Higher-order invariants in the nonlinear dynamics of nematics

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The effect of higher-order gradient terms in the free energy of nematic liquid crystals on the dynamics of the director is studied. These terms lead to a nonlinear equation of motion for the director, which has soliton solutions in several cases. The nature of the solutions is studied as a function of the initial conditions and the parameters of the problem. The possibility of an experimental observation of the predicted effects is discussed.

I. INTRODUCTION

Recent years have seen increased interest in the nonlinear dynamics of liquid crystals because of the numerous liquid-crystal systems which have been synthesized, with characteristic parameters over a broad range, making it possible to choose types of structures with the characteristics most suitable for observing nonlinear effects.

Of particular interest are systems which permit soliton solutions in the distribution of the director of the liquid crystal, since such formations can easily be observed experimentally by optical methods. Lin *et al.*[2] have offered a qualitative theory to explain the experimentally observed[1] structure of a soliton type which arises in the distribution of the director of a nematic liquid crystal subjected to a shear flow. That study, however, was restricted to description of the structure; it does not relate the resulting solutions to the initial data of the problem. The nonlinear dynamics of the director of a nematic liquid crystal in a static magnetic field, excited by a pulse of an electric or magnetic field, was studied in Ref. 3. It was shown there that, for certain relations among the parameters of the problem, the director distribution is of a soliton type.

In the present paper we propose a new mechanism for the formation of solitons in a nematic liquid crystal on the basis of higher-order terms in the expansion of the free energy in the gradients of the director **n**. The need for considering these terms has been demonstrated[4–6] in several cases for other systems. With certain modifications, the results derived below can be used to describe the nonlinear dynamics of magnetic systems and of superfluid He^3. We restrict the discussion here to the geometric situation in which all the changes in the director **n** are uniform in the plane perpendicular to some y axis. In this case the problem becomes effectively one-dimensional and analogous to the linear problem of the propagation of a twist-orientation wave. This formulation of the problem was studied in Ref. 3 and had the advantage that in this case the so-called reverse flows, which seriously complicate the problem, do not arise (but even in a geometry with reverse flows, the picture remains qualitatively the same[7]). Furthermore, this geometry is simple to arrange experimentally.

2. EQUATION OF MOTION

Descriptions of nematic liquid crystals are usually restricted to those terms in the expansion of the free energy which are quadratic in the spatial derivatives. This approximation is fully justified in a description of static effects and linear dynamic effects. In a description of the dynamics of the director, however, it may be necessary to consider higher-order gradient terms in cases of very nonuniform initial distributions of the director or for the long-term dynamics, where even corrections which are small at the beginning of the motion lead to significant changes in the final distribution.

For nematic crystals of symmetry $D_{\infty h}$ the energy must be invariant under rotations arround the preferred direction of the director and under the replacement of **n** by $-\mathbf{n}$, so that they energy cannot contain third-order gradient terms. It is necessary to consider those terms of fourth order which are invariant under these transformations. Analysis[9] has shown that there are generally 52 such terms in the free energy. The situation simplifies considerably in the geometry assumed here. In this case the number of independent invariants reduces to seven, which can be written explicitly:

$$(\Delta\mathbf{n})^2,\quad (\mathbf{n}\Delta\mathbf{n})^2,\quad (\nabla_i n_j)^4,\quad (\mathbf{n}\Delta\mathbf{n})(\nabla_i n_j)^2,\quad (\mathbf{n}\,\mathrm{rot}\,\mathbf{n})^2(\mathbf{n}\Delta\mathbf{n}),$$
$$(\mathbf{n}\,\mathrm{rot}\,\mathbf{n})^2(\nabla_i n_j)^2,\quad (\mathbf{n}\,\mathrm{rot}\,\mathbf{n})^4.$$

Consequently, in the case $\mathbf{n} = (\sin\varphi, 0, \cos\varphi)$, within gradient terms of fourth order, the free energy is given by

$$F=\frac{K}{2}(\varphi')^2+\frac{A}{2}(\varphi')^4-\frac{B}{2}(\varphi'')^2. \qquad (1)$$

Here K is the Frank rotational constant; A and B are combinations of coefficients of the fourth-order invariants; and the derivatives are taken with respect to the coordinate y.

