The structure of the nonlinear equations of a magnetized plasma and the problem of the stability of magnetosonic solitons

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We discuss the problem of the generalization of the nonlinear Korteweg–de Vries equation for weakly dispersive waves in a magnetized plasma. We obtain a three-dimensional nonlinear equation for magnetosonic waves in such a plasma. We show that in the case of waves with frequencies of the order of or higher than the ion cyclotron frequency the structure of this equation differs significantly from the structure of the three-dimensional Kadomtsev-Petvisashvili equation. We study the three-dimensional stability of one-and two-dimensional magnetosonic solitons. We show that two-dimensional high-frequency magnetosonic solitons are stable under long-wavelength three-dimensional perturbations.

1. INTRODUCTION

Magnetosonic (MS) solitons play an important role in the problem of collective methods for heating a plasma and some other problems. Sagdeev[1] predicted them in the Fifties when studying the problem of collisionless shock waves in a magnetized plasma. Reference 1 and a subsequent review article[2] dealt with large-amplitude solitons (the magnetic field of the wave was assumed to be comparable with the equilibrium magnetic field) corresponding to strongly dispersive MS waves. Later the main attention was paid in theoretical studies to small-amplitude MS waves with a weak dispersion. Such waves turned out to be interesting not only from a physical but also from a mathematical point of view. This was to a large degree connected with the fact that weakly nonlinear, weakly dispersive MS waves are in the one-dimensional approximation described by the well known Korteweg–de Vries (KdV) equation[3] and because of this are among the plasma applications of the general results of an analysis of that equation.

The aim of the present paper is the derivation of two- and three-dimensional (i.e., of so-called multidimensional) equations for weakly nonlinear, weakly dispersive MS waves and the study of soliton effects described by those equations. We are thus in our work dealing, in particular, with multidimensional generalizations of the KdV equation for the case of MS waves. Kadomtsev and Petviashvili[4] were the first to state in 1970 the problem of multidimensional generalizations of the KdV equation. These authors constructed a two-dimensional generalization of the KdV equation, which later was named the Kadomtsev–Petviashvili (KP) equation. This equation has a canonical form (see Eq. (3.9) of the review by Danilov and Petviashvili[5]).

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^3}\right)=\pm\frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

Here $u=u(t,x,z)$ is the required function which describes the wave field; t, x, z are the time and the coordinates (we assume that all quantities are suitably made dimensionless) and the " + " and " − " signs refer, respectively, to waves with positive and negative dispersions. In this notation the KdV equation means:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^3}=0. \tag{1.2}$$

It was noted in Ref. 4 that the KdV equation describes a wide class of waves in weakly dispersive media, amongst them MS waves in plasma. However, whether Eq. (1.1) describes MS waves was not elucidated in Ref. 4.

Reference 6 was the first paper about the problem of a multidimensional generalization of the KdV equation for waves in a magnetized plasma. In that paper ion-acoustic waves considered which propagated along the magnetic field. The authors of Ref. 6 showed that in the approximation of a sufficiently strong magnetic field such waves are described by a three-dimensional equation of the form (see Eq. (3.12) in the review by Danilov and Petviashvili[5])

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^3}=-\Delta_\perp\frac{\partial u}{\partial z}, \tag{1.3}$$

where z is the direction of the magnetic field, $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$; x, y are the coordinates at right angles to the magnetic field.

Reference 6 also stimulated studies of three-dimensional ion-acoustic waves in a plasma with a rather weak magnetic field (in the unmagnetized plasma approximation) and of similar kinds of waves in other weakly dispersive media. Along those line, an equation was obtained of the form (see, for instance, Ref. 5)

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+\frac{\partial^3 u}{\partial z^3}\right)=\pm\Delta_\perp u, \tag{1.4}$$

which by analogy with (1.1) was called the three-dimensional KP equation.

The earlier history of multidimensional generalizations of the KdV equation was summarized in the above mentioned review article.[5] However, the authors of Ref. 5 did not touch upon the case of MS waves.

The problem of multidimensional genralizations of the KdV equation for MS waves did not receive attention in earlier investigations because of its seeming triviality, in view of

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the similarity between the dispersion laws of MS waves and isotropic ion sound. An impression was therefore gained that MS waves, like ion-acoustic waves in a plasma with a weak magnetic field, are described by the two- and three-dimensional KP equations. This point of view was reflected in Refs. 7 and 8. Let us see how far this point of view is justified. One can get some idea about the problem of whether the two- and three-dimensional nonlinear equations for MS waves reduce to KP equations by comparing the linear parts of the corresponding equations. To construct the linear part of the equations for MS waves we turn to the linear dispersion equation of these waves, which in the simplest case of a cold plasma has the form[3]

$$\omega=\frac{c_A k}{2s^{1/2}}\left\{\left[(1+\cos\theta)^2+\frac{k^2c^2\cos^2\theta}{s\omega_{pi}^2}\right]^{1/2}+\left[(1-\cos\theta)^2+\frac{k^2c^2\cos^2\theta}{s\omega_{pi}^2}\right]^{1/2}\right\}. \tag{1.5}$$

Here $s\equiv 1+k^2c^2/\omega_{pe}^2$; ω and k are the frequency and wave number, $c_A=B_0/(4\pi n_0 m_i)^{1/2}$ is the Alfvén velocity,

$$\omega_{pe}^2=4\pi e_e^2 n_0/m_e,\qquad \omega_{pi}^2=m_e\omega_{pe}^2/m_i$$

are the squared electron and ion plasma frequencies, c is the velocity of light, θ is the angle between the wave vector and the direction of the magnetic field $\mathbf{B}_0$ which is assumed to be parallel to the z-axis; m_i, m_e are the ion and electron masses, and e_e is the electron charge. We put $k^2=k_x^2+k_y^2+k_z^2$ approximation $k_y\gg(k_x,k_z)$ we get

$$\frac{\omega}{c_A}=k_y-\frac{c^2k_y^3}{2\omega_{pe}^2}+\frac{k_x^2}{2k_y}+\frac{k_z^2}{2k_y}\left(1+\frac{c^2k_y^2}{\omega_{pi}^2}\right). \tag{1.6}$$

Replacing ω and $\mathbf{k}$ by the operators $\omega\to i\partial/\partial t$, $\mathbf{k}\to -i\nabla$ acting on a function φ which characterizes the field, we construct using (1.6) the required linear equation for φ:

$$\frac{\partial}{\partial y}\left(\frac{2}{c_A}\frac{\partial}{\partial t}+2\frac{\partial}{\partial y}+\frac{c^2}{\omega_{pe}^2}\frac{\partial^3}{\partial y^3}\right)\varphi=-\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2}{\partial z^2}\left(1-\frac{c^2}{\omega_{pi}^2}\frac{\partial^2}{\partial y^2}\right)\varphi. \tag{1.7}$$

Instead of y we introduce the self-similar variable

$$\eta\equiv y+\alpha z-ut, \tag{1.8}$$

where u is the propagation velocity of the waves along y [do not confuse this u with the function u occurring in the canonical nonlinear Eqs. (1.1) to (1.4) and u/α is the wave velocity along z (we assume that $\alpha\ll 1$). In that case (1.7) reduces to the form

$$\frac{\partial}{\partial\eta}\left[\frac{2}{c_A}\frac{\partial}{\partial t}-\varepsilon\frac{\partial}{\partial\eta}+\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^3}{\partial\eta^3}\right]\varphi=-\frac{\partial^2\varphi}{\partial x^2}-\left(2\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}\left(\varphi-\frac{c^2}{\omega_{pi}^2}\frac{\partial^2\varphi}{\partial\eta^2}\right). \tag{1.9}$$

Here $\varepsilon=1-c_A^2(1+\alpha^2)/u^2$ is a small parameter which has the meaning of the appropriately normalized permittivity (we assume $u\approx c_A$), and $\sigma^2=\alpha^2 m_i/m_e$.

