Inelastic tunneling by a particle interacting with oscillations

M. Yu. Sumetskiĭ

Leningrad Electrotechnical Institute for Communications
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Tunneling of a particle that interacts with harmonic oscillations is investigated. Account is taken of the particle action on a quantum oscillator in the external-force approximation, and of the influence of the oscillations on the particle in first-order quasiclassical perturbation theory. The method employed is valid for long-range local oscillations and long-wave phonons. The tunneling probability and the amplitudes of the multiquantum tunneling transitions are calculated for a quasistationary state in a well and also for scattering by a potential barrier. Applications considered are the tunneling of a particle interacting with acoustic phonons at low temperature and inelastic tunneling of an electron in field emission from a metal.

Interaction with oscillations of a medium (local oscillations, phonons) determines the temperature dependence of the tunneling and of the amplitude of the inelastic tunneling transitions. Allowance for this interaction may turn out to be substantial in the study of the tunneling of light[1–7] as well as heavy[8–11] particles. A theory of inelastic tunneling has recently been developed in connection with investigations of dissipative effects in the region below the barrier.[12,13]

The tunneling-transition amplitudes are usually calculated by using quantum perturbation theory or the tunnel-Hamiltonian method (see Refs. 1–3 and 14). To determine the tunneling probability we must know the wave functions, at least accurate to the terms quadratic in the interaction, but their calculation is quite laborious. The functional (path)-integration method permits calculation of the tunneling probability by integrating over the oscillation variables, and reduces the problem in the quasiclassical approximation to a solution of a single-particle Newton equation with a nonlocal potential.[9] This approach cannot be used to calculate the amplitudes of tunneling transitions in which a fixed number of vibrational-energy quanta is acquired (or lost).

We develop below a method, based on quasiclassical perturbation theory, for calculating the tunneling probability and the tunneling-transition amplitudes. In the analysis of the interaction between the particle and the oscillations the starting point is the Schrödinger equation.[1)]

$$\frac{1}{2}\left(\Psi_{xx}+\sum_{j=1}^{N}\Psi_{u_j u_j}\right)+\left[E-V(x)-\sum_{j=1}^{N}\left(W_j(x)u_j+\frac{\omega_j^2 u_j^2}{2}\right)\right]\Psi=0, \qquad (1)$$

where x is the particle coordinate, u_j are the oscillation coordinates, $V(x)$ is the initial potential in which the particle moves, and $W_j(x)$ are the interaction forces.

We assume that the interaction energy and the oscillator frequencies are much lower than the particle energy. We assume also, unless otherwise stipulated, that the potential $V(x)$ and the interaction $W_j(x)$ are quasiclassically slow functions of the coordinate x. The asymptotic solution obtained for Eq. (1) takes into account the influence of the oscillators on the particle motion in first order of quasiclassical perturbation theory, as well as the influence of the particle on quantum oscillators within the framework of the external-force approximation.

Quasiclassical perturbation theory is more convenient than the quantum theory for the investigation of tunneling, since it yields the increment of the exponential in the expression for the wave function. As a result, the tunneling probability and the transition amplitudes can be changed by several times even in first order of such a theory. On the other hand, we are restricted to the analysis of rather smooth perturbations (long-range local oscillations, or long-wave phonons).

We consider below the probability of an inelastic tunneling transition from a bound or from a free state into a free one. The form of the potential $V(x)$ is shown in Fig. 1. The method proposed can be used to treat similarly a transition between two bound states.

In Sec. 1 we obtain an asymptotic solution of Eq. (1), while in Sec. 2 we present the conditions for its validity. We use the solution first to consider highly excited states in a well, after which we calculate the tunneling probability and the transition amplitudes. If the adiabatic approximation holds in the well, or if the potential can be approximated by a quadratic one, the resultant expressions can be extended to include low-lying states.

In Secs. 3–7 it is assumed that $N = 1$, a generalization to arbitrary N is given in Sec. 8, and in Sec. 9 the results are generalized, under certain assumptions, to the case of three-

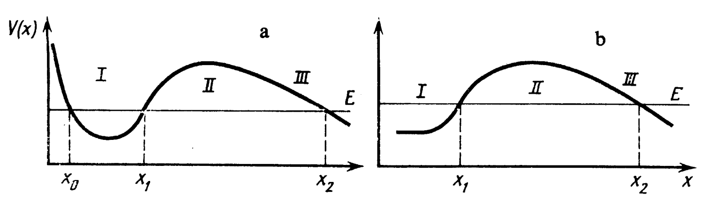

FIG. 1

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dimensional particle motion. Some applications are considered in the last section.

1. SOLUTION OF THE SCHRÖDINGER EQUATION

We separate in Eq. (1) the motion that is quasiclassical in x, by representing the wave function as the product

$$\Psi=p^{-1/2}\exp\left(i\int^{x}p\,dx\right)\Omega(x,u_1,\ldots,u_N),$$

$$p=[2(E^0-V(x))]^{1/2}, \qquad (2)$$

where the energy E^0 will be determined below. Substitution of (2) in (1) leads to an equation for Ω:

$$\left(ip-\frac{p_x}{2p}\right)\Omega_x+\frac{1}{2}\Omega_{xx}+\frac{1}{2}\sum_{j=1}^{N}\Omega_{u_ju_j}$$
$$+\left[E-E^0+\frac{3p_x^2}{8p^2}-\frac{p_{xx}}{4p}-\sum_{j=1}^{N}\left(W_j(x)u_j+\frac{\omega_j^2u_j^2}{2}\right)\right]\Omega=0. \qquad (3)$$

In the quasiclassical approximation in x and for sufficiently small perturbations W_j (the actual conditions are given in Sec. 2), Eq. (3) is transformed into the following nonstationary Schrödinger equation:

$$i\Omega_t+\frac{1}{2}\sum_{j=1}^{N}\Omega_{u_ju_j}$$
$$+\left[E-E^0-\sum_{j=1}^{N}\left(W_j(x(t))u_j+\frac{\omega_j^2u_j^2}{2}\right)\right]\Omega=0, \qquad (4)$$

where x is replaced by the time variable

$$t(x)=\int^{x}(dx/p). \qquad (5)$$

It can be easily noted that the oscillations with different u_j separate. We seek the solution of this equation in the form

$$\Omega(x,u_1,\ldots,u_N)=\prod_{j=1}^{N}\Omega^{(j)}(x,u_j).$$

Equation (1) is satisfied if

$$i\Omega_t^{(j)}+\frac{1}{2}\Omega_{u_ju_j}^{(j)}+\left[\lambda_j-W_j(x(t))u_j-\frac{\omega_j^2u_j^2}{2}\right]\Omega^{(j)}=0, \qquad (6)$$

and the arbitrary constants satisfy the condition

$$\sum_{j=1}^{N}\lambda_j+E-E^0=0.$$

Equation (6) coincides formally with the exactly solvable nonstationary Schrödinger equation for an oscillator in a field of variable strength.[15] We arrive in the upshot to the following asymptotic expression for the wave function:

