Hierarchical structure of an Edwards-Anderson spin glass

L. B. Ioffe and M. V. Feĭgel'man

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

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A three-dimensional Edwards-Anderson model of spin glass with a large but finite interaction radius is analyzed. A phase transition completely different from that which occurs in a spin glass in a space of higher dimensionality ($d \geqslant 4$) is shown to occur in this model. The hypothesis of scale invariance does not apply to a phase transition of this sort, so that at the phase transition the physical properties have singularities which are more complex than power-law singularities. The behavior of the nonlinear susceptibility $\tilde{\chi}(T)$, of the maximum relaxation time $t_{\max}(T)$, and of the relaxation law $\overline{\langle \mathbf{S}_i(0)\mathbf{S}_i(t)\rangle}$ at $t \ll t_{\max}$ are calculated in the critical-temperature region above the transition point.

1. INTRODUCTION

The surprising spin-glass properties that stem from the formation of a frozen random state of the magnetic system at a sufficiently low temperature were discovered by Canella and Mydosh[1] (see Ref. 2 for a review of the present experimental situation). The first theory of this state was offered by Edwards and Anderson in Ref. 3, where it was shown that in the mean-field approximation a lowering of the temperature results in a phase transition from a paramagnetic phase to a frozen random state (the "spin glass"). Edwards and Anderson[3] studied a model with a random, alternating-sign interaction between localized magnetic moments [see expression (2.1) below]. A very simple version of the Edwards-Anderson model, with an infinite interaction radius (the Sherrington-Kirkpatrick model[7]), has by now been studied in detail,[4–6] but we have essentially no theory for the Edwards-Anderson model with a finite interaction radius in three dimensions. In particular, the basic theoretical question of whether a thermodynamic phase transition occurs at a finite temperature has not been resolved. The present opinion regarding this problem, based on experimental results and numerical simulations, is as follows:

1. a) Precise magnetic[8] and magnetocalorimetric[9] measurements in the paramagnetic phase indicate a critical anomaly in the nonlinear magnetic susceptibility $\tilde{\chi} = \partial^3 M / \partial h^3$ in the alloy $Cu_{1-x}Mn_x$ ($x \sim 0.01$). The critical index is $\gamma \simeq 3.5$ [$\tilde{\chi} \propto (T - T_f)^{-\gamma}$], quite different from the prediction of the mean field theory, $\gamma_{\mathrm{MFA}} = 1$. b) A numerical simulation on a two-dimensional Edwards-Anderson model[10] demonstrates that there is no phase transition in it at a finite temperature [in particular, $\tilde{\chi} \propto T^{-4} (d = 2)$], while a numerical simulation for a three-dimensional model indicates that there is a phase transition.[11] A numerical analysis of the high-temperature series in an Edwards-Anderson model in a space of arbitrary dimensionality d indicates[12] a phase transition only at $d \geqslant 4$.

2. The experimental results on a wide variety of spin glasses[13] can be described by assigning the freezing temperature $T_f(h)$ (i.e., the temperature at which an irreversibility of the magnetic response appears) a dependence $T_f(h) - T_f(0) = -Ah^{2/3}$ on the external magnetic field h. The exponent 2/3 agrees with the prediction of the mean field theory,[14] while the coefficient A is in all cases much greater (e.g., by an order of magnitude) than the predicted value and increases as the scale time of the measurements increases.

3. As the temperature is lowered and approaches the freezing point, the spectrum $g(\tau)$ of spin relaxation times, broadens rapidly. The maximum time $t_{\max}$ can be described satisfactorily by a Vogel-Fulcher law $t_{\max} \sim t_0 \exp[E_\alpha / (T - T_f)]$, while the mean relaxation time t_m has no anomalies of any sort near T_f (Refs. 15–17). At $T < T_f$, the time $t_{\max}$ turns out to be comparable to the duration of the experiment, $t_{\exp}$; it has not been found possible to detect a saturation in the increase in $t_{\max}$ with increasing $t_{\exp}$ (Ref. 18).

4. A spin anisotropy, even very slight, substantially increases T_f in real substances.[19] In numerical (Monte Carlo) simulations based on a model with isotropic vector spins, the freezing is not observed at all, while the inclusion of a slight dipole interaction leads to a freezing.[20,21]

In the present paper we analyze a three-dimensional Edwards-Anderson model with an interaction of large but finite radius. We show that all the facts listed above can be understood (at least qualitatively) on the basis of a common microscopic theory. In particular, the ordinary phase transition of the type which occurs in the Sherrington-Kirkpatrick model[7] actually occurs only at $d \geqslant 4$, but a phase transition of a completely different type occurs at $2 < d < 4$. The other phase transition is unrelated to the macroscopic population of one delocalized state, and it is apparently not discernible by the standard analysis of high-temperature series.

The basic idea of our approach is to systematically single out critical variables in analogy with the construction of a transformation of the renormalization group in the theory of ordinary phase transitions. Roughly speaking, we single out groups of variables (clusters) which are strongly correlated with each other, and we construct an effective free energy for these groups. The resulting free energy turns out to be equivalent to an Edwards-Anderson model with renormalized parameters (the effective temperature and the number of neighbors, Z). In contrast with the known phase transitions, the transformation of the renormalization group is fundamentally discrete; i.e., the permissible change in scale is dic-

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tated by the initiating parameters of the model, and for this reason there is a discontinuity of the scale invariance at the point of the transition (in $3D$ space). The apparent reason is that it is not possible to parametrize the spin configurations by a finite number of slowly varying fields (order parameters).

As the temperature is lowered, correlations of variables within one group appear first; then the variables corresponding to groups as a whole start to become correlated; etc. As the temperature is lowered, a discrete hierarchy of correlations thus arises in the system. At the transition point this hierarchy becomes infinite. We stipulate at the outset that either "group" or "cluster" is a poor word to use here, since it gives the incorrect impression that each variable is part of only a single group. Actually, these "clusters" overlap greatly, so that the value of the spin at each site is determined by a set of a large number of variables corresponding to different clusters.

In Section 2 of this paper we derive an effective Hamiltonian of the slow degree of freedom (by "slow" we mean a slow relaxation near the transition point). In Section 3 we analyze the condensation of localized critical modes, and we derive a renormalized Hamiltonian of the same form as the original Hamiltonian of the Edwards-Anderson model. An explicit construction of the renormalization procedure allows us to determine the nature of the nonlinear susceptibility near the transition point. This is the subject of Section 4. It turns out that in this case the nonlinear susceptibility becomes infinite not in accordance with a power law but oscillating between two power-law envelopes. In Section 5 we use the same method to study the purely dissipative dynamics of the Edwards-Anderson model, and we derive the maximum relaxation time as a function of the temperature (an analog of the Vogel-Fulcher law) and the behavior of the correlation function $\overline{\langle \mathbf{S}_i(0)\mathbf{S}_i(t)\rangle}$ near the transition. In the last section of this paper we discuss the results.

2. DERIVATION OF AN EFFECTIVE INTERACTION OF THE SLOW DEGREES OF FREEDOM

1. Let us put the Edwards-Anderson model in a form convenient for our purposes. We consider a system of classical spins $S_\alpha(i)$ ($\alpha = 1, \ldots, n$; $S_\alpha^2 = 1$) which are scattered at random with a density c in a d- dimensional space (we will be concerned for the most part with the case $d = 3$). The interaction between spins is assumed isotropic and uncorrelated for different pairs of spins:

$$H=-\frac{1}{2}\sum_{i,j} J_{ij}\mathbf{S}_i\mathbf{S}_j, \quad \bar{J}_{ij}=0, \qquad \overline{J_{ij}^2}=K(\mathbf{r}_i-\mathbf{r}_j), \quad \int K(r)\,d^3r=K_0. \tag{2.1}$$

We assume that each spin interacts with a large number (Z) of neighbors; i.e., we assume

$$Z=cK_0^{-3/2}\left[\int K(r)r^2\,d^3r\right]^{3/2}\gg 1. \tag{2.2}$$

Below we will express the temperature in units of K_0, so that we have $\tilde{K}_0 = 1$ in terms of these new units.

In the case $Z=\mathcal{N}$ (where $\mathcal{N}$ is the total number of spins in the system) we can integrate the partition function over all the spin configurations for the given values of the mean spins at each site, $\langle \mathbf{S}(i)\rangle = \mathbf{m}_i$. We derive an effective free energy $F\{\mathbf{m}(i)\}$; the state of the system in the limit $Z=\mathcal{N}\to\infty$ corresponds to the minimum of $F\{\mathbf{m}\}$ (the TAP equation[22,23]). In this case the functional $F\{\mathbf{m}\}$ replaces the Ginzburg-Landau free energy. At $Z\ll\mathcal{N}$, we need to consider the fluctuations of $F\{\mathbf{m}\}$ near the minimum. The condition $Z\gg 1$ can be used to calculate an effective Hamiltonian for such fluctuations. We wish to stress that the $\mathbf{m}_i$ parametrize poorly the fluctuations of $F\{\mathbf{m}\}$, i.e., they are not correct slow variables. We will make the transformation to these variables later on. The functional $F\{\mathbf{m}\}$ agrees with (see the Appendix) the free energy derived in Refs. 22 and 23:

$$F\{\mathbf{m}_i\}=-\frac{1}{2}\sum_{ij} J_{ij}\mathbf{m}_i\mathbf{m}_j-\frac{1}{4nT}\sum_{ij} J_{ij}^2(1-\mathbf{m}_i^2)(1-\mathbf{m}_j^2) +T'n\sum_i\left\{\frac{1}{2}\mathbf{m}_i^2+\frac{n}{4(n+2)}(\mathbf{m}_i^2)^2 +\frac{1}{6}\frac{n^2}{n+4}\frac{n+8}{(n+2)^2}(\mathbf{m}_i^2)^3\right\}. \tag{2.3}$$

In the last term we have retained only those terms of the expansion in powers of $\mathbf{m}_i^2$, assumed small, which we will be needing (at $T\sim T_c$). Later on we will expand $\mathbf{m}_i$ in the eigenfunctions of the matrix J_{ij}, and describe therefore some of their properties.

2. At $Z=\infty$ all the eigenfunctions of the matrix J_{ij} are delocalized; the magnitude of $\psi_\lambda(i)$ is completely random, with a Gaussian distribution. The different $\psi_{\lambda'}(i')$, $\psi_\lambda(i)$ are not correlated with each other, simply satisfying normalization and orthogonality conditions:

$$\sum_i \psi_\lambda(i)\psi_{\lambda'}(i)=\delta_{\lambda\lambda'}.$$

The eigenvalues E_λ corresponding to these eigenfunctions are distributed over the interval $(-2, 2)$ with a density $\rho(E)=(2\pi)^{-1}(4-E^2)^{1/2}$ (see Refs. 24 and 25, for example).