It is clear that in the case $\partial/\partial\eta\gtrsim\omega_{pi}/c$ (i.e., when $k_y\gtrsim\omega_{pi}/c$) which is of most interest for the problems discussed in Refs. 1, 2, the MS waves are not described by the three-dimensional KP equation. It is thus clearly necessary to derive a three-dimensional equation for MS waves with arbitrary $k_y c/\omega_{pi}$, and this is done in section 2.

Recently[9] a multidimensional generalization of the KdV equation was considered for the case of drift-ion-acoustic waves. In that case not only the usual (scalar) nonlinearity, occurring, for instance, in Eqs. (1.1) to (1.4), but also the so-called vector nonlinearity, i.e., terms of the form $[\nabla a\times\nabla b]_z$, with a, b some functions which characterize the wave field, was taken into account.

A recent paper by Manin and Petviashvili[10] was also devoted to a multidimensional description of nonlinear MS waves. However, that paper is wrong. One of the errors of Ref. 10 was explained in Ref. 11. It concerns the sign of the transverse dispersion of MS waves in a plasma with finite β (β is the ratio of the plasma pressure to the magnetic-field pressure). This sign is positive in Ref. 10 while, in fact, it must be negative. The causes of such an error were discussed in detail in Ref. 11.

The second error of Ref. 10 consists in the fact that when obtaining (although with an incorrect sign) an expression for the transverse dispersion which refers to the case of low-frequency MS waves ($\omega<\omega_{Bi}$), the authors of Ref. 10 automatically extended it to the case of high-frequency MS waves ($\omega>\omega_{Bi}$). In other words, these authors tansferred results obtained by a series expansion in a small parameter (in this case in $1/\omega_{Bi}$ to a region in which this parameter is large; this is, of course, inadmissable.

We give in section 3 a general analysis of the nonlinear equation for MS waves. We study in section 4 the stability of one-dimensional MS solitons and in section 5 that of two-dimensional ones. The results are discussed in section 6.

2. INITIAL EQUATIONS

We study waves with a nonpotential electric field $\mathbf{E}$. In that case the total magnetic field $\mathbf{B}$ differs from the equilibrium field by a wave part $\tilde{\mathbf{B}}$ so that $\mathbf{B}=\mathbf{B}_0+\tilde{\mathbf{B}}$. We assume that the fields $\mathbf{E}$ and $\mathbf{B}$ depend both on the transverse coordinates x, y and on the longitudinal one z, so that $\partial/\partial z\neq 0$. It is then necessary to take into account both the longitudinal component $\tilde{B}_z$ of the wave magnetic field and the transverse components, denoted by $\mathbf{B}_\perp$, i.e., to assume that $\tilde{\mathbf{B}}=\tilde{B}_z\mathbf{e}_z+\mathbf{B}_\perp$ where $\mathbf{e}_z$ is a unit vector along z. We assume for the sake of simplicity that the plasma is cold, i.e., we neglect the ion and electron temperatures. We discuss in section 6 the limits of applicability of the cold-plasma approximation.

To describe the ions we use the gyromagnetic equation of motion

$$4\pi n m_i d\mathbf{V}_i/dt=(\mathbf{B}\nabla)\mathbf{B}-\nabla\mathbf{B}^2/2, \tag{2.1}$$

obtained by adding the usual ion and electron equations of motion and neglecting the electron inertia ($\mathbf{V}_i$ is the ion velocity). It follows from (2.1) that

$$\frac{\partial}{\partial\eta}\operatorname{div}\mathbf{V}_i=\frac{c_A{}^2}{uB_0}\left[\frac{\partial^2}{\partial\eta^2}+\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)^2+\frac{\partial^2}{\partial x^2}\right]\tilde{B}_z$$

$$+\frac{1}{u}\frac{\partial}{\partial\eta}\left[\frac{\partial}{\partial\eta}\frac{(V_{yi}^{(0)})^2}{2}+\frac{\partial}{\partial t}V_{yi}^{(0)}\right], \qquad (2.2)$$

where $V_{yi}^{(0)}=c_A^2(1+\alpha^2)\tilde{B}_z/uB_0$ is the main part of V_{yi}. Equation (2.2) has been written in terms of the variables η, x, z, t which were introduced in section 1. We neglect the longitudinal motion.

Substituting (2.2) into the equation of continuity for the ions we get an expression for the wave part $\tilde{n}_i$ of the ion density:

$$\tilde{n}_i=\frac{n_0c_A{}^2}{u^2}\left[(1+\alpha^2)h+\frac{3}{2}h^2+2\hat{L}^{-1}\left(\alpha\frac{\partial h}{\partial z}+\frac{1}{u}\frac{\partial h}{\partial t}\right)+\hat{L}^{-2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)h\right], \qquad (2.3)$$

where $\tilde{h}\equiv\tilde{B}_z/B_0$, $\hat{L}\equiv\partial/\partial\eta$. We neglected terms of order α^2 in the correction terms in (2.3).

Turning to the evaluation of the wave part $\tilde{n}_e$ of the electron density we write their equation of continuity in the form

$$\frac{\partial\tilde{n}_e}{\partial t}-u\frac{\partial\tilde{n}_e}{\partial\eta}+\operatorname{div}(n\mathbf{V}_{\perp e})+\frac{1}{e_a}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)j_z=0, \qquad (2.4)$$

where $n=n_0+\tilde{n}_e$ is the total electron density, $\mathbf{V}_{\perp e}$ the transverse electron velocity, and j_z the longitudinal current connected with the perturbed magnetic field through the Maxwell equation

$$j_z=c\operatorname{rot}_z\mathbf{B}_\perp/4\pi. \qquad (2.5)$$

We used the fact that because we neglect the longitudinal wave motion of the ions the longitudinal electron velocity is

$$V_{ze}=j_z/e_en_0. \qquad (2.6)$$

Moreover, from the equation for the transverse electron motion we get, using (2.6),

$$\mathbf{V}_{\perp e}=\frac{c[\mathbf{E}\times\mathbf{e}_z]}{B_z}+\frac{\mathbf{B}_\perp}{e_en_0B_0}j_z+\frac{e_e}{m_e\omega_{Be}{}^2}\frac{d\mathbf{E}_\perp}{dt}. \qquad (2.7)$$