$$\Psi_{\mathbf{n}k}(x,\mathbf{u})=p^{-1/2}\exp\left(i\int_{x_{k-1}}^{x}p\,dx\right)\Omega_{\mathbf{n}k},$$
$$\mathbf{n}=(n_1,\ldots,n_N),\qquad \mathbf{u}=(u_1,\ldots,u_N), \qquad (7)$$
$$\Omega_{\mathbf{n}k}=\prod_{j=1}^{N}\left(2^{n_j}n_j!\left(\frac{\pi}{\omega_j}\right)^{1/2}\right)^{-1/2}H_{n_j}(\omega_j^{1/2}(u_j-f^{(j)}))$$
$$\times\exp\left\{-\frac{\omega_j}{2}(u_j-f^{(j)})^2+ipf_x^{(j)}u+\frac{i}{2}\right.$$
$$\left.\times\int_{x_{k-1}}^{x}\left[(\omega_jf^{(j)})^2-(pf_x^{(j)})^2\right]\frac{dx}{p}\right\},$$
$$E^0=E-\sum_{j=1}^{N}\omega_j\left(n_j+\frac{1}{2}\right), \qquad (8)$$

where E^0 is chosen to satisfy the condition that Eq. (7) become the usual solution that is quasiclassical in x if the variables separate ($W=f\equiv 0$; $H_n(z)$ is a Hermite polynomial, and the functions $f^{(j)}(x)$ satisfy the differential equation

$$p(pf_x)_x+\omega_j^2f^{(j)}+W_j(x)=0, \qquad (9)$$

which reduces via the substitution (5) to Newton's equation for an harmonic oscillator with an external force $-W_j$.

Using the asymptotic expression for high-index Hermite polynomials, we easily obtain from (7) and (8) the classical action corresponding to the wave function (7):

$$S=\int^{x}p\,dx+\sum_{j=1}^{N}\left\{pf_x^{(j)}u_j+\frac{1}{2}\int^{x}[(\omega_jf^{(j)})^2-(pf_x^{(j)})^2]\frac{dx}{p}\right.$$
$$\left.\pm\int^{\omega_j^{1/2}(u_j-f^{(j)})}(2n_j-z_j^2)^{1/2}dz_j\right\}. \qquad (10)$$

This expression satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial S}{\partial u_j}\right)^2$$
$$+V(x)+\sum_{j=1}^{N}\left(W_j(x)u_j+\frac{\omega_j^2u_j^2}{2}\right)=E$$

accurate to terms quadratic in the ratio of the interaction and kinetic energy of the oscillators to the particle kinetic energy.

From the viewpoint of diffraction theory, solution (8) is a Gaussian beam localized in configuration space $(x,u_1,\ldots,u_N)$ near the axis. The asymptotic character of such solutions as $\hbar\to 0$ and at finite transverse quantum numbers n_j was proved in general form in Ref. 16.

2. CONDITIONS FOR THE VALIDITY OF THE APPROXIMATION

To determine the actual conditions under the asymptotic form (7) is valid, we assume for simplicity that the characteristic length x_0 defines in the region considered the distance over which the functions $V(x)$ and $W_j(x)$ vary. We introduce also the characteristic values of the oscillator coordinates u_{0j}, of the time $\tau_0=x_0/p_0$, and of the interaction W_{0j}. We note that under the condition

$$p_0x_0\gg 1 \qquad (11)$$

the terms $3p_x^2/8p^2$ and $p_{xx}/4p$ in the square of (3) can, just as in the one-dimensional WKB method, be discarded in view of their relatively small [of the order of $(p_0x_0)^{-1}$] contribution to the wave function. Under the same condition, $p_x/2p$ can be neglected compared with ip in the first term. For (3) to go over into (4) it suffices to neglect Ω_{xx} compared with the remaining terms. To determine the conditions under which this is permissible, we rewrite (3), with the indicated small terms neglected), in the form

$$ip\Lambda_x+\frac{1}{2}(\Lambda_{xx}+\Lambda_x^2)+\frac{1}{2}\sum_{j=1}^{N}(\Lambda_{u_ju_j}+\Lambda_{u_j}^2)+\left[E-E^0-\sum_{j=1}^{N}\left(W_j(x)u_j+\frac{\omega_j^2u_j^2}{2}\right)\right]=0, \qquad (12)$$

where the function Λ is obtained from $\Omega=e^{\Lambda}$, Let $\Omega_n=e^{\Lambda_0}$ be the solution (8) of Eq. (4) and let $\Lambda=\Lambda_0+\Lambda_1$. The condition for the transition from (3) to (4) is then determined by the inequality

$$\Lambda_1\ll 1. \qquad (13)$$

We seek the value of Λ_1 by iterating Eq. (12) with respect to $\Lambda_{xx}+\Lambda_{0x}^2$. In first order, we arrive at the equation

$$ip\Lambda_{1x}+\frac{1}{2}\sum_{j=1}^{N}(\Lambda_{1u_ju_j}+2\Lambda_{0u_j}\Lambda_{1u_j})=-\frac{1}{2}(\Lambda_{0xx}+\Lambda_{0x}^2). \qquad (14)$$

We consider now two cases.

a) $\omega_j\tau_0\lesssim 1$ or $\omega_jx_0\lesssim p_0$. It follows from this and from (11) that

$$\omega_j\ll p_0^2. \qquad (15)$$

According to (9) we have then

$$|f^{(j)}|\sim W_{0j}/\omega_j^2,\qquad |f_x^{(j)}|\sim W_{0j}/(\omega_j^2x_0).$$

In addition, it follows from (8) and (10) that in the region of interest to us

$$u_{0j}\lesssim(n_j/\omega_j)^{1/2}+|f^{(j)}|. \qquad (16)$$

As a result we obtain from (14) and (13)

$$\Lambda_1\lesssim\sum_{j=1}^{N}\left(\frac{n_j^{1/2}W_{0j}}{\omega_j^{5/2}x_0^2}+\frac{Nn_jp_0W_{0j}^2}{\omega_j^5x_0^3}+\frac{Np_0W_{0j}^4}{\omega_j^8x_0^3}\right)\ll 1. \qquad (17)$$

b) $\omega_j\tau_0\gtrsim 1$. Consider first the classical forbidden region. From (9) we get the estimates:

$$|f^{(j)}|\sim W_{0j}\exp(\omega_j\tau_0)/\omega_j^2,\qquad |f_x^{(j)}|\sim W_{0j}\exp(\omega_j\tau_0)/(\omega_jp_0).$$

Using them together with (16) we obtain in this case

$$\Lambda_1\lesssim\sum_{j=1}^{N}\left\{\frac{n_j^{1/2}W_{0j}\exp(\omega_j\tau_0)}{\omega_j^{1/2}p_0^2}+\frac{Nn_jW_{0j}^2\exp(2\omega_j\tau_0)}{\omega_j^2p_0^2}+\frac{NW_{0j}^4\exp(4\omega_j\tau_0)}{\omega_j^5p_0^2}\right\}\ll 1. \qquad (18)$$

It can be similarly shown that the condition for the validity of the solution (7) in the classically allowed region is obtained from (18) by replacing each exponential in it by unity.

