Fermi-liquid approach to the anomalous elasticity of ferromagnetic metals

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The principles of a theory of magnetoelastic properties of ferromagnets with mobile collectivized electrons are formulated. The theory is based on allowance for the exchange interaction of the electrons and for the electron interaction with the lattice. A theory free of the assumption that the exchange energy depends on the volume is developed for the temperature anomaly of the elastic moduli. It is shown that, in agreement with experiment, an anomalous change of the elastic moduli can occur both near the Curie point and relatively far from it. The conditions under which the magnetic induction determines the change of the elastic moduli are found.

1. INTRODUCTION

The theory of magnetoelastic properties of ferromagnetic metals has long attracted much attention, mostly in connection with engineering applications of these properties.[1,2] Particular attention is paid lately to the development of a magnetoelastic-phenomena theory based on the model of collectivized electrons (see the review by Wohlfarth[3]). It has become clear that use can be made of the earlier[4–6] successful general approach to the elasticity of normal metals, which points, as follows from Refs. 7–11, to a dependence of the elasticity of a metal on its magnetization.

We generalize here the approach of Refs. 4 and 5 to include the case of ferromagnetic metals. The principles of the magnetoelasticity theory set forth here shed light on the nature of the magnetoelastic properties of metals with collectivized electrons, and show how the magnetoelastic parameters are determined by the exchange interaction of the lattice with the electrons. In Sec. 1 we formulate the general premises of the theory of dynamic elastic moduli of a ferromagnet with mobile electrons. In Sec. 2 is discussed the magnetodeformation connection between the spin and sound waves. The third section is devoted to an analysis of the elastic moduli. With a weak ferromagnet as the example, we consider explicit relations between the elastic moduli and the magnetic induction, the temperature, and the magnetization. These relations, containing no magnetoelastic constants (cf. Ref. 12), correspond to the Invar anomaly (see, e.g., Ref. 13). We show here that the anomaly of the elastic moduli can set in near temperatures that differ substantially from the Curie temperature. The conditions are found under which the induction in a ferromagnet determines the values and temperature dependences of the elastic moduli. The Appendix contains expressions for the coefficients that govern the excitation of sound by a high-frequency electromagnetic field (cf. Ref. 14), and other electromagnetic effects.

2. EQUATION FOR SOUND OSCILLATIONS IN A FERROMAGNET WITH MOBILE ELECTRONS, AND THE DYNAMIC ELASTIC MODULI

Our purpose being formulation of an elasticity theory for a ferromagnetic metal in which the states of the mobile electrons are changed by lattice deformations, we use the ideas of Ref. 4, in which the corresponding theory was developed for normal metals. In contrast to normal metals, we take consistent account of the effect of magnetization on the susceptibilities that determine the dynamic elastic moduli.[7–11] The exposition in the present section follows a generalization of the premises of Ref. 5.

In a ferromagnet in a nonequilibrium state corresponding to propagation of a sound wave having a frequency ω and a wave vector $\mathbf{k}$, the nonequilibrium increment to the electron-density matrix is given by

$$\delta f(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k})=\frac{f_F[\varepsilon(\mathbf{p}-\hbar\mathbf{k},\sigma')]-f_F[\varepsilon(\mathbf{p},\sigma)]}{\hbar\omega+\varepsilon(\mathbf{p}-\hbar\mathbf{k},\sigma')-\varepsilon(\mathbf{p},\sigma)+i0}$$
$$\times W(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k})\equiv\Gamma(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k})W(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k}).\qquad(2.1)$$

Here $f_F(\varepsilon)$ is the Fermi distribution function of the electrons, $\varepsilon(\mathbf{p},\sigma)=\varepsilon(\mathbf{p})-\sigma b$, where $\sigma=\pm1$ and b is the energy of the magnetic splitting of an electron band with dispersion $\varepsilon(\mathbf{p})$.[15] In contrast to Ref. 5, we neglect the effect of the magnetic field on the orbital motion of the electrons; this is reasonable at not too low temperatures and for metals that are not very pure.

We examine now the form of the energy W corresponding to perturbation of the electron state. We express the energy in the form

$$W(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k})=W_e(\sigma,\sigma',\omega,\mathbf{k})+W_c(\sigma,\sigma',\omega,\mathbf{k})$$
$$+W_d(\mathbf{p},\sigma,\sigma',\omega,\mathbf{k})+W_{ds}(\sigma,\sigma',\omega,\mathbf{k}).\qquad(2.2)$$

The first term

$$W_e(\sigma,\sigma',\omega,\mathbf{k})$$
$$=\delta_{\sigma\sigma'}\left[e\Phi(\omega,\mathbf{k})-\frac{e}{c}\mathbf{v}\mathbf{A}(\omega,\mathbf{k})\right]-2\beta\hat{\mathbf{s}}_{\sigma\sigma'}\mathbf{b}(\omega,\mathbf{k})\qquad(2.3)$$

is here the electron energy in the self-consistent electromagnetic field that accompanies the sound-wave propagation. $\Phi(\omega,\mathbf{k})$ and $\mathbf{A}(\omega,\mathbf{k})$ are the scalar and vector potentials, $\mathbf{b}(\omega,\mathbf{k})$ is the nonequilibrium electromagnetic induction, and $\hat{\mathbf{s}}$ and β are the spin operator and the magnetic moment of the electron. Using the gauge condition $\mathbf{k}\cdot\mathbf{A}(\omega,\mathbf{k})=0$, we define the scalar potential by the Poisson equation

$$k^2\Phi(\omega,\mathbf{k})=4\pi[en(\omega,\mathbf{k})+q(\omega,\mathbf{k})].\qquad(2.4)$$

The nonequilibrium electron-charge density is accordingly

$$n(\omega,\mathbf{k})=\sum_{\sigma=\pm}\int d\tau\delta f(\mathbf{p},\sigma,\sigma,\omega,\mathbf{k}),$$

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where $d\tau = d\mathbf{p}(2\pi\hbar)^{-3}$. For the lattice nonequilibrium charge density we have[4]

$$q(\omega, \mathbf{k}) = -Qi\mathbf{k}\mathbf{u}(\omega, \mathbf{k}),$$

where $\mathbf{u}(\omega,\mathbf{k})$ is the Fourier component of the lattice local displacement, and Q is the charge per unit volume of the lattice.