Substituting (2.7) into (2.4) we obtain

$$\left(\frac{\partial}{\partial t}-u\frac{\partial}{\partial\eta}+\mathbf{V}_{\perp e}\nabla\right)\left(\tilde{n}_e-\frac{n_0}{B_0}\tilde{B}_z+\frac{n_0e_e}{m_e\omega_{Be}}\frac{\partial E_y}{\partial\eta}\right)+\frac{1}{e_e}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}+\frac{\mathbf{B}_\perp}{B_0}\nabla\right)j_z=0. \qquad (2.8)$$

We use for the function φ characterizing the wave field [see (1.7), (1.9)] the transverse electric potential which is defined by the relation

$$E_y=-\partial\varphi/\partial y. \qquad (2.9)$$

We find a connection between $\tilde{B}_z$ and φ by using the Maxwell equation $\operatorname{curl}_x\mathbf{B}=4\pi j_x/c$ which in the framework of our approximations reduces to the form

$$\partial\tilde{B}_z/\partial y=4\pi e_en_0E_y/B_0. \qquad (2.10)$$

It follows from (2.9) and (2.10) that

$$\tilde{B}_z=-4\pi e_en_0\varphi/B_0. \qquad (2.11)$$

In contrast to the linear problem we take into account also the z-component of the wave electric field E_z (it will become clear in what follows that E_z is connected with the vector nonlinearity). Similar to Ref. 12 we express E_z in terms of the longitudinal electric potential ψ defined by the relation

$$E_z=-(\alpha\partial/\partial\eta+\partial/\partial z)\psi. \qquad (2.12)$$

We then get for $\mathbf{B}_\perp$ from the Maxwell equations the expression (cf. Ref. 12)

$$\mathbf{B}_\perp=\frac{c}{u}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)[\nabla\hat{L}^{-1}(\varphi-\psi),\mathbf{e}_z]. \qquad (2.13)$$

We find the connection between ψ and φ by using the equation for the longitudinal motion of the electrons, neglecting their longitudinal inertia, i.e., the condition for perfect electron conductivity along the total magnetic field:

$$E_z+(1/c)[\mathbf{V}_e\mathbf{B}]_z=0. \qquad (2.14)$$

Using (2.7) and (2.12) to (2.14) we express $\mathbf{B}_\perp$ in terms of φ:

$$\mathbf{B}_\perp=\frac{c}{u}\left[\nabla\hat{D}^{-1}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)\varphi,\mathbf{e}_z\right]. \qquad (2.15)$$

Here

$$\hat{D}\equiv\frac{\partial}{\partial\eta}-\frac{c}{uB_0}[\nabla\varphi,\nabla]_z. \qquad (2.16)$$

Substituting (2.15) into (2.15) and using the fact that $\partial/\partial x\ll\partial/\partial\eta$ we get

$$j_z=-\frac{c^2}{4\pi u}\frac{\partial^2}{\partial\eta^2}\hat{D}^{-1}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)\varphi. \qquad (2.17)$$

The formulae given here allow us to express $\tilde{n}_i$ and $\tilde{n}_e$ in terms of φ. Finding these expressions and using the quasineutrality condition $\tilde{n}_i=\tilde{n}_e$ we arrive at the required nonlinear equation for φ:

$$\frac{2}{c_A}\frac{\partial}{\partial t}\hat{L}^{-1}\varphi+\frac{c^2}{\omega_{pe}{}^2}\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi+q\varphi^2+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\hat{L}^{-2}\varphi=\frac{c^2}{\omega_{pi}{}^2}\hat{D}^{-1}\frac{\partial}{\partial\eta}\hat{D}_\parallel\frac{\partial\chi}{\partial\eta}. \qquad (2.18)$$

Here

$$q=3e_i(2m_iu^2)^{-1}, \qquad (2.19)$$

$$\chi=\hat{D}^{-1}\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\right)\varphi, \qquad (2.20)$$

$$\hat{D}_\parallel=\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}-\frac{c}{uB_0}[\nabla\chi,\nabla]_z, \qquad (2.21)$$

and the expression for ε is given in section 1. To obtain (2.18) we changed to a frame of reference moving along the magnetic field with a velocity αc_A, i.e., we changed from a variable z to the variable $\zeta = z - \alpha c_A t$ and after that, for the sake of convenience denoted ζ by z.

3. GENERAL ANALYSIS OF THE NONLINEAR EQUATION

We consider some general consequences of Eq. (2.18). Acting upon that equation with the operator $\hat{D}$, multiplying the result by φ and then integrating over space we get after a number of transformations

$$\partial I_1/\partial t=0, \tag{3.1}$$

$$I_1 = \int \varphi^2 \, d\mathbf{r}. \tag{3.2}$$

Equation (3.1) is the law of energy conservation for the case considered of MS waves and at the same time is a well defined guarantee for the self-consistency of the approximations made above.

Neglecting the vector nonlinearity, Eq. (2.18) takes the form

$$\frac{2}{c_A}\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial\eta}\Big[\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi+q\varphi^2 +\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\Big)\hat{L}^{-2}\varphi-\frac{c^2}{\omega_{pi}^2}\Big(2\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\Big)\frac{\partial\varphi}{\partial z}\Big]=0, \tag{3.3}$$

where σ was introduced in section 1. It follows from (3.3) that together with (3.1) we have, when we neglect the vector nonlinearity, also the conversation law

$$\partial I_2/\partial t=0, \tag{3.4}$$

$$I_2 = \int\Big[\Big(\frac{\partial}{\partial x}\hat{L}^{-1}\varphi\Big)^2+\Big(\frac{\partial}{\partial z}\hat{L}^{-1}\varphi\Big)^2+\frac{c^2}{\omega_{pi}^2}\Big(\alpha\frac{\partial\varphi}{\partial\eta}+\frac{\partial\varphi}{\partial z}\Big)^2 -\frac{c^2}{\omega_{pe}^2}\Big(\frac{\partial\varphi}{\partial\eta}\Big)^2+\frac{2}{3}q\varphi^3\Big]d\mathbf{r}. \tag{3.5}$$

To analyze the other consequences of Eq. (2.18) we introduce the concept of high-frequency and low-frequency magnetosonic (HFMS and LFMS) solitons connected, respectively with HFMS and LFMS waves, i.e., waves with characteristic frequencies large or small compared to the ion cyclotron frequency $\omega_{Bi} \equiv e_i B_0/m_i c$ ($e_i = -e_e$ is the ion charge). According to (1.6) HFMS waves correspond to characteristic wave numbers $k_y > \omega_{pi}/c$ and LFMS waves to $k_y < \omega_{pi}/c$. The analog of the characteristic wave number is in the soliton problem the reciprocal of the characteristic width of the soliton $1/l_s$ and the analog of the characteristic frequency is the reciprocal of the characteristic soliton time u/l_s. HFMS waves thus correspond to solitons with $l_s < c/\omega_{pi}$ (small scale solitons) and LFMS waves to solitons with $l_s > c/\omega_{pi}$ (large scale solitons).