Conditions (17) and (18) can be simplified by assuming that all the W_{0j} are of the same order, and that N, n_j, and the adiabaticity parameters $\omega_j\tau_0$ are of the order of unity. Equations (11), (17), and (18) are then equivalent to the inequalities

$$\omega_j\ll p_0^2,\qquad W_{0j}^4\ll\omega_j^5p_0^2. \qquad (19)$$

Conditions (17) and (18) can be regarded, on the one hand, as constraints on the interactions W_j. On the other hand, according to the result of Ref. 16, they will be satisfied in the classical limit if p_0 and x_0 are large enough.

The conditions obtained become invalid near the turning points x_k of the momentum p. In the vicinity of x_k it is necessary to find the matching conditions for the solutions such as (7). This is the subject of the next section.

3. MATCHING RULES AND CERTAIN DESIGNATIONS

For brevity, we assume for the time being that $N=1$ and omit the subscript j.

The matching near the turning point x_k can be effected by Zwaan's method.[17] In analogy with the one-dimensional case, we bypass the turning point x_k in the complex x plane, where the conditions for quasiclassical behavior are formally satisfied. As a result we find that in Eq. 7 the substitution $p\to\pm i|p|$ (on going below the barrier) or $p\to\pm p$ (in reflection) must be made not only in the factor preceding the exponential in the first integral, but also in the expansion of the function f near the turning point x_k:

$$f(x)\approx D_1+D_2\int_{x_k}^{x}(dx/p)$$

as well in the analogous expansion of the integral of $(\omega^2f^2-p^2f_x^2)$.

Let, for example, the regions to the left and to the right of x_1 be classically allowed and forbidden, respectively. We express the solution (7) at $x\approx x_1$ in the form $\Psi_{n2}(p)$, denoting explicitly the functional dependence on $p(x)$. The matching rule for the transition into the below-barrier region is then

$$[\Psi_{n2}(p)+\Psi_{n2}(-p)]|_{x<x_1}\to\Psi_{n2}(i|p|)|_{x>x_1}. \qquad (20)$$

The other matching rules are obtained similarly.

We introduce some designations. We express the solution of (9) in the region of k in the form

$$f_k(x)=\frac{1}{2i\omega}\left[\exp\left(-i\omega\int^{x}\frac{dx}{p}\right)\int_{x_{k-1}}^{x}\exp\left(i\omega\int^{x}\frac{dx}{p}\right)W\frac{dx}{p}-\exp\left(i\omega\int^{x}\frac{dx}{p}\right)\int_{x_{k-1}}^{x}\exp\left(-i\omega\int^{x}\frac{dx}{p}\right)W\frac{dx}{p}\right]+A_k^+\exp\left(i\omega\int_{x_{k-1}}^{x}\frac{dx}{p}\right)+A_k^-\exp\left(-i\omega\int_{x_{k-1}}^{x}\frac{dx}{p}\right), \qquad (21)$$

where $A_k^{\pm}$ are free parameters. We introduce also the quantities

$$G_k^{\pm}=\mp\frac{1}{2i\omega}\int_{x_{k-1}}^{x_k}\exp\left(\mp i\omega\int_{x_1}^{x}\frac{dx}{p}\right)W\frac{dx}{p},\qquad k=1,2,$$

$$G_3^{\pm}=\mp\frac{1}{2i\omega}\int_{x_2}^{\infty}\exp\left(\mp i\omega\int_{x_2}^{x}\frac{dx}{p}\right)W\frac{dx}{p}, \qquad (22)$$

$$\gamma_2=\frac{1}{2\omega}\int_{x_1}^{x_2}\int_{x_1}^{x}\frac{dx}{|p(x)|}\frac{dx'}{|p(x')|}$$

$$\times\exp\left(\omega\int\limits_{x}^{x'}\frac{dx''}{|p(x'')|}\right)W(x)W(x'), \tag{23}$$

$$\tau_k=\int\limits_{x_{k-1}}^{x_k}\frac{dx}{|p|}. \tag{24}$$

Here τ_1 is half the period of the oscillations in the well, and τ_2 is the tunneling time. The parameters $G_k^{\pm}$ related the solutions of Eq. (9) near the turning point, x_{k-1} and x_k. For example, if we have in the vicinity of x_1, according to (21), the function

$$f_2\approx(A_2^{+}+A_2^{-})+(A_2^{+}-A_2^{-})\,\omega\int\limits_{x_1}^{x}\frac{dx}{|p|},$$

this function must take near x_2 the form

$$f_2\approx[(A_2^{+}+G_2^{+})e^{\omega\tau_2}+(A_2^{-}+G_2^{-})e^{-\omega\tau_2}]$$

$$+[(A_2^{+}+G_2^{+})e^{\omega\tau_2}-(A_2^{-}+G_2^{-})e^{-\omega\tau_2}]\,\omega\int\limits_{x_2}^{x}\frac{dx}{|p|}.$$

4. EIGENFUNCTIONS IN A POTENTIAL WELL

We seek the normalized quasiclassical wave function with energy $E_l^0+\omega(n+1/2)$ in a potential well in the form

$$\Phi_{ln}=(2\tau_1)^{-1/2}(\Psi_{n1}+\Psi_{n1}^{*}). \tag{25}$$

The continuation of Φ_{ln} beyond the turning points x_0 and x_1 must decreases exponentially. According to the matching rules this is equivalent, as in the one-dimensional case, to the requirement that the function Ψ_{n1} remain unchanged after tracing the contour C on the Riemann surface of the function

$$p(x)=[2(E_l^0-V(x))]^{1/2}$$

with a cut on the interval (x_0, x_1) (Fig. 2). This rule, when applied to the function f_1, determines uniquely the coefficients A_1^{+} and A_1^{-}:

$$A_1^{+}=A_1^{-}=A_1=(G_1^{-}-G_1^{+})/(e^{i\omega\tau_1}-e^{-i\omega\tau_1}), \tag{26}$$

and leads also to a rule for quantizing the particle energy E_l^0:

$$\int\limits_{x_0}^{x_1}[2(E_l^0-V(x))]^{1/2}dx=\pi\left(l+\frac{1}{2}\right)+\frac{1}{2}\int\limits_{x_0}^{x_1}(p^2f_{1x}^2-\omega^2f_1^2)\frac{dx}{p}, \tag{27}$$

from which we obtain for the particle energy shift on account of the interaction

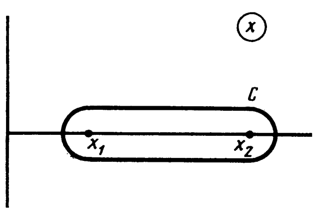

FIG. 2

$$\Delta E_l=\frac{1}{2\tau_1}\int\limits_{x_0}^{x_1}(p^2f_{1x}^2-\omega^2f_1^2)\frac{dx}{p}. \tag{28}$$