We use for the correlation energy of the electron-electron interaction the very simple approximation

$$W_e(\sigma, \sigma', \omega, \mathbf{k}) = \delta_{\sigma\sigma'}\varphi n(\omega, \mathbf{k}) + 2\psi \hat{\mathbf{s}}_{\sigma\sigma'}\mathbf{s}(\omega, \mathbf{k}), \qquad (2.5)$$

where $\varphi \equiv A_0/2\nu$ and $\psi \equiv B_0/2\nu$ are the Fermi-liquid-interaction constants, ν is the density of the electron states on the Fermi level ε_F in the paramagnetic phase, and

$$\mathbf{s}(\omega, \mathbf{k}) = 2\sum_{\sigma,\sigma'}\int d\tau \hat{\mathbf{s}}_{\sigma\sigma'}\delta f(\mathbf{p}, \sigma', \sigma, \omega, \mathbf{k})$$

is the nonequilibrium spin density of the electrons.

Even Eqs. (2.3)–(2.5) alone provide a definite description, corresponding to the so-called "jellium" model, of the interaction between the electrons in the lattice. This model is refined, first, by taking into account the deformation interaction

$$W_d(\mathbf{p}, \sigma, \sigma', \omega, \mathbf{k}) = \delta_{\sigma\sigma'}\Lambda_{ij}(\mathbf{p})ik_j u_i(\omega, \mathbf{k}), \qquad (2.6)$$

where $\Lambda_{ij}(\mathbf{p})$ is the symmetric tensor of the deformation potential,[16] and, second, by taking into account the magnetodeformation potential $\Lambda_{ij,kl}(\mathbf{p},\mathbf{p}')$ of the electrons,[7] for which we use in the present paper a very simple approximation that does not depend on the electron quasimomenta:

$$W_{ds}(\sigma, \sigma', \omega, \mathbf{k}) = 2(\hat{s}_l)_{\sigma\sigma'}S_k\Lambda_{ij,kl}ik_j u_i(\omega, \mathbf{k}), \qquad (2.7)$$

where $\mathbf{S} = \mathbf{M}/\beta$ is the component of the doubled equilibrium spin density determined by the equilibrium magnetization and oriented along the z axis. Equations (2.2), (2.3), and (2.5)–(2.7) yield the following set of equations:

$$n(\omega, \mathbf{k}) = [e\Phi(\omega, \mathbf{k}) + \varphi n(\omega, \mathbf{k})]\langle\Gamma(+,+) + \Gamma(-,-)\rangle - \frac{e}{c}$$
$$\times \mathbf{A}(\omega, \mathbf{k})\langle \mathbf{v}[\Gamma(+,+) + \Gamma(-,-)]\rangle + ik_j u_i(\omega, \mathbf{k})\langle\Lambda_{ij}[\Gamma(+,+)$$
$$+\Gamma(-,-)]\rangle + [-\beta b^z(\omega, \mathbf{k}) + \psi s^z(\omega, \mathbf{k}) + ik_j u_i(\omega, \mathbf{k})S_m$$
$$\times \Lambda_{ij,mz}]\langle\Gamma(+,+) - \Gamma(-,-)\rangle, \qquad (2.8)$$

$$s^z(\omega, \mathbf{k}) = [1 - \psi\langle\Gamma(+,+) + \Gamma(-,-)\rangle]^{-1}$$
$$\times\Big\{[e\Phi(\omega, \mathbf{k}) + \varphi n(\omega, \mathbf{k})]\langle\Gamma(+,+)$$
$$-\Gamma(-,-)\rangle - \frac{e}{c}\mathbf{A}(\omega, \mathbf{k})\langle\mathbf{v}[\Gamma(+,+) - \Gamma(-,-)]\rangle$$
$$+ik_j u_i(\omega, \mathbf{k})\langle\Lambda_{ij}$$
$$\times[\Gamma(+,+) - \Gamma(-,-)]\rangle + [-\beta b^z(\omega, \mathbf{k}) + ik_j u_i(\omega, \mathbf{k})S_m\Lambda_{ij,mz}] \qquad (2.9)$$
$$\times\langle\Gamma(+,+) + \Gamma(-,-)\rangle\Big\},$$

$$s^\pm(\omega, \mathbf{k}) = 2[1 - 2\psi\langle\Gamma(\mp, \pm)\rangle]^{-1}\{-\beta b^\pm(\omega, \mathbf{k})\langle\Gamma(\mp, \pm)\rangle$$
$$+ik_j u_i(\omega, \mathbf{k})S_m\Lambda_{ij,m\pm}\langle\Gamma(\mp, \pm)\rangle\}. \qquad (2.10)$$

Here

$$s^\pm = s^x \pm is^y, \quad \Lambda_{ij,m\pm} = \Lambda_{ij,mx} \pm i\Lambda_{ij,my},$$

$$\langle\Gamma(\sigma, \sigma')V\rangle \equiv \int d\tau\Gamma(\mathbf{p}, \sigma, \sigma', \omega, \mathbf{k})V(\mathbf{p}), \quad \mathbf{v} = \partial\varepsilon/\partial\mathbf{p}.$$

The perturbed state of the lattice is described by the equation of motion (cf. Ref. 4)

$$-\omega^2\rho_m u_i(\omega, \mathbf{k}) + \lambda^{(0)}_{ij,kl}k_j k_l u_k(\omega, \mathbf{k}) = F_i(\omega, \mathbf{k}), \qquad (2.11)$$

where ρ_m is the lattice mass density, $\lambda^{(0)}$ is the ion contribution to the elastic-moduli tensor, and $\mathbf{F}$ is the force exerted on the lattice by the electrons. An explicit equation for this force can be expressed, in accord with (2.2), as the sum

$$\mathbf{F}(\omega, \mathbf{k}) = \mathbf{F}^{(e)}(\omega, \mathbf{k}) + \mathbf{F}^{(d)}(\omega, \mathbf{k}) + \mathbf{F}^{(ds)}(\omega, \mathbf{k}). \qquad (2.12)$$

The first term corresponds here to the Lorentz force

$$\mathbf{F}^{(e)}(\omega, \mathbf{k}) = -iQ\{\mathbf{k}\Phi(\omega, \mathbf{k}) - (\omega/c)\mathbf{A}(\omega, \mathbf{k}) + (\omega/c)[\mathbf{u}(\omega, \mathbf{k})\times\mathbf{B}]\},$$

where $\mathbf{B}$ is the equilibrium magnetic induction. The remaining two terms of (2.12) corresponds to the deformation interaction

$$F_i^{(d)}(\omega, \mathbf{k}) = ik_j\sum_{\sigma=\pm}\int d\tau\Lambda_{ij}(\mathbf{p})\delta f(\mathbf{p}, \sigma, \sigma, \omega, \mathbf{k})$$

and to the magnetodeformation interaction

$$F_i^{(ds)}(\omega, \mathbf{k}) = ik_j\Lambda_{ij,kl}s_l(\omega, \mathbf{k})S_k.$$