In the case of LFMS waves Eq. (2.18) reduces to the form

$$\frac{\partial}{\partial\eta}\Big[\frac{2}{c_A}\frac{\partial\varphi}{\partial t}+\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^3\varphi}{\partial\eta^3}-\varepsilon\frac{\partial}{\partial\eta}(\varphi-q\varphi^2)\Big] =-\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\Big)\varphi. \tag{3.6}$$

Apart from the notation, this equation is the same as (1.4), i.e., it is one of the variants of the three-dimensional KP equation. The integral I_2 for such waves (see (3.5)) means

$$I_2 = \int\Big[\Big(\frac{\partial}{\partial x}\hat{L}^{-1}\varphi\Big)^2+\Big(\frac{\partial}{\partial z}\hat{L}^{-1}\varphi\Big)^2 -\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\Big(\frac{\partial\varphi}{\partial\eta}\Big)^2+\frac{2}{3}q\varphi^3\Big]d\mathbf{r}. \tag{3.7}$$

On the other hand, for the case of HFMS waves instead of (3.6) it follows from (2.18) that

$$\frac{2}{c_A}\frac{\partial}{\partial t}\hat{L}^{-1}\varphi+\frac{c^2}{\omega_{pe}^2}\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi+q\varphi^2+\frac{\partial^2}{\partial x^2}\hat{L}^{-2}\varphi =\frac{c^2}{\omega_{pi}^2}\hat{D}^{-1}\frac{\partial}{\partial\eta}\hat{D}_{\parallel}\frac{\partial\chi}{\partial\eta}. \tag{3.8}$$

It is clear that the three-dimensional equation for HFMS waves does not reduce to the three-dimensional KP equation.

According to Refs. 1, 2 (see also Ref. 13) in the case of strongly nonlinear solitons $l_s \approx c/\omega_{pe}$ so that such solitons correspond to waves with frequencies of the order of the lower hybrid one: $\omega \approx (m_i/m_e)^{1/2}\omega_{Bi}$, i.e., with frequencies much higher than the ion cyclotron frequency. Therefore, if we use the assumption of weakly nonlinear MS waves and aim at an application of the results to the problem of collisionless shock waves, discussed in Refs. 1, 2, and 13, we must bear in mind that for this problem we can only be interested in HFMS waves which are not described by the three-dimensional KP equation. The authors of Ref. 8 ignored this fact when they stated, in their study of wave collapse in media with positive dispersion in the framework of the three-dimensional KP equation, that the problem considered by them is particularly important for the above mentioned problem. From what we have said it is clear that, indeed, the collapse problem studied in Ref. 8 bears no direct relation to this problem.

It follows also from (3.8) that when speaking of two-dimensional HFMS waves one must distinguish two variants of such waves: η, z waves and η, x waves, the field of which depends, respectively on the coordinates mentioned. Putting in (3.8) $\partial/\partial x = 0$ we find that η, z waves are described by the equation

$$\frac{2}{c_A}\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial\eta}\Big[\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi +q\varphi^2-\frac{c^2}{\omega_{pi}^2}\Big(2\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}\Big)\frac{\partial\varphi}{\partial z}\Big]=0. \tag{3.9}$$

This equation does not reduce to the two-dimensional KP equation. However, the case of η, x waves ($\partial/\partial z = 0$) it follows from Eq. (3.8) that

$$\frac{\partial}{\partial\eta}\Big\{\frac{2}{c_A}\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial\eta}\Big[\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi+q\varphi^2\Big]\Big\}+\frac{\partial^2\varphi}{\partial x^2}=0. \tag{3.10}$$

Apart from the notation and a transition to the appropriate frame of reference, this equation is the same as the two-dimensional KP equation (1.1). Hence, for HFMS η, x waves the results of the general analysis of the two-dimensional KP equation are applicable. Amongst such results is the conclusion of Ref. 4, 14 that one-dimensional solitons have a two-dimensional (bending) instability for media with a positive dispersion (this case means $\sigma^2 > 1$) and that there are no instabilities in media with a negative dispersion (when $\sigma^2 < 1$). For the problem of η, x waves with $\sigma^2 > 1$ the conclusion in Ref. 15 that there exist two-dimensional rational solitons (with a wave field decreasing as r^{-2} with distance) is of considerable interest.

Although HFMS and LFMS waves in the cold plasma case considered by us have almost the same dispersion law $\omega \approx k_y c_A$ [see (1.6)] the nature of the particle motion in them is essentially different. The difference in the nature of the ion motion in those and other waves is well known (notwithstanding the fact that Eq. (2.3) for $\tilde{n}_i$ is independent of the ratio of the characteristic wave frequency to the ion cyclotron frequency!). It consists in the fact that in the case of HFMS waves the equilibrium magnetic field affects the ion wave motion very little, whereas in the case of LFMS waves this effect is very important. In the problem of the HFMS waves, in contrast to the LFMS wave case, the unmagnetized ion approximation is thus valid with some degree of accuracy. The difference in the nature of the electron motion is not so apparent. One can use (2.7) to check that in the case of HFMS waves this motion is approximate rotational in the sense that the rotational part of the transverse electron velocity $\mathbf{V}_{\perp e}$ is large compared to the compressible part, i.e.,

$$|\mathrm{rot}_z \mathbf{V}_{\perp e}| > |\mathrm{div}\, \mathbf{V}_{\perp e}|. \qquad (3.11)$$

From this it is clear that the HFMS structures discussed by us have rotational properties. Hence, in our problem we are, generally speaking, dealing not with a trivial multidimensional generalization of the KdV equation through adding to it linear terms with derivatives with respect to one or two additional coordinates, as was done in Refs. 4 and 6, but with generalizing it to the case of rotational structures. A reflection of the rotational nature of the HFMS structures are the terms in Eq. (3.8) with the vector nonlinearity.

The estimates to be given in what follows indicate that the vector nonlinearity is important when $l_s \gtrsim (m_i/m_e)^{1/8} c/\omega_{pe}$. This corresponds to the case of HFMS solitons with rather large amplitudes $h \sim (m_e/m_i)^{1/4}$.

It is also interesting to note that in contrast to the case of drift-ion-acoustic waves[9] the vector nonlinearity occurs only in the essentially three-dimensional equations for the HFMS waves, but drops out of the two-dimensional equations. Hence, the analysis of the role of the vector nonlinearity turns out to be necessary only in the case of three-dimensional problems.