In writing free energy (1), we omitted nonlocal corrections which arise when fluctuations are taken into account. The role of fluctuations in the dynamics of nematic liquid crystals was studied in Ref. 4. It was shown that they cause a slight renormalization of the elastic constants and of the viscous coefficients. However, these corrections to the harmonic terms in the free energy must be smaller than the anharmonic effects which are taken into account, for otherwise there might be substantial changes in the nature of the equation of motion.

It can be concluded from an analysis of these results that incorporating fluctuations changes the picture of events, slightly at least for high initial frequencies.

It should also be noted that the very choice of a geometry for the problem presupposes that it is one-dimensional

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and essentially eliminates from consideration the fluctuations δn_y which are the most "dangerous" in the sense explained above. The physical meaning may be a suppression of fluctuations by virtue of boundary conditions or by an orienting magnetic or electric field.

The equation of motion for the director **n** which corresponds to a free energy in the form in (1) is

$$J\ddot{\varphi}+\gamma_1\dot{\varphi}=K\varphi''+6A\varphi'^2\varphi''+B\varphi^{(IV)}, \tag{2}$$

where J is the moment of inertia of the director, and γ_1 is the rotational viscosity. The dissipative term significantly complicates analysis of Eq. (2), so we will ignore it below, assuming that the scale times (τ_x) of the problem are quite small ($\tau_x \ll J/\gamma_1$).

Generally speaking, in typical nematic liquid crystals the parameters are such that this condition holds only for an estimate of high frequencies (at the applicability limit of the hydrodynamic approximation).

The value of J may be significantly larger in lyotropic liquid crystals, where the structural units responsible for the nematic order are not individual molecules but molecular clusters of cylindrical shape containing a larger number of molecules. It can also be expected that Eq. (2) may be applicable to dilute nematics. At any rate, even if the dissipative term is small in comparison with the inertial term in a purely numerical sense, this approximation can be used to derive a qualitative picture. Dissipation can be dealt with by perturbation theory or numerical methods.

Transforming to the dimensionless coordinates $\tilde{y}=|\beta|^{-1/2}y$, $\tilde{t}=c|\beta|^{-1/2}t$, and introducing $u(\tilde{y},\tilde{t})=|\alpha/\beta|^{1/2}\varphi(\tilde{y},\tilde{t})$, where $c=(K/J)^{1/2}$, $\beta=B/K$, $\alpha=A/K$, we find

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial y^2}\left[1+6\,\mathrm{sign}\,\alpha\left(\frac{\partial u}{\partial y}\right)^2\right]+\mathrm{sign}\,\beta\,\frac{\partial^4 u}{\partial y^4}. \tag{3}$$

Here and below, where it will cause no confusion, we will omit the tilde indicating the dimensionless variables.

Equation (3) is very similar in form to the well-known equation of a nonlinear string (see Ref. 10 and the bibliography there) which has soliton solutions. A difference is that the nonlinearity in the equation for a nonlinear string, which stems from the second term in brackets, is proportional to the first power of $\partial u/\partial y$. It seems quite likely that Eq. (3) may also have soliton solutions.

3. SOLITON SOLUTIONS

The nature of the solutions of Eq. (3) is determined by the initial conditions of the problem. Let us assume that at the time $t=0$ an initial perturbation which is uniform in the xz plane arises in some region of the sample. Depending on the magnitudes of the initial nonlinearity and the dispersion, Eq. (3) may correspond to either a very nonlinear regime (the case $\partial u/\partial y|_{t=0}\sim\partial^4 u/\partial y^4|_{t=0}\sim 1$), or a nearly linear regime (the case of small derivatives), with the nature of the solution being determined by the usual wave equation over a rather long time.

In the first case, Eq. (3) has solutions of the solitary-wave type, as can be shown by direct substitution:

$$u=2\,\mathrm{arctg}\exp[\pm(y-wt+y_0)|w^2-1|^{1/2}]+\mathrm{const} \tag{4}$$

under the condition sign α = sign β.

The velocity w of the solitary wave can have either sign and must satisfy the conditions

$|w|<1$ for $\alpha,\ \beta<0$,

$|w|>1$ for $\alpha,\ \beta>0$.

The characteristic size of the wave is $\sim|w^2-1|^{-1/2}$ (in dimensional units, the wave velocity is cw, and its size is $|\beta/(w^2-1)|^{1/2}$). The sign and magnitude of the velocity in (4) are determined by the initial conditions from $\partial u/\partial t|_{t=0}$.