Equations (2.8)–(2.10) allow us to eliminate from (2.12) the electronic nonequilibrium quantities $n(\omega,\mathbf{k})$ and $\mathbf{s}(\omega,\mathbf{k})$. Equation (2.12) takes therefore the form

$$-\omega^2\rho_m u_i(\omega, \mathbf{k}) + \lambda_{ij,kl}(\omega, \mathbf{k})k_j k_l u_k(\omega, \mathbf{k})$$
$$+iQ(\omega/c)[\mathbf{u}(\omega, \mathbf{k})\times\mathbf{B}]_i = -i\beta k_l[b^+(\omega, \mathbf{k})R_{il}^+ + b^-(\omega, \mathbf{k})R_{il}^-$$
$$+b^z(\omega, \mathbf{k})R_{il}^z] + (i/c)P_{it}A_t(\omega, \mathbf{k}). \qquad (2.13)$$

We then obtain for the tensor of the dynamic elastic moduli

$$\lambda_{ij,kl}(\omega, \mathbf{k}) = \lambda^{(0)}_{ij,kl} + \lambda^{\parallel}_{ij,kl}(\omega, \mathbf{k}) + \lambda^{\perp}_{ij,kl}(\omega, \mathbf{k}), \qquad (2.14)$$

$$\lambda^{\parallel}_{ij,kl}(\omega, \mathbf{k}) = \langle[\Gamma(+,+) + \Gamma(-,-)]\Lambda_{ij}\Lambda_{kl}\rangle$$
$$+\frac{M}{\beta}\langle[\Gamma(+,+) - \Gamma(-,-)]$$
$$\times[\Lambda_{ij}\Lambda_{kl,zz} + \Lambda_{kl}\Lambda_{ij,zz}]\rangle$$
$$+\left(\frac{M}{\beta}\right)^2\langle\Gamma(+,+) + \Gamma(-,-)\rangle\Lambda_{ij,zz}\Lambda_{kl,zz}$$
$$+\frac{\varphi\mathfrak{A}_{ij}\mathfrak{A}_{kl}}{1+\varphi\chi(\omega, \mathbf{k})} + \frac{\psi(a_{ij}^+ - a_{ij}^-)(a_{kl}^+ - a_{kl}^-)}{1 - \psi\langle\Gamma(+,+) + \Gamma(-,-)\rangle}$$
$$+\frac{4\pi e^2[1 + \varphi\chi(\omega, \mathbf{k})]}{k^2\varepsilon_e(\omega, \mathbf{k})} \qquad (2.15)$$
$$\times\left[\frac{Q}{e}\delta_{ij} - \frac{\mathfrak{A}_{ij}}{1+\varphi\chi(\omega, \mathbf{k})}\right]\left[\frac{Q}{e}\delta_{kl} - \frac{\mathfrak{A}_{kl}}{1+\chi\varphi(\omega, \mathbf{k})}\right],$$

$$\lambda^{\perp}_{ij,kl}(\omega, \mathbf{k}) = \left(\frac{M}{\beta}\right)^2\Big\{(\Lambda_{ij,zx} + i\Lambda_{ij,zy})(\Lambda_{kl,zx} - i\Lambda_{kl,zy}) \qquad (2.16)$$
$$\times\frac{\langle\Gamma(+,-)\rangle}{1 - 2\psi\langle\Gamma(+,-)\rangle}$$

$$+(\Lambda_{ij,zx}-i\Lambda_{ij,zy})(\Lambda_{kl,zx}+i\Lambda_{kl,zy})\frac{\langle\Gamma(-,+)\rangle}{1-2\psi\langle\Gamma(-,+)\rangle}\bigg\}.$$

We have used here the notation

$$a_{ij}^{\sigma}=\langle\Gamma(\sigma,\sigma)[\Lambda_{ij}+\sigma(M/\beta)\Lambda_{ij,zz}]\rangle,$$

$$\mathfrak{A}_{ij}=[1-\psi\langle\Gamma(+,+)+\Gamma(-,-)\rangle]^{-1}[a_{ij}^{+}(1-2\psi\langle\Gamma(-,-)\rangle)+a_{ij}^{-}(1-2\psi\langle\Gamma(+,+)\rangle)],$$

$$\varepsilon_e(\omega,\mathbf{k})=1+[(4\pi e^2/k^2)+\varphi]\chi(\omega,\mathbf{k}),$$

$$\chi(\omega,\mathbf{k})=[4\psi\langle\Gamma(+,+)\rangle\langle\Gamma(-,-)\rangle-\langle\Gamma(+,+)+\Gamma(-,-)\rangle]\times[1-\psi\langle\Gamma(+,+)+\Gamma(-,-)\rangle]^{-1},$$

where ε_e is the longitudinal dielectric constant of the electrons, and χ is the electron polarizability and takes into account the exchange interaction. The right-hand side of (2.13) determines the connection between the acoustic field and the solenoidal electromagnetic field (cf. Ref. 14). The corresponding coupling coefficients are given in the Appendix.

In the next two sections we obtain relatively simple corollaries of Eqs. (2.15) and (2.16).