It is clear from what we have said that together with the case of sufficiently small-scale HFMS waves when it is necessary to take into account the vector nonlinearity there are HFMS solitons with not too small l_s in the study of which one can neglect the vector nonlinearity. In the case Eq. (3.8) reduces to the form

$$\frac{\partial}{\partial \eta}\left[\frac{2}{c_A}\frac{\partial \varphi}{\partial t} + \frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^3 \varphi}{\partial \eta^3} - \varepsilon \frac{\partial}{\partial \eta}(\varphi - q\varphi^2)\right] = -\frac{\partial^2}{\partial x^2}\varphi + \frac{c^2}{\omega_{pi}^2}\left(2\alpha\frac{\partial}{\partial \eta} + \frac{\partial}{\partial z}\right)\frac{\partial^2 \varphi}{\partial z^2} = 0. \qquad (3.12)$$

One can also obtain this equation from (3.3) by dropping from it the term with $\partial^2 \hat{L}^{-2}\varphi/\partial z^2$. It is clear that Eq. (3.12) differs radically from the three-dimensional KP Eq. (3.6). This difference is caused by the fact that whereas the term with the derivative with respect to z on the right-hand side of Eq. (3.6) is connected with the main terms in the dispersion Eq. (1.6), the "block" with derivatives with respect to z in Eq. (3.12) is connected with the longitudinal dispersion. To avoid misunderstandings we note that the dispersion occurs in Eq. (3.12) in the form of a combination of two terms: one with the coefficient $1-\sigma^2$ on the left-hand side of Eq. (3.12), and the second with the coefficent c^2/ω_{pi}^2 on the right-hand side of this equations. In the theory of one-dimensional MS waves only the first of these terms occurs.[3] However, in the problem of non-one-dimensional MS waves it is necessary to take into account also the second of these terms and this is done in the present paper. We note also that such kinds of terms are also taken into account in the above mentioned Ref. 10.

4. STABILITY OF ONE-DIMENSIONAL SOLITONS

In the present section we consider the stability of one-dimensional solitons characterized by a potential $\varphi_0(\eta)$, where $\varphi_0(\eta)$ is a soliton solution of the one-dimensional stationary equation

$$\frac{c^2}{\omega_{pe}^2}(1-\sigma^2)\frac{\partial^2 \varphi_0}{\partial \eta^2} - \varepsilon\varphi_0 + q\varphi_0^2 = 0. \qquad (4.1)$$

According to (4.1) the function $\varphi_0(\eta)$ has the form

$$\varphi_0(\eta) = m_i c_A^2 \varepsilon \Phi_0(\xi)/e_i, \qquad (4.2)$$

$$\Phi_0(\xi) = 1/\mathrm{ch}^2(\xi/2), \quad \xi = \eta/\lambda, \quad \lambda = [(1-\sigma^2)/\varepsilon]^{1/2} c/\omega_{pe}. \qquad (4.3)$$

In section 2 we also introduced the function $h \equiv B_z/B_0$ characterizing the relative deviation of the longitudinal magnetic field from its equilibrium value. Using (2.11), (4.2) we find that the potential $\varphi_0 = \varphi_0(\eta)$ corresponds to a function $h = h_0(\eta)$ which is equal to

$$h_0(\eta) = \varepsilon \Phi_0(\xi). \qquad (4.4)$$

It is clear from (4.3) that the quantity λ plays the role of the characteristic size l_s of the soliton which was introduced in section 3, i.e.,

$$l_s \sim \lambda. \qquad (4.5)$$

In accordance with what was said in section 3 we are dealing with HFMS solitons, if

$$(1-\sigma^2) m_e/m_i \varepsilon \ll 1, \qquad (4.6)$$

and with LFMS solitons in the case of an inequality with the opposite sign. Comparing (4.4) and (4.6) we conclude that in the case of HFMS solitons

$$\bar{h}_0 \gg |1-\sigma^2| m_e/m_i, \qquad (4.7)$$

where $\bar{h}_0$ is a characteristic value of the function $h_0(\eta)$. On the other hand, in the case of LFMS we have an inequality which is the opposite of (4.7).

We consider a solution of Eq. (2.18) in the form

$$\varphi=\varphi_0(\eta)+\tilde{\varphi}(\eta, z, x, t), \tag{4.8}$$

where $\tilde{\varphi}$ is a small correction to φ_0 (perturbation of the soliton potential). Putting

$$\tilde{\varphi}\sim\exp(-i\omega t+ik_\parallel z+ik_\perp x),$$

and using (2.18) we get the following expression for $\tilde{\varphi}$:

$$\tilde{\varphi}''-\tilde{\varphi}+3\Phi_0\tilde{\varphi}=2i\left[\Omega\hat{l}^{-1}\tilde{\varphi}+\frac{K_\parallel\sigma}{1-\sigma^2}\hat{d}_0^{-1}\frac{\partial}{\partial\xi}(\tilde{\varphi}'-ib\Phi_0''\hat{d}_0^{-1}\tilde{\varphi})\right]$$
$$-\frac{K_\parallel^2}{1-\sigma^2}\hat{d}_0^{-1}\frac{\partial^2}{\partial\xi^2}\hat{d}_0^{-1}\tilde{\varphi}+\left[K_\perp^2(1-\sigma^2)+\frac{m_e}{m_i\varepsilon}K_\parallel^2\right]\hat{l}^{-2}\tilde{\varphi}. \tag{4.9}$$

The prime indicates here derivatives with respect to ξ,

$$\Omega=\lambda\omega/\varepsilon c_A,\quad K_\parallel=k_\parallel\lambda(m_i/m_e)^{1/2},\quad K_\perp=k_\perp c/\varepsilon\omega_{pe},$$
$$b=K_\perp\mu,\quad \mu=(m_i/m_e)^{1/2}\varepsilon^2/(1-\sigma^2), \tag{4.10}$$
$$\hat{l}=\partial/\partial\xi,\quad \hat{d}_0=\partial/\partial\xi+ib\Phi_0'.$$

Similarly to Refs. 4, 9 we look for $\tilde{\varphi}$ in the form of a series in powers of Ω, $K_\parallel$, and $K_\perp$

$$\tilde{\varphi}=\varphi_1+\varphi_2+\varphi_3+\ldots\,. \tag{4.11}$$

For the sake of generality we assume here that the parameter b which characterizes the vector nonlinearity is a quantity of the order of unity. In what follows we shall be dealing with both cases where $b\ll1$ and with cases when $b\gg1$. We note that the assumption that $b>1$ does not contradict the condition that $K_\perp$ is small, provided that $\mu\gg1$. Using (4.6) it is clear that the vector nonlinearity is important only in the problem of the HFMS solitons. We discuss this problem in more detail in what follows.