In the case at hand, however, we can neither offer a systematic description of the derivation of solutions of type (4) from arbitrary initial conditions nor (especially) describe the interaction of the solitary waves. All that we can assert is that, if the initial data of the problem constitute a profile of u similar to that constructed from solutions (4), and if they satisfy auxiliary conditions (the initial coordinates of the centers of the waves, y_{0i}, must be sufficiently far apart, and the corresponding velocities w_i must be arranged in order of increasing $|y|$), then the solution of Eq. (3) will have the form of an initial profile which evolves in accordance with (4).

Such initial conditions are extremely special, of course, and they could hardly be arranged experimentally in the case of several solitary waves.

It is thus interesting to examine the opposite case of small initial derivatives. Let us assume that the initial conditions are such that we have $(\partial u/\partial y)^2_{t=0}\ll 1$, $L^{-2}\ll 1$ (L is the region in which the initial values are specified). In this case the non-linear term is small in comparison with unity, and the dispersive term is small in comparison with $\partial^2 u/\partial y^2$. In this approximation, Eq. (3) becomes an ordinary wave equation, and its initial solution can be written as two waves: $u=u_1(y-t)+u_2(y+t)$.

If the solution is to remain of this nature over large distances, u_1 and u_2 must vary slowly with y and t. If the conditions $\partial u_1/\partial y$, $\partial u_2/\partial y\ll 1$ (and thus $\partial u_1/\partial t$, $\partial u_2/\partial t\ll 1$) hold at distances $\gtrsim L$, the meaning is that the waves have already moved apart and can subsequently be treated as independent. For an initial perturbation which is symmetric along y, it is obviously sufficient to consider only one of the propagating waves. We consider that which is propagating in the positive y direction. Using the change of variables $\xi=\delta(y-t)$, $\tau=\varepsilon t$ (the parameters δ, $\varepsilon\ll 1$ make the derivatives small), we find from (3)

$$\frac{\partial v}{\partial \tau}+6\,\mathrm{sign}\,\alpha v^2\frac{\partial v}{\partial \xi}+\mathrm{sign}\,\beta\frac{\partial^3 v}{\partial \xi^3}=0, \tag{5}$$

where $v=\partial u_1/\partial\xi$. In the derivation of this equation we must assume $\varepsilon\sim\delta^3$, since other relations between these parameters are given by equations which are of no physical interest.

With no loss of generality we can set $\varepsilon=\delta^3/2$. By virtue of the arguments above, the parameter δ is none other than the reciprocal scale length of the initial perturbation: $\delta=L^{-1}$. Equation (5) is the well-known modified Korteweg–de Vries equation, which can have soliton solutions and which can be solved by the method of the inverse scattering problem. As initial data we must know $v(\xi, \tau=0)$.

We consider the following initial data for Eq. (3):

$$u(y, t=0) \approx 0,$$

$$\dot{u}(y, t=0)=\begin{cases} 0, & y<-l \\ 2g_0, & -l \leqslant y \leqslant l. \\ 0, & y>l \end{cases} \quad (6)$$

This choice is useful both for simplifying the calculations below and because it approximates the actual situation of a possible experiment.

Let us assume that these initial data satisfy the restrictions in terms of small derivatives specified above. From the condition

$$u(y, 0)=u_1(y, 0)+u_2(y, 0)=0$$

we then find

$$\left.\frac{\partial u_1}{\partial y}\right|_{t=0}=-\left.\frac{\partial u_2}{\partial y}\right|_{t=0},$$

and from the relation

$$\frac{\partial u}{\partial t}=\frac{\partial u_1}{\partial t}+\frac{\partial u_2}{\partial t}=-\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial y}$$

we find

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=2\left.\frac{\partial u_2}{\partial y}\right|_{t=0}=-2\left.\frac{\partial u_1}{\partial y}\right|_{t=0}=2g_0.$$

The conditions that the nonlinearity and the dispersion for u_1 be weak in this case are $(\partial u_1/\partial y)^2=g_0^2 \ll 1$, $(2l)^{-2} \ll 1$. We thus have $v(\xi, \tau=0) \equiv \partial u_1/\partial \xi|_{\tau=0}=2l\partial u_1/\partial y|_{t=0}=-2g_0 l$ for $-1 \leqslant \xi \leqslant 1$ and 0 for other ξ.