In the adiabatic limit $\omega\tau_1\ll1$, expanding (26), (22), and (21) in powers of $\omega\tau_1$, we obtain

$$A_1=-\frac{\langle W\rangle}{2\omega^2}(1+O(\omega^2\tau_1^2)),\quad f_1=-\frac{\langle W\rangle}{\omega^2}(1+O(\omega^2\tau_1^2)), \tag{29}$$

$$f_x=-\frac{1}{p}\int\limits_{x_0}^{x}\frac{dx}{p}(W-\langle W\rangle)(1+O(\omega^2\tau_1^2)),$$

$$\langle W\rangle=\frac{1}{\tau_1}\int\limits_{x_0}^{x_1}W\frac{dx}{p}. \tag{30}$$

Substituting these expressions in (8), we arrive in the approximation linear in $\omega\tau_1$ at an adiabatic wave function in the Condon form:

$$\Psi_{n1}(x,u)=\left[2^n n!\,p\left(\frac{\pi}{\omega}\right)^{1/2}\right]^{-1/2}H_n\left(\omega^{1/2}\left(u+\frac{\langle W\rangle}{\omega^2}\right)\right)\times\exp\left\{-\frac{\omega}{2}\left(u+\frac{\langle W\rangle}{\omega^2}\right)^2+i\int\limits_{x_0}^{x}\left[2\left(E+\frac{\langle W\rangle^2}{2\omega^2}\right)-V(x)-(W(x)-\langle W\rangle)u\right]^{1/2}dx\right\}. \tag{31}$$

In the opposite adiabatic limit $\omega\tau_1\gg1$, integrating (22) and (21) by parts, we get

$$A_1=-\frac{W(x_0)}{2\omega^2}(1+O(\omega^{-2}\tau_1^{-2})),$$

$$f_1=-\frac{W(x)}{\omega^2}(1+O(\omega^{-1}\tau_1^{-1})). \tag{32}$$

We have hence in the zeroth approximation in $(\omega\tau_1)^{-1}$

$$\Psi_{n1}(x,u)=\left[2^n n!\,p\left(\frac{\pi}{\omega}\right)^{1/2}\right]^{-1/2}H_n\left(\omega^{1/2}\left(u+\frac{W(x)}{\omega^2}\right)\right)\times\exp\left\{-\frac{\omega}{2}\left(u+\frac{W(x)}{\omega^2}\right)^2+i\int\limits_{x_0}^{x}\left[2\left(E-V(x)+\frac{W^2(x)}{2\omega^2}\right)\right]^{1/2}dx\right\}. \tag{33}$$

Since it follows from (17) and (18) that W^2/ω^2 and Wu are small compared with p^2, we moved these quantities into the intregrals of (31) and (33). We note that at constant W these expressions coincide and yield for Eq. (1) a solution that is quasiclassical with respect to x.

Equation (33) acquires the known effective increment $-W^2(x)/2\omega^2$ to the potential $V(x)$ (Refs. 9 and 12), and the particle motion in the renormalized potential is elastic at $\omega\tau_1\gg1$ in the classically allowed region. The next terms of the expansion in $(\omega\tau_1)^{-1}$ are responsible for the inelastic corrections.

According to (26), the quantity A_1 and with it the probability of tunneling decay of the state (see below) increases strongly, generally speaking, near the classical resonance

$\omega\tau_1 = \pi q$, $q = 1,2,\ldots$. In the immediate vicinity of the resonance the approximation considered no longer holds.

5. ASYMPTOTE OF WAVE FUNCTION AT LARGE x

We continue now the wave function from region I into region III (Fig. 1). We consider first the case of a highly excited bound state. We indicate next the variation of the wave function if the states is not necessarily highly excited, but in a certain vicinity of the well the potential can be regarded as quadratic, or else the adiabatic approximation is valid. In the end of the section we consider the case of a free-free transition.

Assume that the final states of the particles correspond to x tending to infinity, where the interaction $W(x)$ becomes negligibly small. This condition is not important for the determination of the tunneling probability, since the latter is independent of the behavior of $V(x)$ and $W(x)$ at $x > x_2$. It determines, however, the final states of a particle with fixed energy, and hence also the transition amplitudes.

The continuation of the wave function (25) into the below-barrier region decreases exponentially. In the region $x_1 < x < x_2$ it takes the form[2)]

$$\Phi_{ln} \approx C\Psi_{n2}(x, u), \quad C = (2\tau_1)^{-1/2}, \tag{34}$$

where the unknown constants A_2^+ and A_2^- of the function $f_2(x)$ in (21) are determined by the matching rules. According to Sec. 3, we obtain

$$A_2^+ = A_2^- = A_2 = G_1^\pm + A_1 e^{\pm i\omega\tau_1}$$
$$= (G_1^- e^{i\omega\tau_1} - G_1^+ e^{-i\omega\tau_1})(e^{i\omega\tau_1} - e^{-i\omega\tau_1})^{-1}. \tag{35}$$

Continuing the wave function (34) into the region III, we obtain at large $x \to \infty$, apart from inessential constant phase factors,

$$\Phi_{ln} \approx \left[2^n n! \left(\frac{\pi}{\omega}\right)^{1/2}\right]^{-1/2} C H_n(\omega^{1/2}(u - B_- e^{-i\omega t} - B_+ e^{i\omega t}))$$
$$\times \exp\left\{-\int_{x_1}^{x_2} |p|\,dx + \varphi + 2\omega u B_- e^{-i\omega t} - \omega B_-^2 e^{-2i\omega t} - \frac{\omega u^2}{2}\right\}, \tag{36}$$

where

$$t = \int_{x_2}^{x} \frac{dx}{p}, \quad B_\pm = A_3^\pm + G_3^\pm, \quad A_3^\pm = (A_2^\pm + G_2^\pm) e^{\pm\omega\tau_2}, \tag{37}$$
$$\varphi = \gamma_2 - \omega(|A_2^-|^2 + |G_3^-|^2) - 2\omega \,\mathrm{Re}\,(A_2^- G_2^+ + A_3^- G_3^+).$$

For low-lying states the quasiclassical approximation (25) no longer holds in the well, but the continuation (36) retains the same form. It is necessary here to redefine the normalization factor C and the constants $A_2^\pm$. We consider first the case when $V(x)$ can be regarded as quadratic near the potential well, and $W(x)$ as linear. The variables in the initial Schrödinger equation are then exactly separated and it can be shown that the quantization rule (27), as well as expression (35) for $A_2^\pm$, remains valid also for small l. Continuing the solution obtained into region III (we leave out the intermediate algebra), we arrive again at the asymptote (36), where

$$C = \nu_l (2\tau_1)^{-1/2}, \quad \nu_l = \left[\frac{(2\pi)^{1/2}(l + 1/2)^{l+1/2}}{l!\, e^{l+1/2}}\right]^{1/2}. \tag{38}$$

The correction coefficient $\nu_l \to 1$ as $l \to \infty$ and turns out to be close to unity also at small l (namely, $\nu_0 = 1.037$, $\nu_1 = 1.014$, $\nu_2 = 1.008$), so that the asymptote (36) is fully satisfactory at all l.