At a finite $Z\gg 1$, the spectrum of eigenvalues is not restricted to the interval $(-2, 2)$; there are "tails" with $|E_\lambda|>2$. The states in the tail are localized, while the states with eigenvalues in the interior of the interval $(-2, 2)$ are delocalized. Everything we said above about the case $Z=\infty$ applies to the latter states. The width of the transition region between these two regimes decreases with increasing Z. The events which unfold in this region are studied in localization theory, and we will need to draw on the results and representations of that theory. We first examine the properties of delocalized wave functions far from the localization threshold.

We use the replica method, which reduces the problem of the behavior of an eigenfunction of a random matrix to problems of field theory. For example, the state density is expressed in terms of the field correlation functions[26]:

$$\rho(E)=-\frac{1}{\pi}\operatorname{Im}\lim_{l\to 0} i\int \varphi_j^1\varphi_j^1 \exp\left\{-\frac{i}{2}\sum_{i,j,a}[J_{ij}-(E+i\delta)\delta_{ij}]\varphi_i^a\varphi_j^a\right\}D\varphi_i^a, \tag{2.4}$$

where l is the number of replicas, i.e., the number of components of the field φ. After taking an average over the random components, we can conveniently write the functional integral as

$$\begin{aligned}\rho(E)=&-\frac{1}{\pi}\operatorname{Im}\lim_{l\to 0} i\int \varphi_j^1\varphi_j^1\exp\left\{+\frac{i}{2}\sum_{i,a}(E+i\delta)(\varphi_i^a)^2\right.\\&\left.-\frac{1}{4}\sum_{i,j,a,b}\varphi_i^a\varphi_i^b K(\mathbf{r}_i-\mathbf{r}_j)\varphi_j^a\varphi_j^b\right\}D\varphi_i^a\\=&-\frac{1}{\pi}\operatorname{Im}\lim_{l\to 0} i\int \varphi_j^1\varphi_j^1\exp\left\{\frac{i}{2}\sum_{i,a}(E+i\delta)(\varphi_i^a)^2\right.\\&\left.+\frac{i}{2}\sum_{i,a,b}\varphi_i^a Q_i^{ab}\varphi_i^b-\frac{1}{4}\sum_{i,j,a,b}Q_i^{ab}Q_j^{ab}\hat{K}^{-1}(\mathbf{r}_i-\mathbf{r}_j)\right\}D\varphi_i^a DQ_i^{ab}\\=&-\frac{1}{\pi}\operatorname{Im}\lim_{l\to 0}\int[(E+i\delta)\delta^{ab}+Q_j^{ab}]_{a,b=1}^{-1}\\&\times\exp\left\{-\frac{1}{4}\sum_{i,j,a,b}Q_i^{ab}\hat{K}^{-1}(\mathbf{r}_i-\mathbf{r}_j)Q_j^{ab}\right.\\&\left.-\frac{1}{2}\sum\operatorname{Sp}\ln[(E+i\delta)\delta^{ab}+Q_i^{ab}]\right\}DQ_i^{ab}.\end{aligned} \tag{2.5}$$

As we will see below, the characteristic values of Q^{ab} vary only slightly from point to point, so we can take the continuum limit and use the replacement

$$\hat{K}^{-1}(\mathbf{r}_i-\mathbf{r}_j)\to(1-\nabla^2/\varkappa^2)\delta(\mathbf{r}_i-\mathbf{r}_j),$$

where $\varkappa$ is the reciprocal interaction radius: $4c\varkappa^{-3}\approx Z$. In the region of eigenvalues corresponding to delocalized states in the interior of the interval $(-2, 2)$, we can completely ignore the variations of $\hat{Q}_i$ from point to point, and we can determine $\hat{Q}$ from the condition for a minimum of the argument of the exponential function in (2.5):

$$\hat{Q}_i=-[\hat{Q}_i+\hat{1}(E+i\delta)]^{-1}. \tag{2.6}$$

We find

$$\begin{aligned}&Q^{ab}=\delta^{ab}Q_0,\\&Q_0=-E/2+i(1-E^2/4)^{1/2},\quad |E|\leqslant 2,\end{aligned} \tag{2.7}$$

$$Q_0=-E/2+\operatorname{sgn}E\,(E^2/4-1)^{1/2},\quad |E|\geqslant 2. \tag{2.8}$$

Substituting solutions (2.7) and (2.8) into (2.5), we find the result mentioned above for the state density:

$$\rho(E)=(4-E^2)^{1/2}(2\pi)^{-1}\theta(4-E^2). \tag{2.9}$$

At $Z\gg 1$, expression (2.9) for the state density is inapplicable near the ends of the energy interval $(-2, 2)$. To determine the range over which it is applicable, we will calculate the correction to it from the spatial fluctuations $\hat{Q}$. Expanding the argument of the exponential function in (2.5) for $S\{\hat{Q}\}$ near the maximum Q_0 in (2.8), $\hat{Q}=\hat{Q}_0+\tilde{Q}$, we find the following expression, which is accurate enough for our purposes near the ends of the energy interval ($\varepsilon=|E|-2\ll 1$):

$$S=\int\operatorname{Sp}\left\{\frac{1}{2}\hat{\tilde{Q}}\left[i(-\varepsilon)^{1/2}-\frac{1}{2\varkappa^2}\nabla^2\right]\hat{\tilde{Q}}+\frac{1}{6}\hat{\tilde{Q}}^3\right\}d^3x. \tag{2.10}$$

Let us determine the first correction $\delta\Sigma$ to the Green's function $\langle\tilde{Q}\tilde{Q}=G_0$. This correction arises in second-order perturbation theory in the interaction $\operatorname{Sp}\tilde{Q}^3$ and is small except in the region $\varepsilon\ll 1$:

$$\delta\Sigma=(\varkappa^3/2\pi)e^{i\pi/4}(-4\varepsilon)^{-1/4}. \tag{2.11}$$

Comparision of (2.10) and (2.11) shows that the correction $\delta\Sigma$ becomes comparable to G_0^{-1} at

$$\varepsilon\approx\varkappa^4/(4\pi)^{4/3}\approx Z^{-4/3}. \tag{2.12}$$

The characteristic width of the transition region must therefore be on the order of $\varepsilon_0\approx Z^{-4/3}$. We can now show that in the tail, $|E|-2\gg\varepsilon_0$, the state density falls off exponentially. The state density calculated by perturbation theory is zero in this region, so that we need to take into account nontrivial extrema of the action S (see Ref. 27, for example), which is given at $\varepsilon>0$, $|\varepsilon|\ll 1$ by

$$S=\int\operatorname{Sp}\left\{\frac{1}{2}\hat{\tilde{Q}}\left[\varepsilon^{1/2}-\frac{\nabla^2}{2\varkappa^2}\right]\hat{\tilde{Q}}+\frac{1}{6}\hat{\tilde{Q}}^3\right\}d^3x. \tag{2.13}$$

We seek a solution by the method of steepest descent for the matrix $\hat{\tilde{Q}}$ in a form which is symmetric with respect to replicas:

$$\tilde{Q}^{ab}(\mathbf{x})=q(\mathbf{x})e^a e^b,$$

where e^a is a unit vector in replica space. For $q(\mathbf{x})$ we have a radially symmetric equation of an extremal:

$$\frac{1}{2\varkappa^2}\left(\frac{d^2q}{dr^2}+\frac{2}{r}\frac{dq}{dr}\right)-\varepsilon^{1/2}q-\frac{q^2}{2}=0. \tag{2.14}$$

As was shown in Ref. 28, Eq. (2.14) has a solution with a finite action S_0. To find S_0, we put Eqs. (2.13) and (2.14) in dimensionless form:

$$\begin{aligned}&q(r)=\varepsilon^{1/2}f(\varkappa\varepsilon^{1/4}r),\\&S_0=\varkappa^3\varepsilon^{3/4}\frac{1}{12}\int_0^\infty 4\pi y^2 f^3(y)\,dy,\end{aligned} \tag{2.15}$$

where $f(y)$ is the solution of the dimensionless equation

$$\frac{1}{2}f''+\frac{1}{y}f'-f-f^2/2=0. \tag{2.16}$$

The state density is given with exponential accuracy by

$$\rho(\varepsilon)\sim\exp(-S_0)=\exp[-a(\varepsilon/\varepsilon_0)^{3/4}],\quad a\sim 1. \tag{2.17}$$

It follows from (2.17) that the state density falls off rapidly with increasing ε far from the transition region. The magnitude of the state density in the transition region [the coefficient of the exponential function in (2.17)] can be found most simply from the condition for the joining of (2.17) and (2.9):

$$\rho(|\varepsilon| \sim \varepsilon_0) \approx \varepsilon_0^{1/3} \approx Z^{-1/3}. \tag{2.18}$$

The scale dimension of the localized states in the region $\varepsilon \gg \varepsilon_0$ is the same as that of the classical solution, (2.15); i.e.,

$$l(\varepsilon) \approx \varkappa^{-1} \varepsilon^{-1/4} \quad (1 \gg \varepsilon \gg \varepsilon_0). \tag{2.19}$$

As ε decreases, the dimension $l(\varepsilon)$ increases, and it becomes infinite at $\varepsilon = \varepsilon_c$ (the localization threshold). The dimension $l(\varepsilon)$ can be expressed in terms of the field correlation functions. Let us consider the density-density correlation function

$$R_E(\mathbf{x}) = \overline{\sum_{n,m} \frac{\psi_n(0)\psi_m(0)\varphi_n(\mathbf{x})\psi_m(\mathbf{x})}{(E-E_n+i\delta)(E-E_m-i\delta)}} \quad (\delta \to 0). \tag{2.20}$$

For values of E in the region of localized states, this correlation function is dominated by terms with $n = m$:

$$R_E(\mathbf{x}) = \frac{1}{\pi\delta} \sum_n \psi_n^2(0)\psi_n^2(\mathbf{x})\delta(E-E_n). \tag{2.21}$$