Using (4.9) we find that the function φ_1 is given by the standard expression

$$\varphi_1=A\varphi_0', \tag{4.12}$$

where A is an arbitrary constant, while the function φ_2 satisfies the equation

$$\varphi_2''-\varphi_2+3\Phi_0\varphi_2=2iAF. \tag{4.13}$$

Here

$$F=\Phi_0\left\{\Omega+\frac{K_\parallel\sigma}{1-\sigma^2}e^{-ig}\left[1-\frac{3}{2}\Phi_0-\frac{ig}{2}(1-\Phi_0)\right]\right\},$$
$$g=b\Phi_0. \tag{4.14}$$

From (4.13) we get

$$\varphi_2=iA\left[\frac{HG}{\Phi_0}-\Phi_0'\hat{l}^{-1}\left(\frac{FG}{\Phi_0}\right)\right], \tag{4.15}$$

$$H(\Phi_0)=\int_0^{\Phi_0}F(\Phi_0)\,d\Phi_0=\frac{\Phi_0^2}{2}\left[\Omega+\frac{K_\parallel\sigma}{1-\sigma^2}e^{-ig}(1-\Phi_0)\right], \tag{4.16}$$

$$G=\frac{15}{2}\Phi_0^2-1-\frac{5}{2}\Phi_0+\frac{15}{4}\xi\Phi_0\Phi_0'. \tag{4.17}$$

According to (4.9) the equation for φ_3 has the form

$$\varphi_3''-\varphi_3+3\Phi_0\varphi_3=2i\left[\Omega\hat{l}^{-1}\varphi_2+\frac{K_\parallel\sigma}{1-\sigma^2}\hat{d}_0^{-1}(\varphi_2'-ib\Phi_0''\hat{d}_0^{-1}\varphi_2)\right]$$
$$-\frac{AK_\parallel^2}{1-\sigma^2}\hat{d}_0^{-1}\Phi_0''+A\left[K_\perp^2(1-\sigma^2)+\frac{m_e}{m_i}K_\parallel^2\right]\hat{l}^{-1}\Phi_0. \tag{4.18}$$

Multiplying both sides of this equation by Φ_0' and integrating the result over ξ with infinite limits we arrive at the required dispersion equation for the perturbations of the soliton:

$$2\int\frac{G}{\Phi_0}(HF^*+H^*F)\,d\xi-\frac{K_\parallel^2}{1-\sigma^2}\int\Phi_0'^2\,d\xi$$
$$-\left[K_\perp^2(1-\sigma^2)+\frac{m_e}{m_i\varepsilon}K_\parallel^2\right]\int\Phi_0^2\,d\xi=0, \tag{4.19}$$

where the asterisk indicates the complex conjugates. It is interesting to note that when we take the vector nonlinearity into account ($b\neq0$) and also when we neglect it, all coefficients in the dispersion equation turn out to be real (compare Ref. 9).

Substituting (4.140, (4.16) into (4.19) we reduce the dispersion equation to the form

$$\Omega^2-\frac{2C}{3}\frac{\Omega K_\parallel\sigma}{1-\sigma^2}-\frac{K_\parallel^2}{3}\left[\frac{1}{5(1-\sigma^2)^2}+\frac{m_e}{m_i\varepsilon}\right]$$
$$-\frac{K_\perp^2}{3}(1-\sigma^2)=0. \tag{4.20}$$

Here

$$C=\left[\int\Phi_0^2\,d\xi\right]^{-1}$$
$$\int\left[G\Phi_0^2\left(\frac{5}{2}\Phi_0-2\right)+\frac{1}{2}(G\Phi_0\Phi_0')'\right]\cos g\,d\xi. \tag{4.21}$$

We note that approximately

$$C\approx1,\ b\ll1;\quad C\approx O(1/b),\ b\gg1. \tag{4.22}$$

It follows from (4.20) that solitons with $\sigma^2<1$, $\varepsilon>0$ corresponding to waves with negative dispersion are stable for any relation between $K_\parallel$ and $K_\perp$ and any vector nonlinearity. According to (4.4) for such solitons $\tilde{B}_z/B_0>0$: the stable solitons are compression solitons. This is also that class of solitons which was first of all studied in Ref. 1 for $\sigma=0$, i.e., for strictly transverse propagation. We need thus consider only the stability of solitons with $\sigma^2>1$, $\varepsilon<0$ (rarefaction solitons). Such solitons were studied in its time in Ref. 2. Using (4.20) we find the criterion for the stability of such solitons:

$$\frac{5C^2\sigma^2+3}{15(\sigma^2-1)^2}>-\frac{m_e}{m_i\varepsilon}+(\sigma^2-1)\frac{K_\perp^2}{K_\parallel^2}. \tag{4.23}$$

As we noted at the beginning of this section, for the case of LFMS solitons we have an equality which is the opposite of (4.6). Using also the fact that in that case $C=1$ [see (4.22)] we conclude that the instability criterion (4.23) is not satisfied for LFMS solitons and that such solitons are thus unstable. To find the growth rate of that instability we turn to the dispersion Eq. (4.20) which in the case of LFMS waves reads

$$\omega^2={}^1/_3\varepsilon c_A{}^2(k_\parallel{}^2+k_\perp{}^2). \tag{4.24}$$

Hence it follows that perturbations of LFMS solitons grow at a rate

$$\gamma\equiv\mathrm{Im}\,\omega=3^{-1/2}(-\varepsilon)^{1/2}c_Ak, \tag{4.25}$$

where $k=(k_\parallel^2+k_\perp^2)^{1/2}$ is the wave number transverse with respect to the soliton front. The instability considered is a variant of the bending instability of a one-dimensional soliton which was first noted in Ref. 4. The condition for the applicability of Eq. (4.25) is found by recalling that, by assumption, $|\varphi_2|<|\varphi_1|$. It follows from that inequality that

$$k<\frac{\omega_{pe}}{c}\frac{|\varepsilon|}{(\sigma^2-1)^{1/2}}. \tag{4.26}$$

For larger k the instability is stabilized.[14] We thus get from (4.25), (4.26) the estimate

$$\gamma<\varepsilon\omega_s, \tag{4.27}$$

where $\omega_s\approx c_A/\lambda$ is a characteristic frequency of the soliton.

When $k_\parallel=0$ the results (4.25) to (4.27) refer not only to LFMS solitons but also to solitons with arbitrary ω_s/ω_{Bi} amongst which are HFMS solitons (as for $k_\parallel=0$ Eq. (2.18) reduces to the KP equation). We must thus elucidate for the case ω_s/ω_{Bi} what new feature emerges when $k_\parallel\neq0$. We consider this problem, first assuming that $k_\perp=0$ and then taking into account finite $k_\perp$.

When $k_\perp=0$ the stability criterion (4.23) reduces to the form

$$(5\sigma^2+3)/15(\sigma^2-1)^2>m_e/m_i|\varepsilon|. \tag{4.28}$$

This inequality does not contain the perturbation wave numbers and thus characterizes a region of stationary soliton parameters which are stable against the perturbations considered. Comparing inequality (4.26) with (4.6) we conclude that the HFMS solitons are stable (in the above-indicated sense). When $\sigma\gg1$ the stability criterion (4.28) takes the form

$$|\varepsilon|>3\alpha^2. \tag{4.29}$$

In terms of the function h_0 [see (4.4)] this means that the solitons are stable if their amplitude is not too small, $\bar{h}_0\gtrsim3\alpha^2$.