Assume now that in a certain vicinity $|x - x_1| \lesssim x^*$ of the well the adiabatic approximation $\omega x^*/p_0 \ll 1$ is valid, with $p_0 x^* \gg 1$. In this case the adiabatic solution near the well in the below-barrier region can also be matched to the wave function Ψ_{2n}. We must now put in the asymptote (36)

$$A_2^+ = A_2^- = \langle\langle W\rangle\rangle / 2\omega^2, \quad C = \langle \Psi_l^0 | \Psi_l^0 \rangle^{-1/2},$$
$$\langle\langle W\rangle\rangle = C^2 \langle \Psi_l^0 | W | \Psi_l^0 \rangle, \tag{39}$$

where Ψ_l^0 is the eigenfunction of the particle in the well at $W = 0$. The effect of the interaction W in the well is taken into account here in first-order quantum perturbation theory. At large l the quantum mean value $\langle\langle W\rangle\rangle$ tends to the classical $\langle W\rangle$.

In the case of a free-free transition (Fig. 1b) we assume that the interaction $W(x)$ is negligibly small as $|x| \to \infty$. We may deal here, for example, with tunneling of an electron that interacts with molecular vibrations.[2,3] We stipulate a wave incident on the barrier from $x = -\infty$ and normalized to unity flux:

$$\Phi_n(x, u) \approx \left[2^n n! p \left(\frac{\pi}{\omega}\right)^{1/2}\right]^{-1/2} H_n(\omega^{1/2} u) \exp\left(i \int^x p\, dx\right). \tag{40}$$

The asymptote of the solution as $x \to +\infty$ is obtained in analogy with the procedure above, and takes the form (36) in which we must put

$$C = 1, \quad A_2^\pm = G_1^\pm, \tag{41}$$

and assume $x_0 \to -\infty$ in Eq. (22) for $G_1^\pm$. All the remaining quantities in (36) are determined from Eqs. (37), just as in the case of tunneling from a bound state.

The asymptote obtained yields explicit expressions for the tunneling probability and for the amplitudes of the inelastic tunneling transitions.

6. TUNNELING PROBABILITY

The tunneling probability or the flux of the wave-functions (36) through the straight line $x = \text{const}$ on the (x, u) plane is defined as

$$P_n = \int_{-\infty}^{\infty} du\, p\, |\Phi_{ln}|^2.$$

Simple calculations yield

$$P_n = C^2 \exp\left\{-2\int_{x_1}^{x_2} |p|\,dx + 2\alpha\right\} L_n^0(-\beta), \tag{42}$$

where $L_n^0(-\beta)$ is a Laguerre polynomial,

$$\alpha = \omega e^{-2\omega\tau_2} |A_2^- + G_2^-|^2 - \omega |A_2^-|^2 - 2\omega G_2^+ \,\mathrm{Re}\, A_2^- + \gamma_2,$$
$$\beta = 2\omega |2A_2^+ \,\mathrm{sh}\, \omega\tau_2 + G_2^+ e^{\omega\tau_2} - G_2^- e^{-\omega\tau_2}|^2. \tag{43}$$

The tunneling probability, as expected, is independent of the behavior of $V(x)$ and $W(x)$ at $x > x_2$.

At large $n \to \infty$ we can use for the Laguerre polynomial an asymptote that contains a modified Bessel function:

$$L_n^0(-\beta)\approx e^{-\beta/2}I_0(2(n+{}^1/_2)^{1/2}\beta^{1/2}). \quad (44)$$

The approximation (44) describes the oscillator quasiclassically, when its action on the particle can be replaced by an alternating increment to the potential $V(x)$:

$$\tilde{W}(x,t)=(n/2\omega)^{1/2}W(x)\cos\omega t.$$

This approximation was used in Ref. 18 to estimate the influence of local oscillations on electron tunneling.[3)] In the case considered there W was concentrated under the barrier, and hence $G_1^{\pm}=0$. In that case Eq. (42), with (44) and (41) taken into account and with terms α and β in the exponential, which are quadratic in the interaction, becomes equal to the result of Ref. 18.

The quantity 2α describes the influence of the zero-point oscillations on the tunneling probability, and can be either positive or negative. At $n>0$, the Laguerre polynomial describes the increase of the tunneling probability on account of the interaction with the excited oscillator: $L_n^0(-\beta)>1$, since $\beta>0$ according to (43).

Assuming that in the initial state the oscillator is in thermal equilibrium with the thermostat, we obtain for the tunneling probability

$$P_T=(1-e^{-\omega/T})\sum_{n=0}^{\infty}e^{-\omega n/T}P_n$$

$$=C^2\exp\left\{-2\int_{x_1}^{x_2}|p|\,dx+2\alpha+\beta(e^{\omega/T}-1)^{-1}\right\}. \quad (45)$$

Separating in the integral in (45) the energy shift ΔE_l due to the interaction in the well, by using the formula

$$p(E_0+\Delta E_l)\approx p(E_0)+\Delta E_l/p(E_0),$$

we find that the interaction increases the tunneling probability at a temperature higher than

$$T^*=\omega[\ln(1-\beta/2(\alpha+\Delta E_l\tau_2))]^{-1}, \quad (46)$$

and decreases it at a lower temperature. For the scattering problem (40) it is necessary to set ΔE_l in this formula equal to zero.

If the interaction with the oscillator is concentrated only in the region ahead of the barrier and can be neglected below the barrier, we get from (43)

$$\alpha=-\omega|A_2^-|^2(1-e^{-2\omega\tau_2}),\quad \beta=8\omega|A_2^+|^2\,\mathrm{sh}^2\,\omega\tau_2. \quad (47)$$

In the adiabatic limit, the value of A_2 for a well is given by (39). Equations (47) coincide then with the analogous expressions of Ref. 4, obtained with the aid of the adiabatic theory of multiphonon transitions. Interaction in the well can also lead to an energy shift that can be obtained from (28) and (30): $\Delta E_l=-\langle W\rangle^2/2\omega^2$ (for low-lying state it is necessary to replace the classical mean value $\langle W\rangle$ by the quantum mean $\langle\langle W\rangle\rangle$. Equation (46) takes in this case the form

$$T^*=\omega\left[\ln\frac{2\omega\tau_2-1+e^{2\omega\tau_2}}{2\omega\tau_2+1-e^{-2\omega\tau_2}}\right]^{-1} \quad (48)$$

and T^* turns out to be independent of the actual form of the interaction W. At $\omega\tau_2\gg1$ we obtain $T^*\approx(2\tau_2)^{-1}$, and at $\omega\tau_2\ll1$ this quantity is of the same order: $T^*\approx\tau_2^{-1}$. For the scattering problem (40) at $G_2^{\pm}=\gamma_2=0$ we obtain $T^*=(2\tau_2)^{-1}$.