3. MAGNETODEFORMATION COUPLING OF SPIN AND SOUND WAVES

We shall be interested hereafter in long-wave perturbations, when $\hbar k$ is small compared with the difference between the Fermi momenta of electrons corresponding to two spin projections ($\sigma=\pm$). The contribution $\lambda^{\perp}$ to the dynamic elastic moduli, which determines the coupling of the spin and sound waves, can then be simplified. We use from now on for a ferromagnet with mobile electrons the equation of state

$$2b\equiv\hbar\Omega_0(B,T)=\hbar\Omega_B-(2\psi/\beta)M(B,T),\qquad(3.1)$$

where

$$\hbar\Omega_B=2\beta B,\qquad M(B,T)=\beta\int d\tau(n^{+}-n^{-}),\qquad n^{\sigma}=f_F[\varepsilon(\mathbf{p},\sigma)].$$

All this enables us to write down, recognizing that $\omega\ll\Omega_0$, the approximate equation

$$\langle\Gamma(\sigma,-\sigma)\rangle=-\frac{M(B,T)}{\hbar\beta[\Omega_0(B,T)-\sigma\omega]}-\frac{\alpha_{ij}k_ik_j}{2\psi\Omega_0(B,T)},\qquad(3.2)$$

where

$$\alpha_{ij}=\frac{\psi}{\Omega_0(B,T)}\int d\tau v_iv_j\left\{\frac{\partial(n^{+}+n^{-})}{\partial\varepsilon}+2\frac{n^{+}-n^{-}}{\hbar\Omega_0(B,T)}\right\}.\qquad(3.3)$$

As a result we obtain from (2.16)

$$\lambda_{ij,kl}^{\perp}(\omega,\mathbf{k})=-\frac{1}{\hbar}\left(\frac{M}{\beta}\right)^3\left\{\frac{(\Lambda_{ij,zx}+i\Lambda_{ij,zy})(\Lambda_{kl,zx}-i\Lambda_{kl,zy})}{\omega_M(B,\mathbf{k})-\omega}+\frac{(\Lambda_{ij,zx}-i\Lambda_{ij,zy})(\Lambda_{kl,zx}+i\Lambda_{kl,zy})}{\omega_M(B,\mathbf{k})+\omega}\right\}.\qquad(3.4)$$

Here $\omega_M(B,\mathbf{k})=\Omega_B+\alpha_{ij}k_ik_j$ is the spin-wave frequency. In the vicinity of the magnetoacoustic resonance, allowance for the spin-wave dissipation (which we neglect in this section) is known to lead to an increase of the sound absorption.[1,2,7] We confine ourselves to Eq. (3.4) which demonstrates the role of the magnetodeformation potential Λ_{ijkl} in the coupling of the spin and sound waves. In the limit as $\omega\to0$ and $k\to0$, Eq. (3.4) yields for the elastic moduli

$$\lambda_{ij,kl}^{\perp}=-\frac{1}{\beta B}\left(\frac{M}{\beta}\right)^3(\Lambda_{ij,zx}\Lambda_{kl,zx}+\Lambda_{ij,zy}\Lambda_{kl,zy}).\qquad(3.5)$$

In the particular case when the magnetic field H is zero and $B=4\pi M$, Eq. (3.5) corresponds to a quadratic dependence on the magnetization $M(B,T)$ (cf. Ref. 1). We point out finally that if $\omega\gg\omega_M(\mathbf{k})$, i.e., the sound frequency exceeds that of the spin waves, we have

$$\lambda_{ij,kl}^{\perp}=-\frac{2}{\hbar}\left(\frac{M}{\beta}\right)^3\left\{\frac{i}{\omega}(\Lambda_{ij,zx}\Lambda_{kl,zy}-\Lambda_{ij,zy}\Lambda_{kl,zx})-\frac{\omega_M(\mathbf{k})}{\omega^2}(\Lambda_{ij,zx}\Lambda_{kl,zx}+\Lambda_{ij,zy}\Lambda_{kl,zy})\right\}.\qquad(3.6)$$

The term with ω^{-1} in the right-hand side of (3.6) describes in this case the rotation of the polarization plane of the sound.[17] We note finally that no account is taken in the equations of the present section of the effects of the anisotropic (Ω_A, Ref. 1) and of the magnetoelastic (Ω_{el}, Ref. 18) gaps of the magnon spectrum. This is equivalent to the assumption that the principal effect that causes the onset of a finite magnon frequency at $k=0$ is the magnetic induction of the ferromagnet ($\Omega_B\gg\Omega_A,\Omega_{el}$).

To conclude this section, we emphasize that in our model the coupling of the spin and sound waves is determined by a phenomenological magnetodeformation potential.

4. TEMPERATURE DEPENDENCE OF THE ELASTIC MODULI OF A WEAK FERROMAGNET

Whereas the magnetodeformation interaction that couples the spin and sound waves is a comparatively weak relativistic effect, the contribution (2.15) to the dynamic elastic moduli, on the contrary, is determined primarily by the nonrelativistic Coulomb and deformation interaction. We consider below this contribution in the limit $\omega=0$ and $\mathbf{k}=0$, when $\omega/k=v_s(\mathbf{k})$ is the speed of sound in the ferromagnet. We then obtain from (2.15)

$$\lambda_{ij,kl}^{\parallel}=\varphi\frac{Q^2}{e^2}\delta_{ij}\delta_{kl}+\langle(F_+'+F_-')\Lambda_{ij}\Lambda_{kl}\rangle-\frac{\langle(F_+'-F_-')\Lambda_{ij}\rangle\langle(F_+'-F_-')\Lambda_{kl}\rangle}{\langle F_+'+F_-'\rangle}+\frac{1}{\chi(0,0)}\left[-\frac{Q}{e}\delta_{ij}+\mathfrak{A}_{ij}(0,0)\right]\left[-\frac{Q}{e}\delta_{kl}+\mathfrak{A}_{kl}(0,0)\right]+\frac{\beta_{ij}\beta_{kl}}{\langle F_+'+F_-'\rangle(1-\psi\langle F_+'+F_-'\rangle)}.\qquad(4.1)$$

Here

$$\chi(0,0)=[4\psi\langle F_+'\rangle\langle F_-'\rangle-\langle F_+'+F_-'\rangle][1-\psi\langle F_+'+F_-'\rangle]^{-1},$$

$$\beta_{ij}=\langle(F_+'-F_-')\Lambda_{ij}\rangle+(M/\beta)\Lambda_{ij,zz}\langle F_+'+F_-'\rangle,$$

$$\mathfrak{A}_{ij}(0,0)=\langle(F_+'+F_-')\Lambda_{ij}\rangle+\langle F_+'-F_-'\rangle[\psi\langle(F_+'-F_-')\Lambda_{ij}\rangle+(M/\beta)\Lambda_{ij,zz}][1-\psi\langle F_+'+F_-'\rangle]^{-1},$$

where

$$F_\sigma'=f_\sigma'[1+i\pi(v_s/v_F)\delta(\mathbf{kv}/kv_F)],\quad f_\sigma'=f_F'(\varepsilon-\varepsilon_f-\sigma b),$$

with ε_f the Fermi energy in the ferromagnetic state and v_F the characteristic electron velocity on the Fermi surface. The imaginary part of the tensor $\lambda^{\parallel}$ corresponds to allowance of an effect well known in the theory of normal metals, that of collisionless absorption of sound by mobile electrons.[4] Collisionless absorption of longitudinal sound in ferromagnetic metals was studied in Ref. 19, where the "jellium" model was used.