If inequality (4.28) is a strong one, i.e., if we are dealing with HFMS solitons, (4.23) means that the perturbations are stable provided their wave numbers satisfy the relation

$$K_\perp{}^2/K_\parallel{}^2<(5C^2\sigma^2+3)/15(\sigma^2-1)^3. \tag{4.30}$$

When $\sigma^2\gg1$ and if we neglect the vector nonlinearity (4.30) reduces to the form [compare (4.29)]

$$k_\perp{}^2/k_\parallel{}^2<|\varepsilon|/3\alpha^2. \tag{4.31}$$

According to (4.22) the coefficient C decreases when the parameter b increases. It is therefore clear from (4.30) that perturbations which are stable when $C=1$ may become unstable when $C\ll1$. In that sense the vector nonlinearity plays a destabilizing role. One can obtain an estimate of the parameter b by taking for $k_\perp$ its upper limit determined by the right-hand side of (4.26). We then have

$$b\approx\frac{\varepsilon^2}{(\sigma^2-1)^{3/2}}\left(\frac{m_i}{m_e}\right)^{1/2}. \tag{4.32}$$

It is clear that $b\propto1/\sigma^3$ when $\sigma^2\gg1$. The vector nonlinearity is thus most important when σ is of the order 1. In that case $b\gtrsim1$, provided that

$$|\varepsilon|\gtrsim(m_e/m_i)^{1/4}. \tag{4.33}$$

The corresponding solitons have a characteristic size $l_s\approx(m_i/m_e)^{1/8}c/\omega_{pe}$. These were just the estimates used in section 3.

In agreement with the general ideas of the theory of soliton stability,[5] two-dimensional solitons may be formed as a result of the instability of one-dimensional solitons. Using this and the analysis given above one may expect that two-dimensional solitons must have a characteristic η-size l_η of the order l_s and a characteristic x-size l_x of the order of the minimum wave length of the perturbation, i.e., according to (4.5), (4.6)

$$l_\eta\sim\lambda,\quad l_x\sim|\varepsilon|^{1/2}\lambda. \tag{4.34}$$

We shall be dealing with such solitons and their three-dimensional stability in section 5.

5. THREE-DIMENSIONAL STABILITY OF TWO-DIMENSIONAL SOLITONS

In the stationary case ($\partial/\partial t=0$) Eq. (3.10) (a two-dimensional KP-type equation) reduces to the form

$$\frac{\partial^2}{\partial\eta^2}\left[\frac{c^2}{\omega_{pe}{}^2}(1-\sigma^2)\frac{\partial^2\varphi}{\partial\eta^2}-\varepsilon\varphi_0+q\varphi_0{}^2\right]+\frac{\partial^2\varphi_0}{\partial x^2}=0, \tag{5.1}$$

where $\varphi_0=\varphi_0(\eta,x)$. We consider the stability of two-dimensional structures of the type (5.1) starting from our three-dimensional Eq. (3.3). A similar problem has been studied before in Ref. 7 in the framework of the three-dimensional KP Eq. (1.1). In that case, as in Ref. 4, the method of a series expansion in powers of the frequency and the wave-number of the perturbations was used (cf. section 4). The same method will be used in our analysis.

We put in (3.3) $\varphi=\varphi_0+\tilde{\varphi}$ where the perturbation $\tilde{\varphi}$ is assumed to depend on t and z in the form $\exp(-i\omega t+ik_\parallel z)$. We then get for $\tilde{\varphi}$ the equation

$$\frac{\partial^2}{\partial\eta^2}\left[\frac{c^2}{\omega_{pe}{}^2}(1-\sigma^2)\frac{\partial^2\tilde\varphi}{\partial\eta^2}-\varepsilon\tilde\varphi+2q\varphi_0\tilde\varphi\right]+\frac{\partial^2\tilde\varphi}{\partial x^2}$$
$$=2i\left(\frac{\omega}{c_A}\frac{\partial\tilde\varphi}{\partial\eta}+\frac{k_\parallel\alpha c^2}{\omega_{pi}{}^2}\frac{\partial^3\tilde\varphi}{\partial\eta^3}\right)+k_\parallel{}^2\left(\tilde\varphi-\frac{c^2}{\omega_{pi}{}^2}\frac{\partial^2\tilde\varphi}{\partial\eta^2}\right). \tag{5.2}$$

We look for $\tilde{\varphi}$ in the form of the series (4.11). As in section 4, we have $\varphi_1=A\partial\varphi_0/\partial\eta$. By analogy with Ref. 7 we find

$$\varphi_2 = 2iA\left(\frac{\omega}{c_A}\frac{\partial\varphi_0}{\partial\varepsilon} + \frac{k_\parallel \alpha m_i}{m_e}\frac{\partial\varphi_0}{\partial\sigma^2}\right). \tag{5.3}$$

The orthogonality condition for φ_3 which follows from (5.2), gives the dispersion relation

$$\nu^2\frac{\partial P}{\partial\varepsilon} + \nu\sigma\left(\frac{\partial P}{\partial\sigma^2} - \frac{\partial Q}{\partial\varepsilon}\right) - \sigma^2\frac{\partial Q}{\partial\sigma^2} - \frac{1}{2}\left(Q + \frac{m_e}{m_i}P\right) = 0. \tag{5.4}$$

Here

$$\nu = \left(\frac{m_e}{m_i}\right)^{1/2}\frac{\omega}{k_\parallel c_A},$$

$$P = \int \varphi_0^2\, dx\, d\eta, \quad Q = \frac{c^2}{\omega_{pe}^2}\int\left(\frac{\partial\varphi_0}{\partial\eta}\right)^2 dx\, d\eta. \tag{5.5}$$

We reduce (5.4) to dimensionless form by making in (5.1) the change of variables

$$\varphi_0(\eta, x) = (4\varepsilon/q)\,\Phi_0(\theta, x). \tag{5.6}$$

Here

$$\theta = 3^{-1/2}\left(\frac{|\varepsilon|}{\sigma^2-1}\right)^{1/2}\frac{\omega_{pe}}{c}\eta, \quad \varkappa = 3^{-1/2}\frac{|\varepsilon|}{(\sigma^2-1)^{1/2}}\frac{\omega_{pe}}{c}x, \tag{5.7}$$

and the dimensionless function Φ_0 satisfies the equation

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{3}\frac{\partial^2\Phi_0}{\partial\theta^2} - \Phi_0 + 4\Phi_0^2\right) - \frac{\partial^2\Phi_0}{\partial\varkappa^2} = 0. \tag{5.8}$$

We assume that $\sigma^2 > 1$, $\varepsilon < 0$. Using (5.6), (5.7) we get from (5.5)

$$P = \frac{48c^2k_P}{q^2\omega_{pe}^2}(\sigma^2-1)(-\varepsilon)^{1/2}, \quad Q = \frac{16c^2k_Q}{q^2\omega_{pe}^2}(-\varepsilon)^{3/2}, \tag{5.9}$$

where the dimensionless coefficients k_P, k_Q are independent of σ^2 and ε and given by the relations

$$k_P = \int \Phi_0^2\, d\theta\, d\varkappa, \quad k_Q = \int\left(\frac{\partial\Phi_0}{\partial\theta}\right)^2 d\theta\, d\varkappa. \tag{5.10}$$

Using (5.9) we reduce (5.4) to the form

$$(\sigma^2-1)\nu^2 + (2 + k_Q/k_P)\varepsilon\sigma\nu + \varepsilon^2 k_Q/3k_P - (\sigma^2-1)\varepsilon m_e/m_i = 0. \tag{5.11}$$

