin{pmatrix} \dot{S}_j \\ S_j \end{pmatrix} = \begin{pmatrix} i\omega \\ 1 \end{pmatrix} a_j(t) \left(\frac{\tilde{\alpha}}{3\beta\Omega^2}\right)^{1/2} e^{i\Omega t} + \text{c.c.}$$

and averaging over the time we arrive at the required model (3). The meaning of the coefficients will in this case be the following:

$$\beta = [k(3A+k)^{1}]\left[4(Ak)^{1/2}\left(4\gamma + \frac{3}{2}\alpha\right)\right]^{-1},$$

$$ce = 2\exp\left(-2\frac{\pi}{k}l\right)$$

$$\left\{(Ak)^{1/2}\left[\left(\frac{8}{3}\gamma + \alpha\right)3\left(\frac{\omega}{k}\right)^2 - \alpha\right]\right\}^{-1},$$

and the dot indicates differentiation with respect to the time

$$t_{new} = 2t\left[\alpha - \left(\frac{8}{3}\gamma + \alpha\right)^3\left(\frac{\omega}{k}\right)^2\right]^{-1}.$$

When the degree of deviation from equilibrium of the medium increases further ($\propto \alpha$) the solitons are no longer "trapped" and can collide with one another. As a result of this strong interaction the number of solitons in the general case is no longer conserved and the description of the dynamics of an ensemble of solitons is thus no longer possible in the framework of the model (3).

APPENDIX 2

We write the system (3) in real form $\dot{\mathbf{u}} = F\mathbf{u}$ where $\mathbf{u}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ and we linearize it near an arbitrary solution $\mathbf{u}_0(t)$. As a result we have for the variation $\xi(t)$

$$\dot{\xi} = B(t)\xi, \quad \|b_{ij}(t)\| = \frac{\partial F_i}{\partial u_j}\bigg|_{\mathbf{u}_0(t)}.$$

One checks easily that the first (maximum) Lyapunov index has as an upper bound the time-average of the maximum eigenvalue of the matrix $\overline{B}(t)$. Indeed, by definition

$$\lambda_1 \leq \lim_{t\to\infty} \frac{1}{t} \ln\|\xi\| = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \ln(\xi^*(t)\xi(t)) \quad (A2.1)$$

(Vazhevskiĭ inequality) or, using the relation

$$\frac{d}{dt}\ln(\xi^*\xi) = (\xi^*(B+B^+)\xi)/(\xi^*\xi),$$

we get

$$\lambda_1 \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{(\xi^*\overline{B}\xi)}{(\xi^*\xi)} d\tau.$$

The maximum value of the integrand at time t is the maximum eigenvalue $\sigma_1(t)$ of the matrix $B(t)$. Therefore

$$\lambda_1 \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t \sigma_1(\tau) d\tau. \quad (A2.2)$$

We obtain a similar upper estimate also for the sum of the first characteristic indexes

$$h_l = \sum_{i=1}^{l} \lambda_i.$$

According to a theorem due to Oseledets[22] h_l is connected with the l-dimensional volume V_l in phase space (almost everywhere):

$$h_l \leq \lim_{t\to\infty} \frac{1}{t} \ln V_l.$$

Choosing the basis vectors ξ_j to be orthonormal we find

$$\ln V_l = \int_0^t \sum_{i=1}^{l} \frac{(\xi_i^*\overline{B}\xi_i)}{(\xi_i^*\xi_i)} d\tau.$$

Using then the Courant-Fisher theorem about mini-maximum relations we finally get (see in this connection Ref. 23)

$$h_l \leq \lim \frac{1}{t} \int_0^t \sum_{i=1}^{l} \sigma_i(\tau) d\tau, \quad (A2.3)$$

where the $\sigma_i(t)$ are the eigenvalues of the matrix $\overline{B}(t)$ ordered as follows: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l$. In many cases Eq. (A2.3) enables us to estimate explicitly the entropy and dimensionality of the stochastic set.

1)As a result of the synchronization of the quasiperiodic motions periodic motions may also be established in some regions of the coupling parameter.

2)The dimensionality D_λ is an upper estimate for the Hausdorff (or fractal) dimensionality of the strange attractor.[18]

3)This assumption is certainly valid for the so-called "phase chaos" which was, in particular, observed in Ref. 16.