If the interaction $W(x)$ is concentrated below the barrier ($A_2^{\pm}=0$), we get $\alpha>0$ [see (50)], which leads at any temperature to an increase of the tunneling probability. In the adiabatic limit $\omega\tau_2\ll1$ this follows from the fact that the particle tunnels through the oscillating barrier predominantly when the barrier is lowered. In the nonadiabatic situation the latter effect suppresses the interaction of the type considered. At zero temperature expression (45) goes over in this case into the result of Ref. 11. In fact, let us introduce the function

$$G_2^-(\tau)=-\frac{1}{2\omega}\int_0^{\tau}e^{\omega\tau}W\,d\tau,\quad \tau=\int_{x_1}^{x}\frac{dx}{|p|},\quad G_2^-(\tau_2)=G_2^-. \quad (49)$$

We then obtain from (43) and (23), integrating γ_2 by parts,

$$\alpha=\omega e^{-2\omega\tau_2}(G_2^-)^2+\gamma_2$$

$$=2\omega e^{-2\omega\tau_2}(G_2^-(\tau_2))^2+2\omega^2\int_0^{\tau_2}e^{-2\omega\tau}(G_2^-(\tau))^2\,d\tau, \quad (50)$$

which, notation aside, coincides at $W(x)=cx$ with Eq. (3.85) of Ref. 11.

If the motion in the below-barrier region is low ($\omega\tau_2\gg1$), the quantity

$$\gamma_2\approx\frac{1}{2\omega^2}\int_{x_1}^{x_2}W^2(x)\frac{dx}{|p|}\gg\omega(G_2^-)^2e^{-2\omega\tau_2}$$

yields, just as in (33), an effective increment $-W^2/(2\omega^2)$ to the potential.

7. AMPLITUDES OF TUNNELING TRANSITIONS

To calculated the amplitude of the tunneling transition we expand the asymptote (36) of the wave function in terms of a set of solutions, normalized to unity flux, of the Schrödinger equation in the region of large $x\to\infty$, where $W(x)$ is assumed equal to zero:

$$\Psi_n(x,u)\approx\sum_{k=0}^{\infty}A_{nk}\Psi_k^0(x,u), \quad (51)$$

$$\Psi_k^0(x,u)=\left[2^kk!p\left(\frac{\pi}{\omega}\right)^{1/2}\right]^{-1/2}H_k(u\omega^{1/2})\exp\left(i\int^x p\,dx-\frac{\omega u^2}{2}\right).$$

This yields for the transition probability

$$|A_{nk}|=\left|\int_{-\infty}^{\infty}du\,e^{-\omega u^2}\Psi_n{}^{\cdot}\Psi_k^0\right|$$

$$=C\left(\frac{k!}{n!}\right)^{1/2}|(2\omega)^{1/2}B_+|^{n-k}|L_k^{n-k}(2\omega B_+B_-)|$$

$$\times\exp\left(-\int_{x_1}^{x_2}|p|\,dx+\varphi\right),\quad n\geqslant k; \quad (52)$$

$$|A_{nk}|=\left|\int_{-\infty}^{\infty}du\,e^{-\omega u^2}\Psi_n{}^{\cdot}\Psi_k^0\right|$$

$$=C\left(\frac{n!}{k!}\right)^{1/2}|(2\omega)^{1/2}B_-|^{k-n}|L_n^{k-n}(2\omega B_+B_-)|$$
$$\times\exp\left(-\int_{x_1}^{x_2}|p|\,dx+\varphi\right),\quad n\leqslant k;$$

where B_+, B_-, and φ are defined in (37).

At low temperature, the particle interacts most strongly with the zero-point oscillations. Equation (52) takes then the simpler form

$$|A_{0k}|=C(k!)^{-1/2}|(2\omega)^{1/2}B_-|^k\exp\left(-\int_{x_1}^{x_2}|p|\,dx+\varphi\right).\quad(53)$$

Neglecting in the exponential the term φ which is quadratic in the interaction, Eq. (53) goes over at $k=1$ into the Fermi golden rule (in which it must be recognized that $\omega\ll p^2$). In this approximation, $|A_{01}|^2$ is obviously not equal to the inelastic part, quadratic in W, of the total tunneling probability, inasmuch as to determine P_0 at the accuracy indicated it is necessary to take into account also the part of $|A_{00}|$ that is quadratic in the interaction.

Assuming that the oscillator was in thermal equilibrium with the thermostat in the initial state, we obtain for the probability of a particle tunneling and acquiring m energy quanta

$$\mathscr{P}_m(E^0)=C\left|\frac{B_+}{B_-}\right|^m I_{|m|}\left(\frac{2\omega|B_+B_-|}{\mathrm{sh}(\omega/2T)}\right)\exp\left\{-\frac{\omega m}{2T}-\frac{4\omega\,\mathrm{Re}(B_+B_-)}{e^{\omega/T}-1}+2\varphi-2\int_{x_1}^{x_2}|p|\,dx\right\}.\quad(54)$$

If the adiabatic approximation $\omega\tau\ll1$ is valid, the main contribution (of zeroth order in $\omega\tau$) to the values $\mathscr{P}_m(E^0)$ is made by the well region, and we can put $B_+=B_-$. Summing $\mathscr{P}_m(E^0)$ over m, we find in this case that the tunneling probability becomes independent of temperature. The same result is obtained from (45), since α and β are quantities of first order in $\omega\tau$. The probabilities of the partial transitions (54) and of the amplitude (52) are thus determined in the adiabatic approximation by the interaction in the well. At the same time, an essential role in the determination of the total tunneling probability is played by the interaction in both the well and in the below-barrier region.

8. GENERALIZATION TO THE CASE OF INTERACTION WITH *N* OSCILLATORS

We consider now the case of interaction of a particle with N oscillators. Owing the factorization of the wave function (7) with respect to j, the tunneling probability and the transition amplitudes reduce to a product of N flux integrals of the same form and the overlap integrals from Secs. 6 and 7, while the contributions from the different oscillations are summed in the quantization rule. In analogy with (28) we obtain

$$\Delta E_l=\frac{1}{2\tau_1}\sum_{j=1}^{N}\int_{x_0}^{x_1}[(pf_{1x}^{(j)})^2-(\omega_jf_1^{(j)})^2]\frac{dx}{p}.\quad(55)$$

The generalization of the results (42) and (45) is given by

$$P_{n_1\ldots n_N}=C^2\exp\left(-2\int_{x_1}^{x_2}|p|\,dx\right)\prod_{j=1}^{N}e^{2\alpha_j}L_{n_j}^0(-\beta_j),\quad(56)$$

$$P_T=C^2\exp\left\{-2\int_{x_1}^{x_2}|p|\,dx+\sum_{j=1}^{N}[2\alpha_j+\beta_j(e^{\omega_j/T}-1)^{-1}]\right\},\quad(57)$$

and the expression for the amplitude of the $(n_1\ldots n_N)\to(k_1\ldots k_N)$ transitions at $k_j\geqslant n_j$ is

$$|A_{n_1\ldots n_N}^{k_1\ldots k_N}|=C\exp\left(-\int_{x_1}^{x_2}|p|\,dx\right)\prod_{j=1}^{N}\left(\frac{n_j}{k_j}\right)^{1/2}|(2\omega_j)^{1/2}B_-^{(j)}|^{k_j-n_j}\times e^{\varphi_j}|L_{n_j}^{k_j-n_j}(2\omega B_+^{(j)}B_-^{(j)})|.\quad(58)$$

If $k_j\leqslant n_j$ for some arbitrary j, we must replace in the product (58) $B_\pm^{(j)}$ by $B_\mp^{(j)}$ and interchange k_j and n_j.