Before we seek solutions of Eq. (5) with these initial values, we have a few general comments. It is clear from the form of (5) that this equation does not change when v is replaced by $-v$, and the solution is determined by the sign of the initial values. If α and β are positive, the equation is

$$\frac{\partial v}{\partial \tau}+6v^2\frac{\partial v}{\partial \xi}+\frac{\partial^3 v}{\partial \xi^3}=0. \quad (7a)$$

If α and β are negative, Eq. (5) is put in form (7a) by the replacement $\tau \rightarrow -\tau$. If $\beta>0$, $\alpha<0$, Eq. (5) is

$$\frac{\partial v}{\partial \tau}-6v^2\frac{\partial v}{\partial \xi}+\frac{\partial^3 v}{\partial \xi^3}=0, \quad (7b)$$

and if $\beta<0$, $\alpha>0$ Eq. (5) is again put in form (7b) by the replacement $\tau \rightarrow -\tau$. It is thus necessary to study Eqs. (7a) and (7b). Equation (7b) may have no soliton solutions at all,[10] so we will first write the complete solution of Eq. (7a) and then summarize the results for the continuous spectrum of (7b). Equation (7a) with $v(\xi, 0) \equiv v_0=-2g_0 l$ can be integrated in the region $-1 \leqslant \xi \leqslant 1$ by means of the system of equations[11]

$$\partial\psi_1/\partial y+i\lambda\psi_1=iv_0\psi_2,$$
$$\partial\psi_2/\partial y-i\lambda\psi_2=iv_0\psi_1, \quad (8)$$

The solution of this system with the initial asymptotic behavior

$$\left.\begin{aligned}\psi_1(\xi,\lambda)&=\exp(-i\lambda\xi)\\ \psi_2(\xi,\lambda)&=0\end{aligned}\right\} \text{ as } \xi\rightarrow-\infty,$$

$$\left.\begin{aligned}\psi_1(\xi,\lambda)&=a(\xi,\lambda)\exp(-i\lambda\xi)\\ \psi_2(\xi,\lambda)&=b(\xi,\lambda)\exp(i\lambda\xi)\end{aligned}\right\} \text{ as } \xi\rightarrow\infty$$

determines the components of the scattering matrix $a(\lambda)$, $b(\lambda)$. Knowing these components along with the zeros λ_n of the function $a(\lambda)$ and the coefficients b_n, determined by the relation

$$\psi_1(\xi, \lambda_n)=b_n\psi_2(\xi, \lambda_n),$$

we can find the solution of Eq. (7a), $v(\xi, \tau)$, from its initial form. The zeros λ_n determine the nature of the solitons.

Spectral problem (8) for initial data of the type in (6) was solved in Ref. 3. It was shown that $a(\lambda)$ and $b(\lambda)$ are determined by the expressions

$$b(\lambda)=-2ig_0 l\eta^{-1}\sin\eta,$$
$$a(\lambda)=\eta^{-1}[\eta\cos\eta-i\lambda\sin\eta]\exp(i\lambda), \quad (9)$$

where $\eta=[\lambda^2+4g_0^2l^2]^{1/2}$. The zeros of $a(\lambda)$ are

$$\lambda_j=i[4l^2g_0^2-x_j^2]^{1/2}=i2g_0l\cos x_j, \quad (10)$$

and the x_j are determined by the equation $\sin x=\pm x/2lg_0$, which has solutions only if $lg_0 \geqslant \pi/4$. The number N of roots which arise is determined by

$$\pi N-\pi/2 \leqslant 2lg_0 \leqslant \pi N+\pi/2.$$

Soliton solutions were found in Ref. 11 for the case of reflectionless potentials, i.e., for $b(\lambda)\equiv 0$. It was shown that for a single soliton the solution is

$$v=2\nu\,\mathrm{sech}\,(8\nu^3\tau-2\nu\xi+\mu), \quad (11)$$

where $\nu=|\lambda_1|$, $\mu=\ln|b^{(0)}(0)|$, and $b^{(0)}(0)$ corresponds to the purely soliton solution. In our case we have $b(\lambda)\neq 0$, so that the solution of Eq. (7a) also contains a continuous spectrum which stretches out toward negative y (Ref. 10) and which may have several solitons, depending on the value of $g_0 l$.