The scale distance over which $R_E(\mathbf{x})$ decreases is thus the same as $l(\varepsilon)$ $(\varepsilon > \varepsilon_c)$. The quantity $R_E(\mathbf{x})$ could be expressed directly in terms of the field correlation functions, in the same manner as $\rho(\varepsilon)$ (Ref. 29):

$$\begin{aligned} R_E(\mathbf{x}) = -\lim_{l\to 0} \int \varphi_{1i}^1 \varphi_{2i}^1 \varphi_{2j}^1 \varphi_{1j}^1 \\ \times \exp\left\{ \frac{-i}{2} \sum_{i,j,a} \left[(J_{ij} - (E+i\delta)\delta_{ij}) \varphi_{1i}^a \varphi_{1j}^a \right.\right. \\ \left.\left. - (J_{ij} - (E-i\delta)\delta_{ij}) \varphi_{2i}^a \varphi_{2j}^a \right] \right\} D\varphi_{pi}^a. \end{aligned} \tag{2.22}$$

After some calculations analogous to the derivation of (2.10), and after the introduction of $Q_{pr}^{ab}(\mathbf{x})$ $(p, r = 1, 2)$, we find the effective action

$$\begin{aligned} S = \int \left\{ \frac{1}{4} Q_{pr}^{ab} (1 - \varkappa^{-2}\nabla^2) Q_{pr}^{ab} \right. \\ \left. + \frac{1}{2} \operatorname{Sp} \ln (\hat{\Lambda}E + \hat{1}i\delta + \hat{Q}) \right\} d^3x, \\ \hat{\Lambda} = \delta^{ab}(\sigma_z)_{pr}, \quad \hat{1} = \delta^{ab}\delta_{pr}. \end{aligned} \tag{2.23}$$

Deep in the region of delocalized states, $\hat{Q}$ is given by the solution of the equation for an extremum of S in (2.23):

$$\hat{Q}_0 = -[\hat{Q}_0 + \hat{1}i\delta + \hat{\Lambda}E]^{-1}, \quad \hat{Q}_0 = -\hat{\Lambda}E/2 + i\hat{1}(1-E^2/4)^{1/2}. \tag{2.24}$$

As E is increased, fluctuations of $\hat{Q}$ around $\hat{Q}_0$ arise at $|\varepsilon| \sim Z^{-4/3}$. A distinction must be made between fluctuations of two types: Goldstone modes, $\hat{Q} = \hat{U}^{-1}\hat{Q}_0\hat{U}$ ($\hat{U}$ is a unitary matrix), and longitudinal fluctuations, which have a gap $\sim \varepsilon^{1/2}$. The longitudinal fluctuations, in contrast with an ordinary phase transition, smear the singularity in $\rho(\varepsilon)$ at $\varepsilon = 0$, so that the singularity [i.e., the point at which $l(\varepsilon)$ becomes infinite] is related only to the Goldstone modes, for which we can find an effective action:

$$\begin{aligned} S(\hat{U}) = \frac{A}{2} \int (\partial_\mu \hat{U}^{-1})(\partial_\mu \hat{U}) d^3x, \quad A \approx \varkappa^{-2}(-\varepsilon), \\ \hat{U} = \hat{U}^{-1}\hat{\Lambda}\hat{U}. \end{aligned} \tag{2.25}$$

We have ignored the longitudinal fluctuations, so that expression (2.25) is valid only over scale distances greater than $L \sim \varkappa^{-1}\varepsilon^{-1/4}$, over which we can separate longitudinal and transverse fluctuations. We can estimate the width of the fluctuation region for interaction (2.25) by using a criterion of the Ginzburg type. We find $AL \sim 1$, i.e., $\varepsilon_0 \sim Z^{-4/3}$, telling us that the width of the fluctuation region agrees with the region over which the state-density edge is blurred. In this region we use the scaling assumption, i.e.,

$$l(\varepsilon) = \left(\frac{\varepsilon_0}{\varepsilon - \varepsilon_c} \right)^{\nu} l_0. \tag{2.26}$$

We can determine l_0 from the condition of the joining of (2.26) and expression (2.19) for $l(\varepsilon)$ at $\varepsilon \gg \varepsilon_0$; this yields $l_0 \approx Z^{2/3}$ (for $d = 3$). With regard to the index ν we assume, following Ref. 30, $\nu = (d-2)^{-1}$, where d is the dimensionality of the space. The quantity $\varepsilon_c \sim \varepsilon_0$, the first-order perturbation-theory correction to ε_c, is positive, so below we will also assume $\varepsilon_c > 0$.

Below we will need averages of the type $\overline{A}_\lambda$, where

$$A_\lambda = \left[\sum \psi_\lambda^4(i) \right]^{-1}$$

in the region of localized states, $\varepsilon > \varepsilon_c$ (the average is calculated over different states with a given energy $\varepsilon = \varepsilon_\lambda$). For states in the tail, $\varepsilon_\lambda - \varepsilon_c \gtrsim \varepsilon_0$, the value of A_λ is determined unambiguously by the steepest-descent solution of (2.15):

$$A_\lambda \sim l^d(\varepsilon_\lambda) \sim \varkappa^{-d} \varepsilon_\lambda^{-d/4}.$$

In the region of localized scaling $(\varepsilon_\lambda - \varepsilon_c \ll \varepsilon_0)$ the eigenfunctions $\psi_\lambda(i)$ may have a fractal structure,[31] which should lead to a wide distribution of A_λ for a given ε_λ. In particular, the mean value $\overline{A_\lambda^{-1}}$ can be described by a nontrivial index: $\overline{A_\lambda^{-1}} \sim l^{-\tilde{d}}(\varepsilon_\lambda) \gg l^{-d}$ $(\tilde{d} < d$; see Refs. 31 and 32). For A_λ, the estimate $A_\lambda \sim l^d$ given above is an upper limit, and we will assume that the mean value $\overline{A}_\lambda$ is determined by precisely this edge of the distribution, so that at $\varepsilon - \varepsilon_c \ll \varepsilon_c$ we have

$$\overline{A}_\lambda = l_0^d \left(\frac{\varepsilon - \varepsilon_c}{\varepsilon_c} \right)^{-d\nu}. \tag{2.27}$$

In addition to $\overline{A}_\lambda$ we will need sums of the type

$$\sum_{ij} J_{ij}^2 \psi_\lambda^2(i) \psi_\lambda^2(j).$$

At $Z = \infty$, the distribution of components $\psi_\lambda(i)$ is Gaussian, so that

$$\sum_i \psi_\lambda^4(i) = 3\sum_{i,j} \psi_\lambda^2(i) J_{ij}^2 \psi_\lambda^2(j).$$

This relation also holds at a finite Z for states in the region of energies E far from the ends of the interval $(-2, 2)$. For states in the scaling region, $|\varepsilon - \varepsilon_c| \ll \varepsilon_0$, it can be shown that an analogous equation holds:

$$\sum_i \psi_\lambda^4(i) = \alpha \sum_{ij} \psi_\lambda^2(i)\psi_\lambda^2(j) J_{ij}^2, \qquad |\alpha - 3| \leqslant Z^{-2/3}. \tag{2.28}$$

3. Let us pursue the derivation of an effective Hamiltonian for the slow degrees of freedom. We write $\mathbf{m}_i$ as a sum over the eigenfunctions of the matrix J_{ij}. We single out the contribution from the eigenfunctions ψ near the upper edge of the spectrum and from the functions for which the eigenvalues lie in the interior of the interval $(-2, 2)$:

$$\mathbf{m}_i = \sum_\lambda \mathbf{a}_\lambda \psi_\lambda(i) + \sum_\mu \mathbf{b}_\mu \Phi_\mu(i). \tag{2.29}$$

The first sum covers the interval $|E_\lambda - 2| \leqslant \varepsilon_d$, and the second the interval $2 - \varepsilon_\mu \geqslant \varepsilon_d$. The separation parameter ε_d can be chosen at any point in the region $1 \geqslant \varepsilon_d \geqslant Z^{-4/3}$: The results will not depend on the choice of ε_d.

The vectors $\mathbf{a}_\lambda$ are the correct "slow" variables. To find an effective Hamiltonian which depends on $\mathbf{a}_\lambda$, we substitute (2.29) into (2.3) and eliminate the variables $\mathbf{b}_\mu$. In deriving (2.3) we already integrated over the fluctuations of the fast degrees of freedom, so that the $\mathbf{b}_\mu$ have the meaning of mean values (at fixed values of $\mathbf{a}_\lambda$) and are determined by the condition $\partial F/\partial \mathbf{b}_\mu = 0$. Substituting in expression (2.3) for F, we find

$$\mathbf{b}_\mu = -\frac{Tn^2}{n+2}\frac{1}{2-E_\mu}\sum_{i,\lambda} \mathbf{a}_\lambda \psi_\lambda(i)\left[\sum_\eta \mathbf{a}_\eta \psi_\eta(i)\right]^2. \tag{2.30}$$

Here we have retained only the lowest order in $\mathbf{a}_\lambda$. Substituting expression (2.30) into (2.29) and (2.3), we find the effective Hamiltonian as a function of the slow variables $\mathbf{a}_\lambda$:

$$H\{\mathbf{a}_\lambda\} = -\frac{1}{2}\sum_{i,j} J_{ij}\tilde{\mathbf{m}}_i\tilde{\mathbf{m}}_j - \frac{1}{4nT}\sum_{i,j} J_{ij}^2 (1-\tilde{\mathbf{m}}_i^2)(1-\tilde{\mathbf{m}}_j^2)\cdot$$
$$+Tn\sum_i\left\{\frac{\tilde{\mathbf{m}}_i^2}{2} + \frac{n}{4(n+2)}(\tilde{\mathbf{m}}_i^2)^2 + \frac{n^2(n+8)}{6(n+4)(n+2)^2}(\tilde{\mathbf{m}}_i^2)^3\right\}$$
$$-\frac{1}{2}\left(\frac{Tn^2}{n+2}\right)^2 \sum_{i,j} \tilde{\mathbf{m}}_i^2 \tilde{\mathbf{m}}_i g(i,j)\tilde{\mathbf{m}}_j\tilde{\mathbf{m}}_j^2, \tag{2.31}$$
$$\tilde{\mathbf{m}}_i = \sum_\lambda \mathbf{a}_\lambda \psi_\lambda(i),$$
$$g(i,j) = \sum_\mu \frac{\Phi_\mu(i)\Phi_\mu(j)}{2-E_\mu}\theta(2-\varepsilon_d - |E_\mu|).$$

The last term in (2.31) arises from the elimination of the variables $\mathbf{b}_\mu$; it is of sixth order in $\mathbf{a}_\lambda$. In the derivation of an effective interaction (the Landau theory) for an ordinary ferromagnet it is sufficient to consider only the terms of fourth order in the slow variables. In the theory of spin glasses, as we will see presently, the coefficient of the terms of fourth order in $\mathbf{a}_\lambda$ is small, so we will retain terms of up to sixth order in $\mathbf{a}_\lambda$.