We consider below the real part of $\lambda^{\parallel}$, which is the contribution to the elastic moduli of the ferromagnet. We write down first a corollary of (4.1) for the paramagnetic state at $B=0$:

$$\lambda^{\mathrm{P}}_{ij,kl}(T)=\varphi\frac{Q^2}{e^2}\delta_{ij}\delta_{kl}+2\langle f_F'\Lambda_{ij}\Lambda_{kl}\rangle-\frac{2}{\langle f_F'\rangle}\left[-\frac{Q}{2e}\delta_{ij}+\langle f_F'\Lambda_{ij}\rangle\right]\times\left[-\frac{Q}{2e}\delta_{kl}+\langle f_F'\Lambda_{kl}\rangle\right],\qquad(4.2)$$

where f_F is the Fermi distribution function of the electrons in the paramagnetic state.

Experiment points to unusual temperature dependences in the case of a number of weak ferromagnets having relatively low magnetization. In this case one can expand in powers of the ratio of the spin-splitting energy to the Fermi energy, and obtain in our theory simple analytic relations. To obtain the temperature dependence of the elastic moduli we must use here an equation of state that follows from (3.1):

$$\frac{2\chi_0B}{M(B,T)}+\frac{M^2(B,T)}{M^2(0,0)}\left\{1+C_1(1+2\psi\nu)\left[1-\frac{M^2(B,T)}{M^2(0,0)}\right]\right\}-\frac{T^2M^2(B,T)}{T_0^2M^2(0,0)}(1+2\psi\nu)C_2=\left(1-\frac{T^2}{T_0^2}\right)\left[1-\frac{T^2}{T_0^2}C_3(1+2\psi\nu)\right],\qquad(4.3)$$

where $|1+2\psi\nu|\ll1$ for a weak ferromagnet, ν is the density of states at $T=0$ on the Fermi level ε_F if induction and magnetization are neglected,

$$\nu^{(n)}=d^n\nu/d^n\varepsilon_F,\quad \chi_0=\beta^2\nu/(1+2\psi\nu),\quad M^2(0,0)=24\beta^2(1+2\psi\nu)[(\nu'/\nu^3)']^{-1}[1+(1+2\psi\nu)C_1],$$

$$T_0^2=6(1+2\psi\nu)[\pi^2\varkappa^2\times(\nu'/\nu)']^{-1}[1+(1+2\psi\nu)C_3],$$

$$C_1=0{,}3[\nu^4(\nu'/\nu^3)']^{-2}[-\nu^3\nu^{(\mathrm{IV})}+15\nu^2\nu'\nu'''+10(\nu\nu'')^2-105\nu(\nu')^2\nu''+105(\nu')^4],$$

$$C_2=[\nu^6(\nu'/\nu)'(\nu'/\nu^3)']^{-1}[-\nu^3\nu^{(\mathrm{IV})}+7\nu^2\nu'\nu'''+4(\nu\nu'')^2-25\nu(\nu')^2\nu''+15(\nu')^4],\quad C_3=0{,}1[\nu^2(\nu'/\nu)']^{-2}[-7\nu^3\nu^{(\mathrm{IV})}+17\nu^2\nu'\nu'''+10(\nu\nu'')^2-35\nu(\nu')^2\nu''+15(\nu')^4].$$

With the aid of the equation of state (4.3) for a weak ferromagnet, we rewrite (4.1) in the form

$$\lambda^{\parallel}_{ij,kl}=\lambda^{\mathrm{P}}_{ij,kl}+\lambda^{\mathrm{f}}_{ij,kl}.\qquad(4.4)$$

In accordance with (4.2),

$$\lambda^{\mathrm{P}}_{ij,kl}(T)=(\varphi Q^2/e^2)\delta_{ij}\delta_{kl}+2\nu Y_{ij}Y_{kl}-2Z_{ij,kl}-2\nu(1+2\psi\nu)\times(T^2/T_0^2)[(Y_{ij}-W_{1,ij}'/o_1')(Y_{kl}-W_{1,kl}'/o_1')-W_{1,ij}'W_{1,kl}'/(o_1')^2+(Z_{ij,kl}'/\nu)'/o_1']\qquad(4.5)$$

is here a collectivized-electron contribution that is independent of magnetization and induction, and is typical of a magnetically ordered phase. In this case

$$\nu(\varepsilon_F)Y_{ij}(\varepsilon_F)=\frac{Q}{2|e|}\delta_{ij}-\int\frac{dS}{(2\pi\hbar)^3}\frac{\Lambda_{ij}(\mathbf{p})}{v},$$

$$Y_{ij}^{(n)}=d^n(\nu Y_{ij})/d^n\varepsilon_F,$$

$$Z_{ij,kl}(\varepsilon_F)=\int\frac{dS}{(2\pi\hbar)^3}\frac{\Lambda_{ij}(\mathbf{p})\Lambda_{kl}(\mathbf{p})}{v},\quad Z_{ij,kl}^{(n)}=d^nZ_{ij,kl}/d^n\varepsilon_F,\qquad(4.6)$$
$$W_{n,ij}\equiv Y_{ij}'/\nu^n,\quad o_n\equiv\nu'/\nu^n.$$

Here dS is an element of the Fermi surface $\varepsilon(\mathbf{p})=\varepsilon_F$ and $v=|\partial\varepsilon/\partial\mathbf{p}|$. We note right away that in the Coulomb model, when $\Lambda_{ij}(\mathbf{p})=0$, the temperature dependent correction to the elastic moduli (4.5), which describes the temperature dependence of the elastic moduli in the paramagnetic phase, is of the form

$$-(Q^2/2e^2\nu)(T/T_0)^2(1+2\psi\nu)\delta_{ij}\delta_{kl}.\qquad(4.7)$$

It follows hence that in metals in which the usual condition $\nu\nu''<(\nu')^2$ for ferromagnets is satisfied, the electronic contribution (4.7) to the elastic moduli decreases as the temperature is lowered; this corresponds to an anomaly of the elastic moduli in the paramagnetic phase.