In the formulas presented, the quantities $\alpha_j,\beta_j,B_\pm^{(j)}$, and φ_j are defined in the same manner as $\alpha,\beta,B_\pm$, and φ, with ω and $W(x)$ replaced by ω_j and $W_j(x)$. The other equations given above are generalized to the case of N oscillators in the same manner. If the frequency spectrum is continuous, the sums in (55) and (57) should be replaced by the corresponding integrals.

If the interaction is concentrated in the below-barrier region, Eq. (57) coincides at zero temperature with that obtained in Ref. 11, where it is shown, in particular, that the contribution, calculated with the aid of (57), of the dissipative terms agrees with the result of Ref. 12.

9. GENERALIZATION OF THE THREE-DIMENSIONAL CASE

When a three-dimensional particle tunnels from a low-lying state in a sufficiently weak external field, the flux of its wave function is localized in the below-barrier region in the vicinity of a certain trajectory. Localization occurs also for the most easily ionized excited states that stretch along the field. This circumstance allows us to generalize our results to the three-dimensional case.

We assume for simplicity that the potential, as a function of the particle coordinates, has a symmetry axis with which the indicated trajectory coincides. We now put in the initial Schrödinger equation (1) $\mathbf{x}=(z,\rho\cos\varphi,\rho\sin\varphi)$, where z, ρ, and φ are cylindrical coordinates.

Equation (1) can be solved in the vicinity of the adiabatic approximation (which is valid for an electron), or else by assuming the potential to be a quadratic function of the coordinates (for a heavy particle). Far from the well, in the below-barrier region, where the wave-function flux is localized, we expand the potential $V(\mathbf{x})=V(z,\rho)$ in the vicinity of the z axis, accurate to terms quadratic in ρ, and neglect the dependence of W_j on ρ. As a result, the variable ρ becomes analogous to the vibrational variables u_j. The frequency $(V_{\rho\rho}(z,0))^{1/2}$ that corresponds to the coordinate ρ is now a function of z.

If the dependence of the potential on the coordinates is quasiclassical, and also if the wave function is localized in the vicinity of the z axis and under the assumptions (17) and (18) are valid, the variables in Eq. (1) can be separated by the procedure described above. The equation is reduced as a result to N exactly solvable two-dimensional equations in z and u_j, and to one equation in z and ρ. Matching the solution obtained to the tail of the bound state, we obtain the wave function of the problem. A similar problem concerning elastic decay of a bound state in an external field was solved in Refs. 20 and 21, and we therefore leave out the details here.

Factoring the solution of the Schrödinger equation with respect to the parameters ρ and u_j changes the inelastic-ionization probability to the factorized form

$$P_T=P^{(0)}(E_0+\Delta E_l)\exp\left\{\sum_{j=1}^{N}\left[2\alpha_j+\beta_j(e^{\omega_j/T}-1)^{-1}\right]\right\},$$

$$P^{(0)}(E_0+\Delta E_l)\approx P^{(0)}(E_0)\exp(2\Delta E_l\tau_2). \qquad (59)$$

Here $P^{(0)}(E)$ is the probability of elastic tunneling ionization from the level E_0 at $W_j\equiv 0$, ΔE_l is the shift of the level by the interaction, and the quantities τ_2, α_j, and β_j are calculated on the z axis by replacing the turning points x_k by z_k. The value of $P^{(0)}$ is known for a small-radius center[22] and for a Coulomb center[23] in a constant electric field. The results of the cited papers were generalized in Refs. 20 and 21 to include arbitrary axisymmetric potentials. Equation (59) permits an estimate of the contribution made to the tunneling probability both by size effects and by effects of interaction with the oscillations.

10. PARTICULAR CASES AND APPLICATIONS

1. We consider first the temperature dependence of the tunneling probability of a one-dimensional bound particle interacting with one-dimensional phonons:

$$W_k(x)=c_k\sin kx,\quad k<0;$$
$$W_k(x)=c_k\cos kx,\quad k>0,$$

after making the substitution

$$\sum_j\to\frac{1}{2\pi}\int dk$$

in (55) and (57).

We assume the potential near the well to be quadratic, $V(x)=\omega_0^2x^2/2$, and the adiabatic approximation $\omega_k\tau_1\ll 1$ to be valid. For the l th state we obtain then

$$\langle\langle W_k\rangle\rangle=c_k\exp\left(-\frac{k^2}{4\omega_0}\right)L_{l-1}^{0}\left(\frac{k^2}{2\omega_0}\right),$$

$$\langle W_k\rangle=c_kJ_0\left(\frac{(2l-1)^{1/2}k}{\omega_0^{1/2}}\right),\quad k>0;$$

$$\langle\langle W_k\rangle\rangle=\langle W_k\rangle=0,\quad k<0. \qquad (60)$$

It follows from (60) that the quantum mean value goes over into the classical one at $l\gg 1$ and $k^2\ll\omega_0 l$. These inequalities are the conditions under which the employed quasiclassical approximation is valid for this region. The second inequality states that the phonon momentum must be small compared with the characteristic momentum of the particle. In the below-barrier region, in analogy with the classically allowed one, the foregoing conditons take the form $p_0x_0\gg 1$, $k\ll p_0$ where p_0 and x_0 are the characteristic momentum and distance below the barrier.

Assume for simplicity that the potential $V(x)$ is constant in the below-barrier region (except in the vicinities of the turning points, whose contributions to the exponential of (57) can be neglected). Assume in addition that the adiabatic approximation is likewise valid under the barrier. Since $\omega_k\tau_2\ll 1$, to determine the temperature dependence of the tunneling probability it suffices to calculate $\beta_k+\beta_{-k}$. Under the assumptions made we get

$$\beta_k+\beta_{-k}=\frac{2}{\omega_kp^2}\Big[a^2\langle\langle W_k\rangle\rangle^2-\frac{2a}{k}(\sin kx_2-\sin kx_1)\langle\langle W_k\rangle\rangle+\frac{2}{k^2}(1-\cos ka)\Big]. \qquad (61)$$

Here $a=x_2-x_1$ is the barrier width, p is the value of the momentum below the barrier, and $x_1=(2l-1)^{1/2}/\omega_0^{1/2}$. At small k, just as at large ones, the expression in the square brackets of (61) tends to zero. For small k this follows from the fact that the interaction independent of x cannot have a tunneling probability.