To take into account the effect of the continuous spectrum and of the other solitons on the characteristics of the soliton determined by the value λ_n, we use the method of Ref. 3. We single out the initial scattering values corresponding to this soliton, and then by virtue of the one-to-one correspondence its evolution follows from Eq. (11).

Let us assume that the time τ is long enough that the solitons are far apart and far from the region of the continuous spectrum. We can then assume, with exponential accuracy, that they are independent. We number the solitons in order of decreasing ν_n (i.e., in order of decreasing velocity). We examine the behavior of the solution $\psi_1(\xi, \lambda_n)$, which has in the limit $\xi\rightarrow-\infty$ the asymptotic form $\psi_1=\exp(-i\lambda_n\xi)$, as we go from left to right through the region of the continuous spectrum and with the solitons numbered from $n+1$ to N. To the left of the nth soliton, ψ_1 is

$$\psi_1=A(\lambda_n)\exp(-i\lambda_n\xi)\prod_{i=n+1}^{N}\frac{\lambda_n-\lambda_i}{\lambda_n-\lambda_i^*},$$

where $A(\lambda_n)$ is the part of $a(\lambda)$ which corresponds to the continuous spectrum. To the right of the nth soliton, by definition, we have

$$\psi_1(\xi,\lambda_n)=A(\lambda_n)\exp(i\lambda_n\xi)\,b_n^{(0)}(\tau)\prod_{i=n+1}^{N}\frac{\lambda_n-\lambda_i}{\lambda_n-\lambda_i^*}.$$

In a similar way we examine the behavior of the function $\psi_2(\xi,\lambda_n)=\exp(i\lambda_n\xi)$ in the limit $\xi\to\infty$. To the right of the nth soliton, this function is

$$\psi_2=\exp(i\lambda_n\xi)\prod_{i=1}^{n-1}\frac{\lambda_n-\lambda_i}{\lambda_n-\lambda_i^*}.$$

Using the definition of the coefficient $b_n(\tau)=b(\lambda_n,\tau)$, and referring the relations derived to the time $\tau=0$, we finally find

$$b_n^{(0)}(0)=b(\lambda_n,0)A^{-1}(\lambda_n)\prod_{i=1}^{n-1}\frac{\lambda_n-\lambda_i}{\lambda_n-\lambda_i^*}\prod_{i=n+1}^{N}\frac{\lambda_n-\lambda_i^*}{\lambda_n-\lambda_i}. \quad (12)$$

The quantity $A(\lambda_n)$ is given by

$$A(\lambda_n)=\exp\left[(i\pi)^{-1}\int_{-\infty}^{\infty}(\lambda-\lambda_n)^{-1}\ln|a(\lambda)|\,d\lambda\right]=\exp\Phi_n, \quad (13)$$

where

$$\Phi_n=\frac{|\cos x_n|}{\pi}\int_0^{\infty}(p^2+\cos^2 x_n)^{-1}$$

$$\ln[1-(1+p^2)^{-1}\sin^2(2lg_0(1+p^2)^{1/2})].$$

Substituting $b(\lambda_n,0)=i(-1)^n$ and (13) into (12), we find

$$b_n^{(0)}(0)=-i\exp(-\Phi_n)\prod_{i=1}^{n-1}\frac{\nu_i-\nu_n}{\nu_i+\nu_n}\prod_{i=n+1}^{N}\frac{\nu_n+\nu_i}{\nu_n-\nu_i}. \quad (14)$$

Knowing $b_n^{(0)}(0)$, and using Eq. (11) with $\mu_n=\ln|b_n^{(0)}(0)|$, we can now describe any soliton.