3. CONDENSATION AND INTERACTION OF LOCALIZED MODES

1. Let us consider interaction (2.31). The coefficients of the powers of the variables $\mathbf{a}_\lambda$ in this expression are terms of the type

$$\sum_i \psi_{\lambda_1}(i)\psi_{\lambda_2}(i)\psi_{\lambda_3}(i)\psi_{\lambda_4}(i).$$

The functions $\psi_\lambda(i)$ are random and independent, so that a sum of this sort with $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$ is much larger than a sum with indices which are not equal in pairs. We therefore single out the part of the Hamiltonian, H_0, in which all the coefficients have indices which are equal in pairs; alternatively, and equivalently, we single out that part of the Hamiltonian which does not depend on the signs of $\mathbf{a}_\lambda$: $H = H_0 + H_1$. As we have already explained, and as we will verify by calculations below, the condition $H_0 \gg H_1$ holds. To determine a_λ^2 it is sufficient to analyze H_0:

$$H_0 = \frac{1}{2}\sum_\lambda (\tau^2 - \varepsilon_\lambda) a_\lambda^2$$
$$-\frac{1}{4(1+\tau)}\sum_{\lambda,\mu} a_\lambda^2 a_\mu^2 \sum_{i,j} \psi_\lambda^2(i)\psi_\lambda^2(j) J_{ij}^2$$
$$+(1+\tau)\frac{n}{4(n+2)}\sum_{\lambda,\mu}[2(\mathbf{a}_\lambda\mathbf{a}_\mu)^2 + a_\lambda^2 a_\mu^2]$$
$$\times \sum_i \psi_\mu^2(i)\psi_\lambda^2(i)\left(1 - \frac{2}{3}\delta_{\lambda\mu}\right)$$
$$+\frac{n^2(n+8)}{6(n+4)(n+2)^2}\sum_{\lambda,\mu,\rho}\{a_\lambda^2 a_\mu^2 a_\rho^2 + 2[a_\rho^2(\mathbf{a}_\lambda\mathbf{a}_\mu)^2 + a_\mu^2(\mathbf{a}_\lambda\mathbf{a}_\rho)^2$$
$$+a_\lambda^2(\mathbf{a}_\mu\mathbf{a}_\rho)^2]$$
$$+8(\mathbf{a}_\lambda\mathbf{a}_\mu)(\mathbf{a}_\mu\mathbf{a}_\rho)(\mathbf{a}_\rho\mathbf{a}_\lambda)\}\sum_i \psi_\lambda^2(i)\psi_\mu^2(i)\psi_\rho^2(i)$$
$$\times\left[1 - \frac{2}{3}(\delta_{\lambda\mu}+\delta_{\mu\rho}+\delta_{\lambda\rho}) + \frac{16}{15}\delta_{\lambda\mu\rho}\right]$$
$$-\frac{1}{2}\left(\frac{n}{n+2}\right)^2\sum_{\lambda,\mu,\rho}[2(\mathbf{a}_\lambda\mathbf{a}_\mu)^2 a_\rho^2 + 4(\mathbf{a}_\lambda\mathbf{a}_\mu)(\mathbf{a}_\mu\mathbf{a}_\rho)(\mathbf{a}_\rho\mathbf{a}_\lambda)]$$
$$\times \sum_{i,j} \psi_\lambda(i)\psi_\mu(i)\psi_\rho(i) g(i,j)\psi_\lambda(j)\psi_\mu(j)\psi_\rho(j)$$
$$\times [1 - (\delta_{\lambda\mu}+\delta_{\mu\rho}+\delta_{\lambda\rho}) + C_1\delta_{\lambda\mu\rho}]. \tag{3.1}$$

Here $C_1 \sim 1$ is a combinatorial numerical factor, whose exact value will not be needed here. In deriving (3.1) we made use of the circumstance that the temperature is near the transition point, and we retained only the second order in τ in the

term quadratic in the amplitudes and the first order in the term of fourth order. We also made use of the orthogonality of $g(i,j)$ and $\psi_\lambda(i)$, i.e., the relation

$$\sum_i g(i,j)\psi_\lambda(i)=0,$$

which follows from the orthogonality of $\psi_\lambda(i)$, $\Phi_\mu(i)$. This circumstance allowed us to discard terms of the type

$$\sum_{\lambda,\mu,\rho} a_\lambda^2 a_\mu^2 a_\rho^2 \sum_{i,j} \psi_\lambda^2(i)\psi_\mu(j)g(i,j)\psi_\mu(j)\psi_\rho^2(j), \qquad (3.2)$$

since, as we will see below, we have

$$\sum_\lambda a_\lambda^2\psi_\lambda^2(i)\approx B,$$

where B is independent of i, and terms of this sort can be ignored in comparison with those retained in the last term in the energy in (3.1).

The functions $\Phi_\mu(i)$ in the definition of $g(i,j)$ are deep in the region of delocalized states, so that we can use the approximation $Z=\infty$ in calculating $g(i,j)$. We find

$$\sum_{i,j}\psi_\mu(i)\psi_\lambda(i)\psi_\rho(i)g(i,j)\psi_\mu(j)\psi_\lambda(j)\psi_\rho(j)$$
$$=\sum_i \psi_\mu^2(i)\psi_\rho^2(i)\psi_\lambda^2(i). \qquad (3.3)$$

Substituting (3.3) into (3.1), we can put H_0 in a more convenient form. Before we proceed with the calculations, we would like to discuss the qualitative behavior of $\langle a_\lambda^2\rangle$ which follows from the form of H_0.

At high temperatures ($\tau^2\gg\varepsilon_0$) the coefficient of a_λ^2 in the first term in the energy in (3.1) it is positive for nearly all states λ except for an exponentially small term in the state-density tail. Only these states acquire $\langle a_\lambda^2\rangle\neq 0$ in the mean field approximation (i.e., when the thermal fluctuations of a_λ near the solution of the equation $\delta H/\delta \mathbf{a}_\lambda=0$ are ignored). In this temperature region, it is of course improper to ignore these fluctuations, but we will show below that in the "critical"region, $-\tau\gg\varepsilon_0^{1/2}$, the thermal fluctuations of a_λ^2 can indeed be ignored, so we will study the equation $\delta H_0/\delta a_\lambda^2=0$. As the temperature is lowered, and τ^2 approaches ε_0, the number of states with $a_\lambda^2\neq 0$ (which we will call "filled") increases, and at $\tau^2\sim\varepsilon_0$ these states begin to overlap each other. The terms corresponding to the interaction of different a_λ^2 become large.

Let us digress for a moment to explain the difference between the transition which we are discussing here and that which occurs in a ferromagnet or superconductor with a random T_c, in which phase-uncorrelated superconducting droplets form above the transition point,[33] and the interaction between these droplets increases as the temperature approaches T_c (the point of the phase transition in the mean field theory). In a ferromagnet or superconductor, all forms of the interaction between droplets are comparable in magnitude, so that a phase transition which orders the phases of the different droplets occurs in the region where the interaction between the droplets reaches a value on the order of T. In a spin glass we have a different situation. Here, only the interaction between the moduli of the order parameters in each droplet becomes large at $\tau^2\sim\varepsilon_0$, while the interaction between the signs of $\mathbf{a}_\lambda$ (in the case of the Ising model) or their directions (in the case of vector fields) remains small. Consequently, no ordering or freezing of any sort has yet occurred at these temperatures.

As the temperature is lowered further, the number of filled states increases, and each state interacts with a large number of neighbors. As long as we are concerned with only H_0, all the interactions are of the same sign, so that we can construct a mean field theory for the interaction of the squares of the amplitudes, a_λ^2, at these temperatures. For this purpose we introduce the molecular field $B_\lambda^{\alpha\beta}$ acting on $a_\lambda^\alpha a_\lambda^\beta$:

$$B_\lambda^{\alpha\beta}=\sum_\mu a_\mu^\alpha a_\mu^\beta\left(1-\frac{2}{3}\delta_{\mu\lambda}\right)\sum_k \psi_\mu^2(k)\psi_\lambda^2(k). \qquad (3.4)$$

The particular form of this field has been chosen so that it is as independent as possible of the index λ. The final result does not depend on the details of this choice (i.e., on the coefficients of $\delta_{\mu\lambda}$), as it should not. Making use of the independence of the ψ_μ^2 with different indices, we find

$$B_\lambda^{\alpha\beta}=\frac{1}{n}\delta_{\alpha\beta}B+b_\lambda^{\alpha\beta}. \qquad (3.5)$$

In the lowest-order approximation, we can ignore the fluctuations $b_\lambda^{\alpha\beta}$ altogether. The subsequent calculations take slightly different paths for the cases of Ising and vector spins, and we will consider these two cases in succession. For Ising spins ($n=1$), the substitution of (3.5) into (3.1) and the variation of H_0 with respect to δa_λ^2 lead to an equation for a_λ^2:

$$(\tau+B)^2-\varepsilon_\lambda+\frac{8}{3}a_\lambda^2 B\sum_i\psi_\lambda^4(i)=0. \qquad (3.6)$$

In deriving (3.6) we have ignored terms of fourth order in a_λ (which are small, as we will see from the solution below), and we have ignored the difference between α and 3 [see (2.28)]; this difference would lead to a change on the order of $Z^{-2/3}$ in the quantity B which is the coefficient in the third term in Eq. (3.6). The self-consistency condition requires $\tau+B\approx 0$, so we have $B\sim-\tau\gg Z^{-2/3}$. Solving Eq. (3.6) for a_λ^2, and substituting the resulting expression into the definition of B in (3.5), we find the self-consistency equation

$$-\tau\approx B=\overline{\sum_\mu a_\mu^2\psi_\mu^2(i)}=\frac{3}{8}\int_{\varepsilon_c+\xi}^{\infty}\rho(\varepsilon)\frac{\varepsilon-\varepsilon_c-\xi}{\tau}l^d(\varepsilon)\,d\varepsilon, \qquad (3.7)$$

where we have used (2.27), and we have set $\xi=(\tau+B)^2-\varepsilon_c$. Solving this equation for ξ, using expression (2.26) for $l(\varepsilon)$, and setting $\rho(\varepsilon)=\rho_0\approx\varepsilon_0^{1/2}$, we find

$$\xi=\varepsilon_0\zeta,\quad \zeta=\left\{\frac{3}{8\tau^2}\rho_0 l_0^d\left(\frac{1}{d\nu-2}-\frac{1}{d\nu-1}\right)\right\}^{(d\nu-2)^{-1}}. \qquad (3.8)$$