It follows from (5.11) that the perturbations considered by us are stable provided that

$$\left(1+\frac{k_Q}{k_P}\right)^2 + (\sigma^2-1)\left[\left(1+\frac{k_Q}{3k_P}\right)^2 + \frac{5}{36}\frac{k_Q^2}{k_P^2}\right] + \frac{m_e}{m_i\varepsilon}(\sigma^2-1)^2 > 0. \tag{5.12}$$

In the case of HFMS waves, i.e., when $|\varepsilon| \gg \alpha^2$ this inequality is satisfied for any k_P, k_Q. The two-dimensionless HFMS structures described by Eq. (5.1) are thus stable in the framework of the assumptions made about the nature of the perturbations. On the other hand, in the case of LFMS waves ($|\varepsilon| \ll \alpha^2$) the stability criterion (5.12) is not satisfied (we recall that $\varepsilon < 0$). According to (5.11) the pertubations then increase with a growth rate given by the relation (compare (4.25))

$$\gamma = k_\parallel c_A(-\varepsilon)^{1/2}. \tag{5.13}$$

The stability of the rational two-dimensional soliton found in Ref. 15 was discussed in Ref. 7. Such a soliton is characterized by the function

$$\Phi_0 = (1+\varkappa^2-\theta^2)/(1+\varkappa^2+\theta^2)^2. \tag{5.14}$$

In that case

$$k_P = \pi/2, \quad k_Q = \pi. \tag{5.15}$$

Substituting (5.15) into (5.12) and assuming that $\sigma^2 \gg 1$ we find that the rational soliton is stable when (cf. (4.29)

$$|\varepsilon| > 3\alpha^2/10. \tag{5.16}$$

According to Ref. 7 the rational soliton is unstable in situations described by the three-dimensional KP equation. This conclusion does not refer to rational solitons of HFMS waves which in accordance with what was said above are stable against three-dimensional perturbations.

6. DISCUSSION OF THE RESULTS

We have studied the problem of the three-dimensional generalization of the KdV equation for MS waves in a magnetized plasma and have found that Eq. (2.18) is such a generalization. In the limiting case of LFMS waves this equation reduces to the three-dimensional KP Eq. (3.6), whereas in the opposite limiting case of HFMS waves it is Eq. (3.8), whose structure is essentially different from the three-dimensional KP equation [see also (3.12)].

In deriving Eq. (2.18) we followed the cold-plasma approximation. In this connection the problem arises about the limit of applicability of our results, i.e., about their sensitivity to the electron and ion temperature. One obtains these limits of applicability easily by turning to the linear theory of magnetosonic waves expounded, for instance, in the monograph by Akhiezer *et al.*[16] We then find that one can neglect the electron temperature, if $\beta_e < 1$, where β_e is the ratio of the electron pressure to the magnetic field pressure. One must then, however, bear in mind that there is a so-called resonance angle $\alpha \approx (m_e/m_i\beta_e)^{1/2}$ for which the damping of the waves is important due to the interaction with resonance electrons. The rate of this damping is proportional to β_e. One must thus exclude waves with $\alpha \approx (m_e/m_i\beta_e)^{1/2}$ from the consideration when $\beta_e > m_e/m_i$ or one must restrict oneself to analyzing processes with characteristic times less than the inverse of the damping rate. The limits of applicability of the cold ion approximation depends in an essential way on whether the waves are high- or low-frequency in relation to the ion cyclotron frequency. In the case of HFMS waves the cold ion approximation is valid when $\beta_i < 1$, where β_i is the ratio of the ion pressure to the magnetic field pressure. In the case of LFMS waves the condition for the applicability of the cold ion approximation is considerably more rigid: $\beta_i < m_e/m_i$ (for details see Ref. 11).

We note also that in deriving our nonlinear equations we assumed for the sake of simplicity that $\alpha < 1$. Moreover, one should bear in mind that we use the weak dispersion approximation. This presupposes that the condition $\alpha < l_s\omega_{pi}/c$ is satisfied. For the smallest-scale solitons with a characteristic frequency $\omega \sim (m_i/m_e)^{1/2}\omega_{Bi}$ when $l_s \sim c/\omega_{pe}$ this inequality means $\alpha < (m_e/m_i)^{1/2}$. When l_s increases the range of admissable α broadens and when $l_s \gtrsim c/\omega_{pi}$ there do not arise any restriction on α except the condition $\alpha < 1$.

Our three-dimensional Eqs. (2.18) and (3.8) supplement the well known set of nonlinear equations for weakly dispersive waves in a magnetized plasma (compare (2.18), (3.8) with (1.1) to (1.3) and the equations of Ref. 9). We used these equations in sections 4 and 5 to analyze the stability of one- and two-dimensional HFMS solitons and showed that the stability problem for such solitons is not as trivial as would have followed from the three-dimensional KP equation. In particular, two-dimensional HFMS solitons turn out to be stable, in contrast to the predictions of Ref. 7, under long-wavelength three-dimensional perturbations. This enhances the practical value of Ref. 15 and of similar studies on two-dimensional structures described in the framework of the usual (two-dimensional) KP equation. It is natural that, as we have studied only long-wavelength pertrubations, the problem of the stability of both one- and two-dimensional HFMS solitons cannot be regarded as finally solved and an analysis is needed of shorter-wavelength perturbations.

One must also bear in mind that soliton perturbations are possible which are not described by our original equation (2.18). Examples of such perturbations and the instabilities connected with them were, in particular, studied in Refs. 17 to 20. According to Ref. 17 such instabilities restrict the lifetime of the solitons, but this time is, generally speaking, long compared to the reciprocal of the characteristic growth rates of the instabilities discussed by us.

We note also that the nonlinear equations introduced by us can be used not only for a study of the stability of magnetosonic solitons, but also in a broader class of problems of magnetosonic waves, such as problems going beyond the framework of the soliton problem (for instance, in problems about periodic waves).

Our three-dimensional nonlinear Eq. (2.18) takes into account besides the traditional effects of the theory of MS waves also the effects of the vector nonlinearity. We note that such an approach is conceptually close not only to the approach of the above mentioned Ref. 9, but also to the approach of Refs. 21, 22, 23 (see also Ref. 24 which is a review) in which the vector nonlinearity was taken into account in the problem of lower-hybrid waves. It follows from our analysis that the vector nonlinearity is important in the highest-frequency part of the MS wave spectrum and that is just the part where the transition from MS waves to lower-hybrid waves begins. We have elucidated the effect of the vector nonlinearity on the stability of one-dimensional HFMS solitons. It turned out that such a nonlinearity affects the growth rate and the condition for the occurrence of an instability, but this effect is basically quantitative rather that qualitative. It is also interesting to study the role of the vector non-linearity in other problems on HFMS waves and this can be done by using our Eqs. (2.18) or (3.8).

We noted in the Introduction problems of applications which started the development of the theory of MS solitons.[1,2] At the moment the concept of MS waves is widely used in the theory of the high-frequency heating of a plasma,[25] in the theory of collective processes in a plasma with α-particles,[26] and other applied sections of plasma theory. For the same problems the analysis given in the present paper is also of interest.

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