4)Taking into account small perturbations of the matrix $\overline{B}$ proportional to $z, \bar{z}, \bar{z}$, removes the degeneracy and leads to a splitting of the eigenvalues. Because of the symmetry of the splitting the result for the sum of the indexes is not changed to first order.

5)Similar equations were obtained in Ref. 10 (see also Ref. 21) for the interaction of two solitons.

1F. H. Busse, in Hydrodynamic Instabilities and Transition to Turbulence (Eds. H. L. Swinney and J. P. Gollub), Springer, Berlin, 1981, p. 97.

2R. C. DiPrima and H. L. Swinney, in Hydrodynamic Instabilities and Transition to Turbulence (Eds. H. L. Swinney and J. P. Gollub), Springer, Berlin, 1981, p. 139.

3S. Ciliberto and J. P. Gollub, Phys. Rev. Lett. 52, 922 (1984).

4E. A. Kuznetsov and M. D. Spektor, Zh. Eksp. Teor. Fiz. 71, 262 (1976) [Sov. Phys. JETP 44, 136 (1976)].

5M. D. Cowley and R. E. Rosensweig, J. Fluid Mech. 30, 671 (1967).

[6]R. J. Donnelly, K. Park, R. Shaw, and R. W. Walden, Phys. Rev. Lett. **44**, 984 (1980).
[7]A. B. Ezerskiĭ, P. I. Korotin, and M. I. Rabinovich, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 129 (1985) [JETP Lett. **41**, 157 (1985)].
[8]D. V. Lyubimov, G. F. Putin, and V. N. Chernatynskiĭ, Dokl. Akad. Nauk SSSR **235**, 554 (1977) [Sov. Phys. Dokl. **22**, 360 (1977)].
[9]A. V. Gaponov-Grekhov and M. I. Rabinovich, Nelineĭnaya fizika; khaos i struktury (Nonlinear Physics; Chaos and Structures) in Fizika XX beka; razbitie i perspektivy (Physics of the Twentieth Century; Developments and Perspectives) (Ed. E. P. Velikhov) Nauka, Moscow, 1985, p. 219.
[10]I. S. Aranson, K. A. Gorshkov, and M. I. Rabinovich, Preprint No. 51, 1982, of Inst. Appl. Phys. Akad. Nauk SSSR, Gor'kiĭ.
[11]V. S. L'vov and A. A. Predtechenskiĭ, in Nelineĭnye volny; stokhastichnost' i turbulentnost' (Nonlinear Waves; Stochasticity and Turbulence) Inst. Appl. Phys. Akad. Nauk SSSR, Gor'kiĭ, 1981, p. 57.
[12]Y. Kuramoto and S. Koge, Progr. Theor. Phys. **66**, 1081 (1981).
[13]I. S. Aranson, M. I. Rabinovich, and I. M. Starobinets, in Nonlinear and Turbulent Processes in Physics (Ed. R. Z. Sagdeev) Gordon and Breach, New York, Vol. 3, p. 1139.
[14]T. S. Akhromeeva *et al.*, Dokl. Akad. Nauk SSSR **279**, 346, 591 (1984) [Sov. Phys. Dokl. **29**, 911, 991 (1984)].
[15]Y. Kuramoto, Progr. Theor. Phys. Suppl. No. 64, 1978.
[16]A. V. Gapanov-Grekhov, M. I. Rabinovich, and I. M. Starobinets, Dokl. Akad. Nauk SSSR **279**, 596 (1984) [Sov. Phys. Dokl. **29**, 914 (1984)].
[17]H. T. Moon, P. Huerre, and L. G. Redekoppe, Phys. Rev. Lett. **49**, 458 (1982).
[18]F. Ledrappier, Commun. Math. Phys. **81**, 229 (1981).
[19]K. Nozakki and N. Bekki, Phys. Rev. Lett. **51**, 2171 (1983).
[20]K. A. Gorshkov and L. A. Ostrovsky, Physica **3D**, 428 (1981).
[21]I. S. Aranson, K. A. Gorshkov, and M. I. Rabinovich, Zh. Eksp. Teor. Fiz. **86**, 929 (1984) [Sov. Phys. JETP **59**, 542 (1984)].
[22]V. I. Oseledets, Trudy Mosk. Mat. Obshch-va **19**, 179 (1968).
[23]D. Ruelle, Characteristic Exponents for a Viscous Fluid Subjected to Time Dependent Forces, Preprint No. 1, IHES, 1984.

Translated by D.ter Haar