It follows from (3.8) that under the condition $d\nu\geqslant 2$ the

quantity ζ remains finite and greater than zero as τ increases deep in the negative region; the delocalized states corresponding to $\zeta \leqslant 0$ are never filled in this case. At $d\nu < 2$ we have a qualitatively different situation. Here the integral on the right side of (3.7) remains finite even at negative values of ξ, so that when τ is negative and sufficiently large a delocalized state is filled, and an ordinary phase transition occurs. Assuming $\nu = (d-2)^{-1}$, we find the critical dimensionality $d_c = 4$. In three-dimensional space we then have

$$\zeta = \varepsilon_0/\tau^2. \tag{3.9}$$

We can also write an expression which we will need below:

$$a_\lambda^2 = \frac{3}{8}\frac{\varepsilon_\lambda - (\varepsilon_c + \varepsilon_0\zeta)}{B\sum_i \psi_\lambda^4(i)} \approx \frac{\varepsilon_\lambda - (\varepsilon_c+\varepsilon_0\zeta)}{|\tau|} l^3(\varepsilon_\lambda). \tag{3.10}$$

We turn now to the model with a vector spin; we wish to derive expressions analogous to (3.9) and (3.10). Even at $Z = \infty$, the phase transition in the vector model ($n \neq 1$) occurs in a slightly different way (from that in the case $n = 1$): One mode (which lies at the edge of the spectrum $E = 2$) does not become unstable at $\tau < 0$, since the coefficient of $\mathbf{a}^4$ remains positive at $\tau < 0$, as can be seen from (3.1), in contrast with the case $n = 1$, in which it is proportional to τ. If we choose n modes lying at the edge of the spectrum, orthogonal in pairs and equal in absolute value to the amplitudes $\mathbf{a}_\lambda$, we find that the coefficient of a^4 becomes proportional to τ, as in the case of Ising spins. This result means that at $n \geqslant 2$ the condensation necessarily occurs to a state with a vector spin, which is directed in a random, spherically isotropic way at different points in space, as expected.

At finite values of Z, the tensor $B_\lambda^{\alpha\beta}$ [see (3.5)] is nearly isotropic ($b_\lambda^{\alpha\beta} \ll B$), for a similar reason. Substituting (3.5) into (3.1), and retaining only the leading terms in τ and a_λ^2, we find

$$(\tau+B)^2 - \varepsilon_\lambda + \frac{2}{3}\left[\frac{n-1}{n+2}\right]\sum_i \psi_\lambda^4(i) a_\lambda^2 = 0. \tag{3.11}$$

Carrying out calculations analogous to the derivation of (3.9) and (3.10), we find

$$\zeta = \varepsilon_0/\tau, \quad \xi = \zeta\varepsilon_0,$$
$$a_\lambda^2 = \frac{3}{2}(n+2)(n-1)^{-1}[\varepsilon_\lambda - (\varepsilon_c+\xi)] \Big/ \sum_i \psi_\lambda^4(i). \tag{3.12}$$

The equation for the critical dimensionality, $d_c \nu(d_c) = 2$, is the same as that for an Ising spin glass, and it gives us $d_c = 4$ if $\nu = (d-2)^{-1}$.

Let us estimate the thermal fluctuations of a_λ^2 near solution (3.10) or (3.12). Making use of the particular form of H_0 in (5.1), we find, for typical states with $\varepsilon_\lambda - \varepsilon_c \sim \xi(\tau)$,

$$\langle (a^2 - a_0^2)^2 \rangle^{1/2} \sim \begin{cases} Z^{4/3}(\tau/\tau_0)^{5/2}, & n=1, \\ Z^2(\tau/\tau_0)^{1/2}, & n \geqslant 2 \end{cases} \tag{3.13}$$

$$\tau_0 = \varepsilon_0^{1/2},$$

where a_0^2 is the solution of (3.10) or (3.12). Comparing (3.13) with (3.10) or (3.12), we see that the thermal fluctuations are small in the critical region, in which we are interested here:

$$\langle (a^2 - a_0^2)^2 \rangle^{1/2} / a_0^2 \sim (\tau_0/\tau)^{1/2}. \tag{3.14}$$

We can also estimate the spatial fluctuations of B_λ which we have discarded: $\delta B \sim B\tilde{Z}^{-1/2}$, where $\tilde{Z} = \zeta^{-2}$ is the number of typical states with $\varepsilon_\lambda - \varepsilon_c \sim \xi(\tau)$, which also dominate the fluctuations over a scale length $l(\xi)$. We find $(\delta B)^2 \sim \varepsilon_0(\tau_0/\tau)^2 \sim \xi$ for Ising spins and $(\delta B)^2 \sim \varepsilon_0^2 \ll \xi$ for vector spins, justifying our neglect of the fluctuations of $(B_\lambda + \tau)^2$ in the calculation of a_λ^2.

We see thus that as the temperature is lowered, a gradual (in the case of Ising spins) filling of states of progressively larger dimensions begins at $\tau \sim -Z^{-2/3}$. In the case of vector spins, this filling begins immediately at states of large size ($\zeta \sim Z^{-2/3} \ll 1$), as follows from (3.12). This result means that there is a transitional temperature region $|\tau| \lesssim Z^{-2/3}$ in this case in which states of small size ($\zeta > Z^{-2/3}$) are filled. In this region, however, the thermal fluctuations are large, and in estimating $\langle a^2 \rangle$ we cannot use simply the solution of the equation $\delta H/\delta a^2 = 0$. Let us evaluate $\langle a^2 \rangle$ in this temperature region from the expression for H_0 derived in the mean field approximation:

$$H_0 = \frac{1}{2}\sum_\lambda [(\tau+B)^2 - \varepsilon_\lambda] a_\lambda^2 + \frac{1}{6}\frac{n-1}{n+2}\sum_\lambda a_\lambda^4 \sum_i \psi_\lambda^4(i). \tag{3.15}$$

In the temperature regions in which we are interested here, either the number of neighbors is large, so that the mean field approximation is legitimate, or we can ignore the interaction of different states altogether (in which case we have $B = 0$), and expression (3.15) will also be valid. We find the estimate

$$\langle a_\lambda^2 \rangle \approx \min\{l^{3/2}(\varepsilon_\lambda),\ (\varepsilon_c + \xi - \varepsilon_\lambda)^{-1}\}, \quad \xi = (\tau+B)^2 - \varepsilon_c. \tag{3.16}$$

At high temperatures, the values of $\langle a_\lambda^2 \rangle$ are small, and the interaction of different states can be ignored. As the temperature is lowered, the values of $\langle a_\lambda^2 \rangle$ increase. At what temperature does the interaction become important? To answer this question, we seek the value of B by perturbation theory, setting $B = B_0 = 0$ in solution (3.16) and evaluating the correction B_1 from definition (3.5) of B:

$$B_1 = \frac{1}{\mathcal{N}}\sum_\mu a_\mu^2 \approx Z^{-4/3}(\tau^2 - \varepsilon_c)^{-1/2}. \tag{3.17}$$

The correction B_1 has little effect on solution (3.16) if $\tau B \ll \tau^2 - \varepsilon_c$, and this condition holds if τ satisfies

$$\tau - \varepsilon_c^{1/2} \gg Z^{-2/3}. \tag{3.18}$$

Even at temperatures $\tau \sim Z^{-2/3}$, we cannot ignore the term of the a_λ^4 type in the energy H_0, so that even at these temperatures the results of the ordinary diagram version of

the mean field theory do not apply. Those results lead, in particular, to a τ^{-1} dependence of the nonlinear susceptibility:

$$\tilde{\chi}=\partial^2\chi/\partial h^2=\int \overline{\langle \mathbf{S}(0)\mathbf{S}(\mathbf{x})\rangle^2} d^3x.$$

At lower temperatures, the interaction between different states cannot be ignored. Analytic expressions can be derived for the temperature region

$$Z^{-2/3}\gg \varepsilon_c^{1/3}-\tau \gg Z^{-5/6}. \tag{3.19}$$

The temperature region $|\varepsilon_c^{1/2}-\tau|\sim Z^{-5/6}$ is a region of transition between two regimes. Let us determine B in region (3.19), using (3.5) and (3.16):

$$B\approx Z^{-4/3}\xi^{-1/3}. \tag{3.20}$$

As before, the self-consistency equation follows from the definition of ξ: $(\tau+B(\xi))^2-\varepsilon_c-\xi=0$. We should substitute (3.20) into it. For B in this temperature region, however, we cannot ignore the term ε_c in the resulting equation (in contrast with the situation in the region $-\tau \gg Z^{-2/3}$). From (3.20) we find $\xi\sim Z^{-4}B^{-3}\ll\varepsilon_c$, so that we can ignore only ξ in the resulting self-consistency equation. We have

$$B=\varepsilon_c^{1/3}-\tau, \quad \xi=Z^{-4}(\varepsilon_c^{1/3}-\tau)^{-3}. \tag{3.21}$$

The states which dominate B (the so-called typical states) have eigenvalues ε such that $\varepsilon-\varepsilon_c\sim Z^{-2/3}\xi^{2/3}$; i.e., the number of nearest neighbors of a given typical state, $\tilde{Z}\sim[(\varepsilon-\varepsilon_c)/\varepsilon_c]^{1-d\nu}$, increases as the temperature is lowered, from $Z^{2/3}$ at $\varepsilon_c^{1/2}-\tau\sim Z^{-5/6}$ to $Z^{4/3}$ at $\varepsilon_c^{1/2}-\tau\sim Z^{-2/3}$. This effect justifies our use above of the mean field approximation. At the lower boundary of temperature region (3.19), the eigenvalues ε of the typical states are such that $\varepsilon-\varepsilon_c\sim\varepsilon_0$. As the temperature is lowered further, the thermal fluctuations of a_λ^2 become smaller than a_0^2 for typical states, while the eigenvalues of the typical states, as we have already mentioned, continue to satisfy the relation $\varepsilon-\varepsilon_c\sim\xi$. There is accordingly a joining of temperature region (3.19) with the region $-\tau\gg Z^{-2/3}$.

2. As we have already mentioned, the interaction of the signs of the amplitudes a_λ (in the case of Ising spins) or the directions of $\mathbf{a}_\lambda$ (in the case of vector spins) is weak in comparison with the interaction of their moduli, a_λ^2, so that at temperatures $-\tau\gtrsim Z^{-2/3}$ there is still no sort of ordering at all in the system as a whole.