For acoustic phonons at low temperature, the dependence of the tunneling probability (57) on T determines the behavior of expression (61) at small k. Let $\omega_k=sk$ and $c_k\approx ck^q$ as $k\to 0$. Under the conditions $lk^2/\omega_0\ll 1$ and $kx_1\ll kx_2\ll 1$, expanding (61) in terms of these small parameters, we then obtain as $k\to 0$

$$\beta_k+\beta_{-k}\approx(c^2a^4/2sp)k^{2q+1}. \qquad (62)$$

The approximation (62) can be used at a temperature much lower than s/a or $s(\omega_0/l)^{1/2}$. This condition is quite restricted for light particles. For a heavy particle, assuming that $\omega_0\sim s\sim 0.003$, $\alpha\sim 1$, $l=1$ and that the particle mass is $M\sim 10^4$ (atomic units), we obtain $T\ll 100$ K. Integrating with respect to k, we arrive at the result

$$P_T=P_0\exp\left\{\frac{\Gamma(2q+2)\zeta(2q+2)c^2a^4}{4\pi ps^{2q+3}}T^{2q+2}\right\}, \qquad (63)$$

where P_0 is the tunneling probability at zero temperature, Γ is the gamma function, and ζ is the Riemann zeta function. The temperature dependence is determined in this case by the interaction in the below-barrier region at $k<0$.

2. Equation (63) can be generalized to include a three-dimensional case when

$$W_{\mathbf{k}}(\mathbf{x})=c_{\mathbf{k}}\sin\mathbf{kx},\quad k_z<0;$$
$$W_{\mathbf{k}}(\mathbf{x})=c_{\mathbf{k}}\cos\mathbf{kx},\quad k_z>0, \qquad (64)$$

$\omega_{\mathbf{k}}=s|\mathbf{k}|$ and $c_{\mathbf{k}}\approx c|\mathbf{k}|^q$ as $|\mathbf{k}|\to 0$ and the momentum p is constant in the below-barrier near the z axis, which coincides with the most probable tunneling path. At $|\mathbf{k}|\ll p$ the main contribution to the wave-function flux under the barrier is made by the interaction on the z axis, and the conclusions of the preceding section are valid. In place of (62) we now obtain $|\mathbf{k}|\to 0$

$$\beta_{\mathbf{k}}+\beta_{-\mathbf{k}}\approx(c^2a^4/2sp)|\mathbf{k}|^{2q-1}k_z^2.$$

As a result we get

$$P_T=P_0\exp\left\{\frac{\Gamma(2q+4)\zeta(2q+4)c^2a^4}{24\pi^2ps^{2q+5}}T^{2q+4}\right\}. \qquad (65)$$

Here a is the width of the barrier along the z axis.

3. Assume that the main contribution to the sum over **k** is made by phonons with $|\mathbf{k}|a \gg 1$ (for light particles). If follows then from (61) that the decisive contribution to the quantity $\beta_{\mathbf{k}} + \beta_{-\mathbf{k}}$ is made by the region of the well. In the general case, if the interaction with the oscillations is significant only in the well region, where the adiabatic approximation is valid, we obtain from (59)

$$P_T = P^{(0)}(E_0)\exp\left\{\sum_{j=1}^{N}\frac{\langle\!\langle W_j\rangle\!\rangle^2}{\omega_j^3}\left[-\omega_j\tau_2 - \frac{1}{2}(1-\exp(-2\omega_j\tau_2)) + 2\,\mathrm{sh}^2(\omega_j\tau_2)\left(\exp\frac{\omega_j}{T}-1\right)^{-1}\right]\right\},\quad \langle\!\langle W_j\rangle\!\rangle = \langle\Psi_{0l}|W_j|\Psi_{0l}\rangle, \tag{66}$$

where Ψ_{0l} is an eigenfunction the particle in the well normalized at $W_j \equiv 0$. Equations (66) can be used for interactions both with phonons and with localized oscillations. A similar result was obtained in Ref. 4 within the framework of the adiabatic theory of multiphoton transitions. Account is taken also in (66) of the shift ΔE_l of the particle energy in the well, due to the interaction that leads in the adiabatic limit $\omega_j\tau_2 \to 0$, together with the remaining terms in the exponential of (66), to a decrease proportional to $(\omega_j\tau_2)^2$ of the argument of the exponential.

4. We consider now the interaction of an electron with local oscillations of an adsorbed molecule in the case of field emission from a metal into a vacuum[3] [a metal-barrier-metal system is treated similarly[2]]. Calculations of the amplitude of the electron inelastic transition, in the zeroth order in ω/p_0^2 (this corresponds to the adiabatic approximation) is inaccurate for broad barriers.[14] The approximation employed, by taking into account the change of the action of the particle in first order in ω/p_0^2, describes quite adequately the inelastic transition in this problem. The emission probability of an electron of energy E, interacting with a vibrational mode ω, is given by the expression[3,24]

$$P(E) = \sum_{m=-\infty}^{\infty}\mathscr{P}_m(E-m\omega)f(E-m\omega), \tag{67}$$

where $\mathscr{P}_m$ is determined by Eqs. (54) and (41), and f is the Fermi distribution function of the metal electrons. Taking into account the most significant elastic and single-phonon transitions, we get from (67) and (54)

$$P(E) = \exp\left\{-2\int_{x_1}^{x_2}|2(E-V(x))|^{1/2}\right\}$$
$$\times\{[1+2\varphi-4\omega\,\mathrm{Re}(B_+B_-)N(\omega)]f(E)$$
$$+2\omega e^{-2\omega\tau_2}|B_+|^2N(\omega)f(E-\omega)$$
$$+2\omega e^{2\omega\tau_2}(N(\omega)+1)f(E+\omega)\},$$
$$f(E) = (e^{(E-E_f)/T}-1)^{-1},\quad N(\omega) = (e^{\omega/T}-1)^{-1}. \tag{68}$$

In the adiabatic approximation it follows from (37) and (41) that $B_+ = -B_-$ and $\varphi = \mathrm{Re}B_+^2$. If the interaction is concentrated below the barrier, B_+ and B_- are real. It is easy to generalize Eqs. (67) and (68) to include interactions with several vibrational modes.

We note in conclusion that the method proposed permits a similar treatment of problems with a larger number of turning points. It is of interest, for example, to develop a theory for a bound-bound transition[10] and for resonant tunneling through a two-hump potential.[25–27]

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[1)]We use a system of units in which Planck's constant and the particle mass are equal to unity.

[2)]In the expression for Ψ_{n2} we must take p to mean $i|p|$.

[3)]A similar problem was solved in Ref. 19.

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