The soliton velocities in the coordinate system ξ,τ are $4\nu_n^2$, and their scale sizes are $\sim(2\nu)^{-1}$. We would also like to determine the distance between the centers of two adjacent solitons:

$$d_{n,n-1}=4\tau(\nu_{n-1}^2-\nu_n^2)+\frac{\mu_{n-1}}{2\nu_{n-1}}-\frac{\mu_n}{2\nu_n}.$$

At estimate of this expression for large $N\gg1$ and for $n<N$ (i.e., for at least the first few solitons) yields, quite accurately,

$$d_{n,n-1}\approx4\tau(x_n^2-x_{n-1}^2)+\nu_n^{-1}\ln\frac{4\nu_n^2}{x_n^2-x_{n-1}^2}$$

$$-(2\nu_n)^{-1}\ln\frac{4(N-n+1)(n-1)^2}{N+n}. \quad (15)$$

Since for $N\gg1$ and at small values of n we can assume $x_n\sim\pi n$, and $\nu_n\sim 2g_0l$, we finally find

$$d_{n,n-1}\approx4\pi^2(2n-1)\tau+(2g_0l)^{-1}\ln\left[\frac{16g_0^2l^2}{\pi^2(2n-1)}\right]. \quad (16)$$

The ratio of the distance between the centers of neighboring solitons to their scale sizes $s\sim(2\nu)^{-1}$, determines the degree of overlap of the solitons. From (16) we find

$$\frac{d_{n,n-1}}{s_n}\approx\frac{d_{n,n-1}}{s_{n-1}}\approx16\pi^2g_0l(2n-1)\tau+2\ln\left[\frac{16g_0^2l^2}{\pi^2(2n-1)}\right],$$

from which we see that for $g_0l\gg1$ $(N\gg1)$ the overlap of solitons is weak even at $\tau\approx0$, and it decreases with increasing τ. This result means that solitons are well separated along the coordinate in this case.

For the solitons with n close to N, we have

$$\frac{d_{n,n-1}}{s_n}\approx16\pi^2g_0l(2n-1)\tau+2\ln\frac{|\cos x_n|+|\cos x_{n-1}|}{|\cos x_{n-1}|-|\cos x_n|}.$$

In this case the overlap is determined primarily by the first term, and it becomes small at $\tau\gg(4g_0l)^{-2}$.

These estimates show that in the case $g_0l\gg1$ there are at least a few first solitons which may be regarded as independent essentially from the beginning of the motion. In the region in which they are localized the form of $u(\xi,\tau)$ can thus be found by integrating (11) over ξ with the corresponding ν_n and then joining the solutions. Integrating (11), we find

$$u_n(\xi,\tau)=2\,\mathrm{arctg}\,\{\exp[8\nu_n^3\tau-2\nu_n\xi+\delta_n]\}. \quad (17)$$

In terms of the dimensional variables y, t we would have, instead of (17),

$$\varphi_n(y,t)=2\left|\frac{\beta}{\alpha}\right|^{1/2}\mathrm{arctg}\{\exp[s_n(tv_n-y+\tilde{\delta}_n)]\}, \quad (18)$$

where

$$s_n=2|\alpha/6\beta|^{1/2}\theta|\cos x_n|,\quad v_n=c\left(1+\frac{\alpha\theta^2\cos^2 x_n}{3}\right),$$

$$\tilde{\delta}_n=\delta_n/s_n,\quad \theta=c^{-1}\frac{\partial\varphi}{\partial t}\bigg|_{t=0}.$$

The final form of the function $\varphi(y,t)$ is thus a series of steps of height $\pi|\beta/\alpha|^{1/2}$ separated by boundaries which are narrow, with a width of order s_n^{-1}. The velocities $(v_n\sim c)$ at which the boundaries of the regions of constant φ propagate are approximately equal, so that the size of the regions depends only slightly on the time and is described by (16).

4. CONTINUOUS SPECTRUM

As we mentioned earlier, Eq. (7a) has, in addition to soliton solutions, solutions corresponding to a continuous spectrum for arbitrary g_0l. In the limit $\tau\to\infty$ the continuous spectrum in the region $-\xi\gg\tau^{1/3}$ is[10]

$$u(\xi,\tau)\sim(3\tau)^{-1/2}\Delta^{-1/4}C\sin\Theta, \quad (19)$$

where

$$C=\left\{\frac{2}{\pi}\ln|a(\Delta/2)|\right\}^{1/2},\quad \Delta=|\xi|^{1/2}(3\tau)^{-1/2},$$

and $\Theta(\Delta)$ is the phase shift, which can be calculated from the scattering data. Since we are not attempting to find a complete description of the continuous spectrum, we will content ourselves with a qualitative analysis of expression (19). Solution (19) is a rapidly oscillating function whose amplitude decreases $\sim\tau^{-1/2}$. Consequently, after a sufficiently long time, the amplitude of the continuous spectrum becomes considerably less than unity throughout this region.