In the case of Ising spins, the weakness of the interaction of the signs of the amplitudes a_λ follows from the fact that the coefficients of the type $\Sigma_i\psi_\lambda(i)\psi_\mu(i)\psi_\nu^2(i)$—coefficients of the terms in the expression for the energy, which depend on the signs of a_λ—are small. In the case of vector spins, Hamiltonian H_0 in (3.1) contains, not terms which appear small at first glance, containing $(\mathbf{a}_\mu\mathbf{a}_\lambda)^2$, or similar terms. However, in the leading order these terms lead to only the isotropy of the tensor $B^{\alpha\beta}\approx n^{-1}B\delta_{\alpha\beta}$, as discussed above. An analogous situation arises in the problem of magnetic moments which are distributed at random in a matrix (each having a large number of neighbors) and which interact with each other in an antiferromagnetic way. A strong antiferromagnetic interaction leads to only a very accurate vanishing of the total moment of the system, while an ordering of the individual spins occurs only because of fluctuations of the interaction between them. Similarly, again in our case the ordering of the directions of $\mathbf{a}_\mu$ is caused exclusively by fluctuations of the interaction, i.e., of the coefficient $\Sigma_i\psi_\lambda^2(i)\psi_\mu^2(i)$. Let us estimate the fluctuations of this quantity, assuming that the individual terms of the sum are random and independent quantities at different points i. We find

$$\Psi_Q=\overline{\Big(\sum_i \psi_\lambda^2(i)\psi_\mu^2(i)\Big)^2}-\Big(\overline{\sum_i \psi_\lambda^2(i)\psi_\mu^2(i)}\Big)^2\sim Z^{-3}\xi^{9/2}. \tag{3.22}$$

The effective interaction can now be estimated by the same method as is ordinarily used to estimate the effective value of a completely random binary interaction: $I_{\text{eff}}^{(Q)}\sim\tilde{Z}^{1/2}I_Q$, where $\tilde{Z}$ is the number of neighbors of the given typical state. In our case we have $I_Q\sim a_0^4\Psi_Q$, where a_0 is the amplitude of a typical state, so that [see (3.12)]

$$I_{\text{eff}}^{(Q)}\sim Z^{-4/3}(|\tau|/\tau_0)^{1/3}\ll 1. \tag{3.23}$$

It follows from (3.23) that the quantity $I_{\text{eff}}^{(Q)}$ can be ignored over the entire temperature region (we recall that $|\tau|\lesssim 1$); i.e., when we take into account only $H_0\{\mathbf{a}_\lambda\}$, no ordering of the directions of the amplitudes $\mathbf{a}_\lambda$ occurs at any temperature. We can therefore move on to a study of the effect of $H_1\{\mathbf{a}_\lambda\}$.

Let us evaluate the magnitude of the sign interaction or of the vector interaction (in the case of vector spins) of the amplitudes $\mathbf{a}_\lambda$ which stems from the part of the Hamiltonian $H_1\{\mathbf{a}_\lambda\}$. The magnitude of the binary interaction between the signs or directions, $\sigma_\lambda=\mathbf{a}_\lambda/|\mathbf{a}_\lambda|$, is determined by a term of the type

$$H_\sigma=\frac{1}{2}\sum_{\lambda\mu} I_{\lambda\mu}\sigma_\lambda\sigma_\mu,$$

$$I_{\lambda\mu}\approx C_2|\tau|\,|\mathbf{a}_\lambda|\,|\mathbf{a}_\mu|\sum_i \psi_\mu(i)\psi_\lambda(i)\sum_\nu \langle a_\nu^2\rangle\psi_\nu^2(i), \tag{3.24}$$

where $C_2\sim 1$. In deriving (3.24) we made use of the isotropy (in the case of vector spins) of the thermal expectation values $\langle a_\nu^\alpha a_\rho^\beta\rangle$. If we use the replacement

$$\sum_\nu \langle a_\nu^2\rangle\psi_\nu^2(i)=B(i)\to B$$

in a first approximation in expression (3.24), then the orthogonality of the different eigenfunctions leads to the vanishing of $I_{\mu\lambda}$, so that only the spatial fluctuations of $B(i)$ (δB_i) contribute to $I_{\mu\lambda}$. The states at the boundary of the spectrum, i.e., the states with $\varepsilon-\varepsilon_c\sim\varepsilon_0$, dominate δB_i. Evaluating δB_i in this manner, we find $|\delta B_i|\sim\tau_0\sim Z^{-2/3}$ in the case of Ising spins or $|\delta B_i|\propto Z^{-1}$ in the case of vector spins.

Substituting this estimate into expression (3.24), we find the effective magnitude of the sign interaction:

$$I \approx \tilde{Z}^{1/2}(\overline{I_{\mu\nu}^2})^{1/2} \sim (|\tau|/\tau_0)^3 Z^{-1} \quad \text{(Ising spins)}, \tag{3.25}$$

$$I \sim (|\tau|/\tau_0)^{5/2} Z^{-1} \quad \text{(vector spins)}. \tag{3.26}$$

At $|\tau| \sim \tau_0$, the interaction I of the signs (or directions) of the a_λ is weak, as expected. As the temperature is lowered, I increases, and at

$$-\tau = \tau_1 \sim Z^{-p} \tag{3.27}$$

(where $p = 1/3$ in the case of Ising spins or $p = 4/15$ in the case of vector spins) reaches a value of order unity. The sign variables $\sigma_\lambda = a_\lambda/|a_\lambda|$ (or the directions $\mathbf{\sigma}_\lambda$) interact with each other in a manner similar to that of spins in the original formulation of the model. Accordingly, at $-\tau \approx \tau_1$ we find an effective Hamiltonian of the type in (2.1) near the transition point for the variables σ_λ; here the quantity $\tilde{\tau} = (\tau + \tau_1)/\tau_1$ serves as the reduced temperature. The effective number ($Z_1 = \tilde{Z}$) of nearest neighbors of the new "spins" is on the order of the number of "typical states" in the volume spanned by this state. It is easy to see that we have $Z_1 \sim \xi^{-2}(\tau_1)$, where ξ is determined by (3.9) and (3.12) in the cases $n = 1$, $n \geqslant 2$, respectively. As a result we find

$$Z_1 \sim Z_0^q \quad (Z_0 \equiv Z), \tag{3.28}$$

where $q = 4/3$ for $n = 1$ or $q = 32/15$ for $n \geqslant 2$. It is very important to note that we have $q > 1$; i.e., at $Z_0 \gg 1$ we find $Z_1 \gg Z_0$. This result means that the accuracy of the expansion in $1/Z_0$ which we used above improves as we go from the original spins σ_i to the "block spins" σ_λ.

Estimating the correction for the four-spin interaction $(\mathbf{a}_1\mathbf{a}_2)(\mathbf{a}_3\mathbf{a}_4)$ by perturbation theory, we can show that this interaction is insignificant near $\tau = -\tau_1$.

Singling out the critical slow variables in the problem of an Ising (or vector) gas has thus led to variables whose interactions is again described by the Hamiltonian of an Ising (or vector) spin glass.

As the temperature is reduced further, the critical slow variables constructed from the spins σ_λ convert into second-level spins, for which we construct an effective Hamiltonian; etc. A spin glass is therefore a hierarchy of superparamagnets. At a certain temperature $T = T_f$ the number of levels of this hierarchy becomes infinite; i.e., the degrees of freedom of an arbitrarily large spatial dimension interact strongly, so that T_f would naturally be considered the temperature of the freezing phase transition. This assumption is confirmed below, where we show that the nonlinear magnetic susceptibility $\tilde{\chi}$ and the maximum spin relaxation time $t_{\max}$ become infinite in the limit $T \to T_f$.

For quantitative estimates of the behavior of the physical quantities we need the number of nearest neighbors at the Nth level of the hierarchy, Z_N, and we also need the value of $T_f - T_N$, where N_N is the physical temperature at which the effective temperature of the Nth level of the hierarchy reaches a value on the order of unity. Using Eqs. (3.27)–(3.28) for a transition to a higher level of the hierarchy N times, we find

$$\ln Z_N \approx q^N \ln Z_0, \tag{3.29}$$

$$-\ln \frac{T_N - T_f}{T_f} \approx \frac{p}{q-1}(q^{N+1} - 1)\ln Z_0. \tag{3.30}$$

At $T = T_N$, the spins of the Nth level of the hierarchy are strongly correlated with each other (this temperature corresponds to the point $T = 1$, $\tau = 0$ for the zeroth level). The temperatures T_N^* corresponding to the point $T = 2$ of the zeroth level behave in an analogous way:

$$\frac{T_N^* - T_f}{T_f} \approx \frac{T_{N-1} - T_f}{T_f} \quad (T_{-1} \equiv 2T_f). \tag{3.31}$$

At T_N^*, we can completely ignore the correlations of the spins of the Nth level of the hierarchy. We recall that Eqs. (3.27)–(3.28) and thus all the subsequent equations are derived at an accuracy to within unknown numerical factors, so that the symbol $\approx$ in (3.29)–(3.30) means that we are retaining the fastest functional dependences on the number N.

4. NONLINEAR SUSCEPTIBILITY

In this section we will determine the nature of the singularity in the nonlinear magnetic susceptibility $\tilde{\chi}$ as $T \to T_f + 0$. We first express $\tilde{\chi}$ in terms of spin correlation functions. We use the definition

$$\tilde{\chi}_{\alpha\beta\gamma\delta} = \frac{\partial^4 F}{\partial h_\alpha \partial h_\beta \partial h_\gamma \partial h_\delta} = \frac{1}{3}\tilde{\chi}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{4.1}$$

Making use of the macroscopic spatial isotropy, we then find

$$\tilde{\chi} = -\frac{1}{\mathcal{N}}\frac{3}{n(n+2)}\sum_{ijkl} \langle\!\langle (\mathbf{\sigma}_i\mathbf{\sigma}_j)(\mathbf{\sigma}_k\mathbf{\sigma}_l) \rangle\!\rangle. \tag{4.2}$$

The terms with indices which are not equal in pairs in sum (4.2) contribute nothing when an average is taken over the realizations, so they do not contribute to sum (4.2). Making use of the definition of the irreducible correlation function and also the identity $\mathbf{\sigma}_i^2 = 1$, and retaining only the terms which grow as $T \to T_f$, we find

$$\tilde{\chi} = \frac{1}{\mathcal{N}}\frac{6}{n^2}\sum_{ij} \langle \mathbf{\sigma}_i\mathbf{\sigma}_j \rangle^2. \tag{4.3}$$