The magnetization- and induction-dependent contribution can be expressed by

$$\lambda^{\mathrm{f}}_{ij,kl}(B,T)=\lambda^{(1)}_{ij,kl}\frac{1+\Xi(B,T)}{[1+\Xi(B,T)]^2+\Theta^2(B,T)}+\lambda^{(2)}_{ij,kl}\frac{M^2(B,T)}{M^2(0,0)}.\qquad(4.8)$$

Here

$$V_{ij}\equiv(Y_{ij}'-2\nu^2\Lambda_{ij,zz})/2\nu',$$

$$\Xi(B,T)=-\chi_0BM^2(0,0)/M^3(B,T)+(1+2\psi\nu)\{C_1[1-2M^2(B,T)/M^2(0,0)]-C_2T^2/T_0^2\},\qquad(4.9)$$

$$\Theta(B,T)=-\frac{\pi v_s}{4v_F(1+2\psi\nu)}\frac{M^2(0,0)}{M^2(B,T)},\qquad(4.10)$$

$$\lambda^{(1)}_{ij,kl}=6\frac{o_1o_2}{o_3'}\left\{(Y_{ij}-2V_{ij})(Y_{kl}-2V_{kl})[1+(1+2\psi\nu)C_1]+(1+2\psi\nu)\left[A_{ij,kl}\frac{M^2(B,T)}{M^2(0,0)}+B_{ij,kl}\frac{T^2}{T_0^2}\right]\right\},\qquad(4.11)$$

$$\lambda^{(2)}_{ij,kl}=-6\nu(1+2\psi\nu)\{(Y_{ij}-W_{3,ij}'/o_3')(Y_{kl}-W_{3,kl}'/o_3')-W_{3,ij}'W_{3,kl}'/(o_3')^2+[(Z_{ij,kl}'/\nu)'-2W_{1,ij}W_{1,kl}]/(\nu^2o_3')\},\qquad(4.12)$$

$$A_{ij,kl}o_1o_3'=2(o_3''-o_1o_3')[(Y_{ij}-V_{ij})(Y_{kl}-V_{kl})-V_{ij}V_{kl}]-(Y_{ij}-2V_{ij})(W_{3,kl}''-o_3'W_{1,kl})-(Y_{kl}-2V_{kl})(W_{3,ij}''-o_3'W_{1,ij}).\qquad(4.13)$$

$$B_{ij,kl}o_1o_1' = 2(o_1'' - o_1o_1')[(Y_{ij} - V_{ij})(Y_{kl} - V_{kl}) - V_{ij}V_{kl}] - (Y_{ij} - 2V_{ij})(W''_{1,kl} - o_1'W_{1,kl}) - (Y_{kl} - 2V_{kl})(W''_{1,ij} - o_1'W_{1,ij}). \qquad (4.14)$$

A feature of relation (4.8), at nonzero induction B, is that it vanishes at zero magnetization. At the same time it must be specially emphasized that at $B = 0$ not only the second term, but also the first differs from zero at $M \neq 0$. At zero induction we have then the parameter $\Xi(0,T) \approx (1 + 2\psi\nu)M^2(0,T)/M^2(0,0) \ll 1$. At the same time the parameter $\Theta(0,T)$ that describes the effect on the elastic moduli by collisionless absorption of sound by mobile electrons is determined by the ratio of two small parameters, v_s/v_F and $(1 + 2\psi\nu)$.

Any discussion of the consequences of (4.8) must reveal the effect exerted on the elastic moduli by the induction and magnetization, on one hand, and by collisionless damping, on the other. We note that the influence of collisionless damping on the velocity of longitudinal sound, with induction neglected, was discussed in the simplest Coulomb-interaction model in Ref. 19, but the peculiarities that characterize weak ferromagnets were not discerned there.

In our discussion of the consequences of (4.8), we shall likewise start with the case $B = 0$. We indicate first that the influence of collisionless damping is particularly strong for anomalously weak ferromagnets, when

$$|1+2\psi\nu| < \pi v_s/4v_F \sim 5\cdot 10^{-3}. \qquad (4.15)$$

Equation (4.8) takes then the form

$$\lambda_{ij,kl}(0,T) = \lambda^{(1)}_{ij,kl}\left[\frac{4v_F(1+2\psi\nu)}{\pi v_s}\right]^2 \frac{M^4(0,T)}{M^4(0,0)} + \lambda^{(2)}_{ij,kl}\frac{M^2(0,T)}{M^2(0,0)}. \qquad (4.16)$$

The first term predominates then if

$$M^2(0,T)/M^2(0,0) > |v_s^2/v_F^2(1+2\psi\nu)|. \qquad (4.17)$$

Disregarding the small quantities $\sim(1 + 2\psi\nu)$ we have the simple equation

$$\lambda^{(1)}_{ij,kl} \approx \Delta\lambda_{ij,kl} = 6(o_1o_2/o_3')(Y_{ij} - 2V_{ij})(Y_{kl} - 2V_{kl}), \qquad (4.18)$$

which in the case of cubic crystals makes it possible to discern directly a relation corresponding to the Invar anomaly. Indeed, since $\Delta\lambda_{ij,kl} = \Delta\lambda\delta_{ij}\delta_{kl}$ for cubic crystals, it is obvious that $\Delta\lambda < 0$, since $o_3' < 0$ for weak ferromagnets. The negative $\Delta\lambda$ causes Eq. (4.16) to describe, near the temperatures (4.17), a decrease of the elastic moduli with decreasing temperature, and this corresponds to the Invar anomaly (see, e.g., Refs. 12 and 13).

If a weak ferromagnet satisfies the condition

$$\pi v_s/4v_F < |1+2\psi\nu| < (1/2)(\pi v_s/2v_F)^{2/3} \sim 10^{-2}, \qquad (4.19)$$

we get at temperatures $T_a \lesssim T < T_0$, where

$$T_a = T_0[1 + \pi v_s/4v_F(1+2\psi\nu)]^{1/2} < T_0, \qquad (4.20)$$

a large change of the contribution $\lambda^f(0,T)$, described by the formula

$$\lambda^f_{ij,kl}(0,T) = \frac{\Delta\lambda_{ij,kl}M^4(0,T)/M^4(0,0)}{M^4(0,T)/M^4(0,0) + [\pi v_s/4v_F(1+2\psi\nu)]^2}. \qquad (4.21)$$