In the region $|\xi|\lesssim(3\tau)^{1/3}$ the solution is an extremely special function, but again its amplitude decreases over time.

Finally, in the region $\xi\gg(3\tau)^{1/3}$ the solution is

$$u\sim b(i\Delta/2)a^{-1}(i\Delta/2)\Delta^{-1/2}\tau^{-1/2}\exp(-2\tau\Delta^3). \tag{20}$$

Analysis of this expression shows that solution (20) is small in the region in which the solitons which form and which have velocities $v_n^2\sim\Delta^2>1$ are localized. In the case $lg_0\gg 1$, essentially all of the solitons which arise, except for the least few, have such velocities. Expression (20) can thus be ignored in a description of the soliton structure.

In the case of Eq. (7b) the continuous spectrum is described by equations analogous to those above, but now $a(\lambda)$ and $b(\lambda)$ in the scattering problem are determined for the system of equations

$$\partial\psi_1/\partial y+i\lambda\psi_1=iv_0\psi_2,$$
$$\partial\psi_2/\partial y-i\lambda\psi_2=-iv_0\psi_1,$$

which has no bound states and thus no soliton solutions.

Figure 1 sketches the solutions of Eqs. (7a) and (7b).

5. CONCLUSION

The function $\varphi(y,t)$, shown in Fig. 1, could in principle be observed quite easily in an experiment. The regions of constant director deflection angles, separated by sharp soliton boundaries, would appear as fringes of different intensity in optical observations. The order and alternation of these fringes would be determined by the quantity $|\beta/\alpha|^{1/2}$ (we recall that a rotation of the director through π does not change the optical pattern). The regions of continuous spectrum would contribute an average background because of the rapid oscillations. Experimental observations can thus provide information on the nature of α and β.

An appropriate initial excitation can be provided by a pulse of a magnetic or electric field applied at some nonzero angle from the original direction of the director. The condition for the appearance of soliton solutions,

$$2\,lg_0=Lc^{-1}|\alpha/6\beta|^{1/4}\dot\varphi(t=0)=L(J/K)^{1/2}|\alpha/6\beta|^{1/4}\dot\varphi(t=0)$$
$$\approx L\cdot 10^{-4}\dot\varphi(t=0)>\pi/2,$$

is satisfied even in a comparatively weak field, $\sim 10^2$ Oe (in the estimates we used the parameter values $J\sim10^{-14}$ g/cm and $\alpha\sim\beta\sim10^{-14}$ cm^2).

However, our model is extremely idealized. We have leaned heavily on the assumption that the dissipative term in the equation of motion of the director is small. In a real system, this condition would not be met. Consequently, for this solution to hold we would need to satisfy the auxiliary condition $\dot\varphi(t=0)>(\gamma_1/J)$. This assertion means that our analysis applies only at high initial director rotation rates and for short time (under the condition that the hydrodynamic approximation has become applicable). The subsequent evolution must incorporate dissipation. Furthermore, our analysis has ignored higher-order gradient terms, which may become important in the final state of formation of the solitons.

In summary, the results derived here should be thought of as basically qualitative results describing the initial stage of the motion of the director of a nematic liquid crystal. This question requires experimental study and numerical simulation.

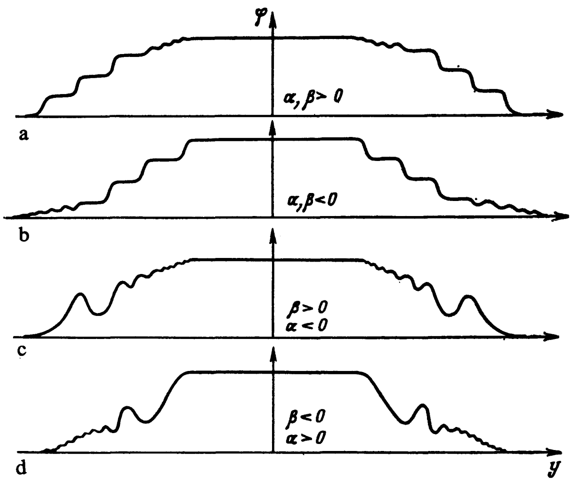

FIG. 1. Sketch of the solutions of Eqs. (7a) (a and b) and (7b) (c and d).

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