In the derivation of (4.3) we made use of the spatial isotropy of all the thermal expectation values above the transition point (e.g., $\langle \sigma_i^\alpha \sigma_j^\beta \rangle = \delta_{\alpha\beta} n^{-1} \langle \mathbf{\sigma}_i\mathbf{\sigma}_j \rangle$), which holds even without an averaging over realizations. Near the transition point the spin correlation function which appears in sum (4.3) can be expressed in terms of the correlation function of the slow variables $\mathbf{a}_\lambda$:

$$\tilde{\chi} = \frac{1}{\mathcal{N}}\frac{6}{n^2}\sum_{\lambda} \langle \mathbf{a}_\lambda \mathbf{a}_\mu \rangle^2. \tag{4.4}$$

In the temperature region $-\tau < \tau_1$, the interaction of

the signs (or directions) of the amplitudes $\mathbf{a}_\lambda$ can be ignored, and we can assume that $\mathbf{a}_\lambda$ and $\mathbf{a}_\mu$ are uncorrelated if $\lambda \neq \mu$. In this case we have

$$\tilde{\chi} = \frac{6}{n^2}\frac{1}{\mathcal{N}}\sum_\lambda \langle a_\lambda^2\rangle^2 = \frac{6}{n^2}\int_\xi^\infty \rho(\varepsilon)\langle a^2(\varepsilon)\rangle^2 d\varepsilon. \tag{4.5}$$

Outside the critical region ($\tau \gg \tau_0$), expression (4.5) is the same as the result of a high-temperature expansion: $\tilde{\chi} = 1/\tau$. At $\tau \sim \tau_0$, we have $\chi \sim Z^{2/3}$. The increase in the susceptibility as the temperature is reduced further takes very different paths in the cases of Ising and vector spins. We consider the Ising spins first. At temperatures $-\tau \gg \tau_0$, we can use expression (3.10) to calculate $\langle a^2\rangle$. We find

$$\tilde{\chi} \approx Z^{10/3}(-\tau)^4. \tag{4.6}$$

Expression (4.6) holds at temperatures to $-\tau \lesssim \tau_1$, at which we can ignore the interaction of the spins of the next level. At the lower boundary of the range of applicability of (4.6), $\tilde{\chi}$ increases to Z^2. As the temperature is lowered further, we go into a region where we can use a perturbation theory in terms of the spins of the next level of the hierarchy. We thus have

$$\tilde{\chi} \sim Z^2\left(\frac{\tau+\tau_1}{\tau_1}\right)^{-1}, \tag{4.7}$$

etc. As the temperature is lowered, and correlations of progressively higher levels appear, the nonlinear susceptibility increases rapidly. In each level, there are two regimes in the behavior of the susceptibility: a regime of the type in (4.6) and a "perturbation-theory" regime ($\tilde{\chi} \sim 1/\tilde{\tau}$). Let us determine the behavior of the susceptibility as we go from the point $T_0^* = 2T_0$ to the analogous point of the next level, T_1^*:

$$\tilde{\chi}(T_1^*) = Z^2 = Z^2\tilde{\chi}(T_0^*). \tag{4.8}$$

For transitions to the following levels we can use

$$\tilde{\chi}(T_{N+1}^*) = Z_N^2\tilde{\chi}(T_N^*);$$

we thus have

$$\ln\tilde{\chi}(T_N^*) \approx 6((4/3)^N - 1)\ln Z_0. \tag{4.9}$$

Comparing (4.9) with (3.30)–(3.31), we find

$$\tilde{\chi}(T_N^*) \sim (T_N^* - T_f)^{-6}. \tag{4.10}$$

Evaluating $\tilde{\chi}(T)$ at the points $T = T_N^{**} \equiv 2T_N - T_f$ in a similar way, and comparing the results with (3.30), we find

$$\tilde{\chi}(T_N^{**}) \sim (T_N^{**} - T_f)^{-19/4}. \tag{4.11}$$

This result means that the function $\tilde{\chi}(T)$ is not described by a common power law but instead oscillates between two power-law envelopes: $\tilde{\chi}_\pm(T) \sim (T - T_f)^{-\gamma_\pm}$; $\gamma_+ = 6$, $\gamma_- = 19/4$. We wish to stress that, despite these oscillations, $\tilde{\chi}(T)$ and the first three of its derivatives increase monotonically in the limit $T \to T_f$. In an actual experiment these oscillations would probably be more reminiscent of smooth deviations of $\tilde{\chi}(T)$ from an approximating power law. The behavior of the thermodynamic quantities near the freezing point of a spin glass thus turns out to be more complicated than an ordinary power-law singularity in the limit $T \to T_f$.

Now let us consider vector spins. At $\tau \gg \tau_0$, as in the case of Ising spins, we have $\tilde{\chi} \sim 1/\tau$, and this susceptibility is of course the same as in perturbation theory. At $-\tau \gg \tau_0$, the expectation value $\langle a^2\rangle$ in (4.5) can be replaced by a^2 from (3.12). Taking this approach, we find an expression analogous to (4.6):

$$\tilde{\chi} \approx Z^{10/3}(-\tau)^3. \tag{4.12}$$

A qualitative distinction from the case of Ising spins is the presence of an intermediate temperature region $Z^{-5/6} \ll \varepsilon_c^{1/2} - \tau \ll Z^{-2/3}$ in which $\langle a^2\rangle$ is determined by thermal fluctuations of $\mathbf{a}_\lambda$ near zero. In this region, we use expressions (3.16) and (3.21) for $\langle a_\lambda^2\rangle$, finding

$$\tilde{\chi} \approx Z^4(\varepsilon_c^{1/2} - \tau)^4. \tag{4.13}$$

At $\tau - \sqrt{\varepsilon_c} \sim Z^{-5/6}$, this expression yields $\tilde{\chi} \sim Z^{2/3}$ and thus joins with the high-temperature behavior $1/\tau$; at the lower boundary of its range of applicability, $\tau \approx -\tau_0$, we find $\tilde{\chi} \sim Z^{-4/3}$ from (4.13), in agreement with the value found from (4.12) at the boundary of its range of applicability. Consequently, in the case of vector spins the nonlinear susceptibility passes through three regimes at each level of the hierarchy as the temperature is lowered: a perturbation-theory regime ($\tilde{\chi} \sim 1/\tau$), a transition regime (4.11), and a "low-temperature regime," (4.10). After arguments analogous to those in the derivation of the expressions for the envelopes $\tilde{\chi}_+$ and $\tilde{\chi}_-$ in the case of Ising spins, we find the following expressions for the envelopes in the case of a vector spin:

$$\tilde{\chi}_\pm \sim (T - T_f)^{-\gamma_\pm}; \quad \gamma_+ = {}^{19}/_2, \quad \gamma_- = {}^{319}/_{64}. \tag{4.14}$$

At temperatures $T = T_N^*$, the nonlinear susceptibility $\tilde{\chi}$ agrees with its upper envelope $\tilde{\chi}_+$, while at $T = T_N^{**}$ it agrees with its lower envelope, $\tilde{\chi}_-$.

Expressions (4.10), (4.11), and (4.14) for the exponents $\gamma_\pm$ of the nonlinear susceptibility $\tilde{\chi}$ have been derived here on the basis of the hypotheses of localization theory, which we believe are the most plausible. In particular, we have assumed $l(\varepsilon) \sim (\varepsilon - \varepsilon_c)^{-\nu}$, $\nu = 1/(d-2) = 1$. Without derivation we now write corresponding expressions for $\gamma_\pm$ for the case of an arbitrary exponent ν (however, we are assuming that this value is greater than 2/3, so that the overall picture of the hierarchy is not changed):

$$\gamma_+ = \begin{cases} 10 - 4/\nu, & n=1 \\ 18\nu - {}^{17}/_2, & n \geq 2 \end{cases}, \tag{4.15}$$

$$\gamma_- = \begin{cases} \dfrac{60\nu^2 - 53\nu + 12}{2\nu(3\nu-1)}, & n=1 \\ \dfrac{1296\nu^2 - 1344\nu + 367}{32(3\nu-1)}, & n \geq 2 \end{cases} \tag{4.16}$$

Small deviations of ν from 1 strongly affect the value of $\gamma_\pm$, especially in the case of vector spins.

5. CRITICAL RELAXATION

As the temperature approaches the transition (from above), the number of levels of the hierarchy increases, so that long relaxation times appear; these times are associated with the relaxation of the "spin" of the highest level of the hierarchy at the given temperature. We will derive quantitative expressions for the maximum relaxation time t_{max} in the simple dynamic generalization of the Edwards-Anderson model with a purely dissipative dynamics. There is the hope that in the region of the critical slowing of the relaxation, which is the only region with which we are concerned, the purely dissipative dynamics will be a reasonable approximation of the actual situation. The purely dissipative dynamics of vector spins is described by the Langevin equation

$$\dot{\boldsymbol{\sigma}}_i=-\Gamma_0\frac{\delta H}{\delta\boldsymbol{\sigma}_i}+\boldsymbol{\eta}_i(t),$$
$$\langle\eta_i^{\alpha}(t)\,\eta_j^{\beta}(t')\rangle=2\Gamma_0T\delta_{ij}\delta^{\alpha\beta}\delta(t-t'), \tag{5.1}$$

where Γ_0 is the reciprocal of the one-spin relaxation time. The purely dissipative dynamics of Ising spins is described by Glauber equations, which are equivalent in meaning:

$$\dot{p}\{\sigma_i\}=\Gamma_0\sum_j\Big[-\exp\Big(-\frac{1}{T}\sum_k J_{jk}\sigma_k\Big)p\{\sigma_1\dots\sigma_j\dots\sigma_n\}$$
$$+\exp\Big(\frac{1}{T}\sum_k J_{jk}\sigma_k\Big)p\{\sigma_1\dots-\sigma_j\dots\sigma_n\}\Big], \tag{5.2}$$

where $p\{\sigma_i\}$ is the probability of the spin configuration $\{\sigma_i\}$. These dynamic equations retain the same form, apart from the replacement $\Gamma_0\to\Gamma_1$, on going to the next level of the hierarchy. Under the condition $\tau_0\lesssim(T-T_0)/T_0\ll1$, we can average over the fast spin fluctuations and reduce Eq. (5.1) or (5.2) to the form

$$\mathbf{m}_i=-\Gamma_0\frac{\delta F}{\delta m_i}+\boldsymbol{\eta}_i(t), \tag{5.3}$$