We emphasize that at $T > T_0$ there is no contribution (4.21) to the moduli of the ferromagnet. The appearance of such a contribution at $T_a \lesssim T < T_0$ changes the elastic moduli by an amount $\Delta\lambda_{ij,kl}$ that is comparable with $\lambda^p_{ij,kl}$. For cubic metals, where the contribution (4.21) is proportional to $\delta_{ij}\delta_{kl}$, the condition $o_3' < 0$ in the vicinity $T_a \lesssim T < T_0$ leads to an anomalously strong decrease of the elastic moduli with decreasing temperature. This agrees with the experimentally observed relations for Invar alloys. It must be particularly emphasized that, in contrast to Ref. 19, according to which the longitudinal-sound velocity undergoes an anomalous jump in the small vicinity $\Delta T = T_0 - T \sim T_0 v_s/v_F \ll T_0$ of the Curie temperature T_0, in our case the presence of the small parameter $1 + 2\psi\nu$ can cause the temperature T_a to differ considerably from the transition temperature T_0. It is precisely this last case which corresponds to the elastic-moduli anomaly observed in $Fe_{1-x}Ni_x$ alloys with low nickel concentration ($x \approx 0.34$).[13] With increasing nickel density in the alloy, the parameter $|1 + 2\psi\nu|$ increases, the temperature T_a tends to T_0, and the anomaly of the elastic moduli of the $Fe_{1-x}Ni_x$ alloys shifts towards the Curie temperature T_0.[13] For stronger ferromagnets that satisfy the condition

$$1 \gg |1+2\psi\nu| \geqslant (1/2)(\pi v_s/2v_F)^{2/3} \qquad (4.22)$$

we must point out, first, that their T_a is quite close to T_0, and second, in the low-temperature region

$$T \ll T_M = T_0\left[1 + \frac{(\pi v_s/2v_F)^{2/3}}{2(1+2\psi\nu)}\right]^{1/2} < T_a \qquad (4.23)$$

Eq. (4.8) can be written in the form

$$\lambda^f_{ij,kl}(0,T) = \lambda^{(3)}_{ij,kl} + \lambda^{(4)}_{ij,kl}M^2(0,T)/M^2(0,0), \qquad (4.24)$$

where

$$\lambda^{(3)}_{ij,kl} = \Delta\lambda_{ij,kl}[1 + (1+2\psi\nu)C_2] + (1+2\psi\nu)B_{ij,kl},$$

$$\lambda^{(4)}_{ij,kl} = \lambda^{(2)}_{ij,kl} + (1+2\psi\nu)[\Delta\lambda_{ij,kl}(2C_1 - C_2) + A_{ij,kl} - B_{ij,kl}].$$

The proportionality of the elastic moduli in (4.24) to $M^2(0, T)$ is analogous to the results of the approach that uses the Heisenberg model for the theory of Invar alloys (see Ref. 13).

The case $B = 0$ considered by us is an idealization. Real ultrasound experiments in which the temperature dependence of the elastic moduli is measured, at any rate at low temperatures, are performed in fields $B > B_0$ ($B_0 = 4\pi M(0,0)$ is the saturation field) that ensure that the ferromagnet is single-domain and eliminate, by the same token, the additional contribution made to the elastic moduli by the domain-wall motion. We begin the analysis of Eq. (4.8) at finite B with the case of sufficiently strong induction in the ferromagnet, when the collisionless damping of sound at

$$B > B_1 = B_0\frac{\pi^{1/2}(v_s/v_F)^{3/2}}{64|1+2\psi\nu|^{1/2}\chi_P} \qquad (4.25)$$

(where $\chi_p \sim \beta^2 \nu$) is insignificant in the discussion of the electronic contribution $\lambda^f(B, T)$. Under our conditions, for a large change of the elastic moduli by an amount comparable with $\lambda^p_{ij,kl}$, we obtain from (4.8), in place of (4.21), the following simple approximation (cf. Ref. 12):

$$\lambda_{ij,kl}(B,T)=\frac{\Delta\lambda_{ij,kl}}{1-\chi_0 BM^2(0,0)/M^3(B,T)}. \quad (4.26)$$

This approximation, just as (4.21), differs substantially from the usually discussed[13] proportionality to $M^2(B, T)$ that arises in the Heisenberg model. For cubic crystals, Eq. (4.26) corresponds to an anomalous decrease of the elastic moduli. We note that in contrast to (4.21), Eq. (4.26) describes the elastic-moduli anomaly observed in Invar alloys at a temperature higher than T_0.[12,13] Equation (4.26) describes at $\Delta\lambda_{ji,kl} \neq 0$ practically the entire temperature dependence of the elastic moduli of weak ferromagnets that satisfy the inequality

$$\pi v_s/4v_F < |1+2\psi\nu| < (\pi v_s/4v_F)^{3/5} \approx 4\cdot 10^{-2}. \quad (4.27)$$

If, however, the condition $|1 + 2\psi\nu| > (\pi v_s/4v_F)^{3/5}$ holds rather than (4.27), we have at low temperatures

$$T \ll T_{B1}=T_0[1-[4\pi\chi_p B/(1+2\psi\nu)^2 B_0]^{2/5}]^{1/2}$$

and at induction values satisfying the inequality $B_1 < B < B_0(4\pi\chi_p)^{-1}(1+2\psi\nu)^2$ the temperature dependence of λ^f is determined by (4.24).

For typical parameters $\chi_p \sim 10^{-5}$, $v_s/v_F \sim 5\cdot 10^{-3}$ and for weak ferromagnets with $|1 + 2\psi\nu| \sim 10^{-1}$, the induction B_1 is found to equal approximately $3B_0$; this is the condition for the ferromagnet to be single-domain at low temperatures. For weaker ferromagnets with $|1 + 2\psi\nu| \ll 0.1$ there can be realized the inequality $B_1 \gg B_0$. We shall therefore discuss hereafter the influence of the magnetic induction on the elastic moduli at $B < B_1$. We note first that the induction plays a substantial role for the contribution $\lambda^f(B, T)$ in the region of relatively high temperatures

$$T^2-T_0^2 \gg T_0^2(-8\pi\chi_0 B/B_0)^{2/3}, \quad (4.28)$$

when we have from (4.8)

$$\lambda^f_{ij,kl}(B,T)=\frac{2(-8\pi\chi_0 B/B_0)^2\Delta\lambda_{ij,kl}}{[(T/T_0)^2-1]^2+[\pi v_s/2v_F(1+2\psi\nu)]^2}\frac{T_0^2}{T^2-T_0^2}. \quad (4.29)$$

Allowance for the finite value of the induction near T_0, when

$$|T-T_0| \ll T_0(-8\pi\chi_0 B/B_0)^{2/3}, \quad (4.30)$$

leads to

$$\lambda^f_{ij,kl}(B)=\frac{3}{2}\left[\frac{4v_F(1+2\psi\nu)}{\pi v_s}\right]^2\left(-8\pi\chi_0\frac{B}{B_0}\right)^{4/3}\Delta\lambda_{ij,kl}. \quad (4.31)$$

If, on the contrary, $B < B_1$, the induction is inessential near the temperature T_a (in this case $T_0^2 - T_a^2 \gg (-8\pi\chi_0 B/B_0)^{2/3}$), where the change of λ^f with temperature is given by (4.21). The condition $B > B_0$ can in this case be satisfied, in view of the inequality (4.19). For temperatures sufficiently lower than T_a the effect of the induction on the elastic moduli can become substantial for ferromagnets with

$$\pi v_s/4v_F < |1+2\psi\nu| < (1/2)(\pi v_s/2v_F)^{2/3}, \quad (4.32)$$

at the values

$$B > B_2 = B_0\frac{\pi(v_s/v_F)^2}{128|1+2\psi\nu|\chi_p}, \quad (4.33)$$

in the temperature region

$$T \ll T_{B2}=T_0\left[1-\left[\frac{\pi(v_s/v_F)^2B_0}{128(1+2\psi\nu)\chi_p B}\right]^2\right]^{1/2}, \quad (4.34)$$

where we get from (4.8)

$$\lambda^f_{ij,kl}(B,T)=\Delta\lambda_{ij,kl}[1+\chi_0 BM^2(0,0)/M^3(0,T)]. \quad (4.35)$$

In this case $B_2 < B_1$.

We note finally that for cubic crystals the anomalous decrease of the transverse-sound velocity is described according to (4.8) and (4.12) by the expression

$$-6(1+2\psi\nu)[(Z'_{ij,kl}/\nu)'/(\nu o_3')][M^2(B,T)/M^2(0,0)], \quad (4.36)$$

which is proportional to the square of the magnetization in the entire range of temperatures where the ferromagnetic state exists.

It can be stated in conclusion that the Fermi-liquid approach permits not only a general description of the magnetoelastic phenomena in ferromagnets with collectivized mobile electrons, based on the use of the exchange interaction of the electrons and of the interaction of the electrons with the lattice, but also a description, without resorting to the magnetoelastic constants, of the anomalous (Invar) dependences of the elasticity of ferromagnetic metals. In this case, first, the character of these anomalous dependences is substantially altered by a change of the parameter $1 + 2\psi\nu$, in accordance with the universally accepted viewpoint. Second, in our analysis the anomalous relations are in no way connected with the usual assumption that the exchange-interaction energy depends on the volume, since a large set of possible anomalous magnetoelastic relations are derived in our approach at ψ = const. This allows us to state that the Fermi-liquid approach, by revealing the anomalous relations that govern the elastic moduli of ferromagnetic metals, provides a new possibility of understanding the Invar anomaly.

APPENDIX

The appearance, in the right-hand side of (2.13), of components of the alternating magnetic field and of the alternating vector potential is due to the coupling, described by our collectivized-electron model, of the sound field with the solenoidal electromagnetic field. The spin coupling is determined here by the coefficients

$$R_{ij}^{\sigma}=(M/\beta)\langle\Gamma(-\sigma,\sigma)\rangle[1-2\psi\langle\Gamma(-\sigma,\sigma)\rangle]^{-1} \times(\Lambda_{ij,zx}\mp i\sigma\Lambda_{ij,zy}), \quad (A.1)$$

$$R_{ij}^{z}=D^{-1}(\varphi,\psi)\{a_{ij}^{+}(1-2\varphi\langle\Gamma(-,-)\rangle) - a_{ij}^{-}(1-2\varphi\langle\Gamma(+,+)\rangle) + 4\pi e k^{-2}\varepsilon_e^{-1}(\omega,\mathbf{k})\langle\Gamma(+,+)-\Gamma(-,-)\rangle \times\{e\mathfrak{A}_{ij}-Q\delta_{ij}[1+\varphi\chi(\omega,\mathbf{k})]\}\}, \quad (A.2)$$

where

$$D(\varphi, \psi) = [1+\varphi\chi(\omega, \mathbf{k})][1-\psi\langle\Gamma(+,+)+\Gamma(-,-)\rangle]. \quad (A.3)$$

For the coefficient of the diamagnetic coupling of the sound with the electromagnetic field we have

$$\begin{aligned} P_{il} = {} & Q\omega\delta_{il} - ek_j\{\langle[\Gamma(+,+)+\Gamma(-,-)]v_l\Lambda_{ij}\rangle + (M/\beta)\Lambda_{ij,zz} \\ & \times\langle[\Gamma(+,+)-\Gamma(-,-)]v_l\rangle + D^{-1}(\varphi,\psi) \\ & \times\{\langle\Gamma(+,+)v_l\rangle[\varphi(a_{ij}^{+}+a_{ij}^{-}) \\ & +\psi(a_{ij}^{+}-a_{ij}^{-}) \\ & -4\varphi\psi a_{ij}^{+}\langle\Gamma(-,-)\rangle] + \langle\Gamma(-,-)v_l\rangle[\varphi(a_{ij}^{+}+a_{ij}^{-}) \\ & +\psi(a_{ij}^{-}-a_{ij}^{+}) - 4\varphi\psi a_{ij}^{-}\langle\Gamma(+,+)\rangle]\} + 4\pi ek^{-2}\varepsilon_e^{-1}(\omega,\mathbf{k}) \\ & \times D^{-1}(\varphi,\psi)[\langle\Gamma(+,+)v_l\rangle(1-2\psi\langle\Gamma(-,-)\rangle) + \langle\Gamma(-,-)v_l\rangle \\ & \times(1-2\psi\langle\Gamma(+,+)\rangle)][e\mathfrak{A}_{ij} - (1+\varphi\chi(\omega,\mathbf{k}))Q\delta_{ij}]\}. \end{aligned} \quad (A.4)$$

Equations (A.1)–(A.4), on the one hand, determine the effects of sound excitation by an electromagnetic field (cf. Ref. 14), and on the other, describe the sound-wave damping due to the self-consistent solenoidal field (cf. Ref. 4), and also the electromagnetic coupling of sound and spin waves (cf. Ref. 7).

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