where the functional $F\{\mathbf{m}_i\}$ is given in (2.3). In deriving (5.3), we have ignored corrections to Γ_0 which are small, on the order of $\mathbf{m}_i^2$. We have not yet completely eliminated the fast degrees of freedom from Eq. (5.3), since the slow relaxation of the variables a_λ leads to a slow relaxation of the variables b_μ, by virtue of the interacton of modes [see (2.29)]. However, the intrinsic relaxation times of these modes are short, so that they can be eliminated, as was done in the static case (Section 2). As a result we find equations for the slow amplitudes:

$$\dot{\mathbf{a}}_\lambda=-\Gamma_0\frac{\delta H\{\mathbf{a}_\lambda\}}{\delta\mathbf{a}_\lambda}+\boldsymbol{\eta}_\lambda(t),$$
$$\langle\eta_\lambda^{\alpha}(t)\,\eta_\mu^{\beta}(t')\rangle=2T\Gamma_0\delta_{\lambda\mu}\delta^{\alpha\beta}\delta(t-t'), \tag{5.4}$$

where $H\{\mathbf{a}_\lambda\}$ is given in (2.31). At temperatures

$$\tau_0\ll(T_0-T)/T_0\lesssim\tau_1$$

we can ignore the part of $H\{\mathbf{a}_\lambda\}$ which depends on the signs of $\mathbf{a}_\lambda$, and the interaction of modes which does not depend on the signs can be taken into account by mean field theory, as explained in Section 3. Equations (5.4) then split up into the independent equations

$$\dot{\mathbf{a}}_\lambda=-\Gamma_0\frac{\partial H_0^{(\lambda)}}{\partial\mathbf{a}_\lambda}+\boldsymbol{\eta}_\lambda(t),$$
$$H_0^{(\lambda)}=\frac{1}{2}\mathbf{a}_\lambda^2[(\tau+B)^2-\varepsilon_\lambda]+(\mathbf{a}_\lambda^2)^2C\sum_i\psi_\lambda^4(i), \tag{5.5}$$

where $C=(n-1)/6(n+2)$ at $n\geqslant2$, and $C\sim|\tau|$ at $n=1$. Our only remaining task is to transform from (5.5) to the equations of motion for the "spins" $\boldsymbol{\sigma}_\lambda=\mathbf{a}_\lambda/|\mathbf{a}_\lambda|$. In the case of a vector spin, it is sufficient to substitute $\mathbf{a}_\lambda=\boldsymbol{\sigma}_\lambda(a_\lambda^2(\tau))^{1/2}$ into (5.5) [where $a_\lambda^2(\tau)$ is determined by the minimum of $H_0^{(\lambda)}$; see (3.12)], since the direction of $\boldsymbol{\sigma}_\lambda$ fluctuates without a change in $|\mathbf{a}_\lambda|$. As a result we find

$$\Gamma_1=\Gamma_0/a_{\lambda_0}{}^2\approx\Gamma_0Z_0^{-2}(\tau_0/\tau_1)^2=\Gamma_0Z_0^{-1/4}, \tag{5.6}$$

where we have taken the $a_{\lambda_0}^2$ to be the values of a_λ^2 for typical states at $-\tau\approx\tau_1$, i.e., for those states which are responsible for the appearance of correlations of the next level. In the case of Ising spins, a change in the sign of a_λ is possible only through a passage of a_λ across a barrier, in a process involving an activation, into the region $a_\lambda\ll a_{\lambda_0}$. The height of this barrier is $\Delta H^{(\lambda)}\approx|\tau|/\tau_0\gg T\approx1$, so that such processes occur exponentially rarely. Estimating their frequency (Γ_1) by means of the ordinary Kramers formula, applied to the potential $H_0^{(\lambda)}$, we find

$$\Gamma_1\approx\Gamma_0Z_0^2\exp(C_3Z_0^{1/4}),\quad C_3\sim1. \tag{5.7}$$

It is the quantity Γ_1 which appears in Glauber equations of the type in (5.2) for the "spins" σ_λ. The recursive transformation which we have been seeking has thus been constructed, and we can write expressions for the quantities Γ_N:

$$\Gamma_N=\Gamma_0\exp\Big\{-\frac{14}{5}\sum_{m=0}^{N-1}\ln Z_m\Big\}\quad(n\geqslant2), \tag{5.8}$$

$$\Gamma_N=\Gamma_0\exp\Big\{-2\sum_{m=0}^{N-1}\ln Z_m\Big\}\exp\Big\{-\sum_{m=0}^{N-1}Z_m^{1/4}\Big\}\quad(n=1). \tag{5.9}$$

Using (3.29), we can put these expressions in the form

$$\Gamma_N=\Gamma_0\exp\Big\{-\frac{42}{17}\Big[\Big(\frac{32}{15}\Big)^N-1\Big]\ln Z_0\Big\}\quad(n\geqslant2), \tag{5.10}$$

$$\Gamma_N=\Gamma_0\exp\Big\{-6\Big[\Big(\frac{4}{3}\Big)^N-1\Big]\ln Z_0\Big\}$$
$$\times\exp\{-Z_0{}^{((4/3)^N/4)}\ F(N)\}. \tag{5.11}$$

The factor $F(N)$ in (5.11) can be put in the form

$$F(N)=\sum_{m=0}^{N-1}Z_{N-1}^{1/4[(3/4)^m-1]}.$$

In the limit $N\to\infty$ we have $Z_{N-1}\to\infty$ and $F(N)\to 1$, but this limit is actually reached only at very large values $Z_{N-1}\sim 10^4$; at smaller values of N, we can use $F(N)\sim 10(3/4)^N/\ln Z_0$ as a rough estimate. It is important to note that $F(N)$ is a relatively slowly varying function of N. The quantity Γ_N^{-1} determines the maximum spin relaxation time t_{max} at the temperature $T=T_N^*$ [see (3.30)–(3.31)]. We thus find

$$t_{max}\sim\left(\frac{T_f}{T-T_f}\right)^{\nu/2} \tag{5.12}$$

for the vector spins and

$$t_{max}\sim\left(\frac{T_f}{T-T_f}\right)^{6}\exp\left\{\left(\frac{Z_0T_f}{T-T_f}\right)^{1/4}F(N(T))\right\} \tag{5.13}$$

for the Ising spins. Expressions (5.12) and (5.13) are similar in meaning to expressions (4.10) and (4.12) for the upper envelope of the function $\tilde\chi(T)$.

At times $t\ll t_{max}$, the relaxation occurs in a nonexponential way. To find this behavior we seek the fraction (w_N) of the total moment of the system which relaxes along with the Nth level of the hierarchy. For this purpose we determine the total moment corresponding to one degree of freedom of the Nth level; we multiply by the number of such degrees of freedom; and we divide by the total number of spins. We find

$$\begin{aligned}-\ln w_N&\approx\frac{9}{2}\left[\left(\frac{4}{3}\right)^N-1\right]\ln Z_0 \quad (n=1),\\ -\ln w_N&\approx\frac{25}{17}\left[\left(\frac{32}{15}\right)^N-1\right]\ln Z_0 \quad (n\geqslant 2).\end{aligned} \tag{5.14}$$

Over the time t, the first $N(t)$ levels of the hierarchy, for which we have $\Gamma_{N(t)}^{-1}\lesssim t$, manage to relax, so that the remaining magnetic moment $M(t)$ is given by (5.14), where N is determined with the help of (5.10)–(5.11), in which we set $t\sim\Gamma_N^{-1}$. For vector spins we finally find

$$\overline{\langle\sigma_i(0)\sigma_i(t)\rangle}\sim M(t)\sim(\Gamma_0 t)^{-25/42}. \tag{5.15}$$

For Ising spins there are two possible cases, depending on the relative roles played by the first and second factors in (5.11). For times which are not too long, the first factor is governing; we find

$$M(t)\sim(\Gamma_0 t)^{-3/4}. \tag{5.16}$$

In the opposite limit we find

$$M(t)\sim(\ln\Gamma_0 t)^{-18}. \tag{5.17}$$

In deriving (5.15)–(5.17) we assumed that t is quite large, so that many levels of the hierarchy participate in the relaxation.

Here are expressions for $t_{max}(T)$ which hold for an arbitrary value of the index ν of the localization theory (see the discussion at the end of Section 4):

$$\begin{aligned}&t_{max}\infty(T-T_f)^{-\alpha_2},\quad a_2=36\nu^2-30\nu+{}^9/_2\quad(n\geqslant 2),\\ &t_{max}\infty(T-T_f)^{-a_1}\exp\left[\left(\frac{Z_0{}^{\nu}T_f}{T-T_f}\right)^{\omega}\right]F(N(T)),\\ &a_1=14-\frac{8}{\nu},\quad \omega=\frac{3\nu-2}{2(3\nu-1)\nu}\quad(n=1),\\ &F(N)=\sum_{m=0}^{N-1}(Z_{N-1})^{\varkappa_m}\xrightarrow[N\to\infty]{}1,\\ &\varkappa_m=\frac{3\nu-2}{3(2\nu-1)}\left[\left(\frac{3(2\nu-1)}{2(3\nu-1)}\right)^m-1\right].\end{aligned} \tag{5.18}$$

6. DISCUSSION OF RESULTS

1. Starting from a microscopic model, we have shown that a phase transition occurs at $T=T_f>0$ in a three-dimensional spin glass with a finite interaction radius. This phase transition is unrelated to the macroscopic filling of any delocalized mode (even if disordered). Instead, at temperatures near T_f the system can be represented as a hierarchy of localized, strongly interacting degrees of freedom. The number of levels of the hierarchy and the spatial scale of the correlations increase without bound as $T\to T_f$, in a process accompanied by strong singularities in the temperature dependence of the nonlinear magnetic susceptibility $\tilde\chi=-\partial^2\chi/\partial h^2$ and the maximum spin relaxation time t_{max}. We have derived $\tilde\chi(T)$ [(4.10)–(4.11), (4.14)–(4.16)], $t_{max}(T)$ [(5.12)–(5.13)], and $\langle\sigma_i(0)\sigma_i(t)\rangle$ [(5.15)–(5.17)] near the transition. We again stress that, in contrast with the behavior at an ordinary phase transition, there is a singularity in $\tilde\chi(T)$ which is more complicated than a power-law singularity: It oscillates between two power-law envelopes (see the end of Section 4).

A hierarchical structure of this sort naturally combines the representations of critical phenomena in spin glasses[3–10,17] with the picture of superparamagnetic clusters.[17,34] We should emphasize, however, that our "degrees of freedom" have a pronounced spatial overlap and are thus quite different from ordinary rigid clusters. The state of a given spin at each level of the hierarchy is determined by a set of variables corresponding to degrees of freedom localized near the given spin.

The spatial dimension of the states increases sharply with the level index N [even the "spins" of the first level contain $\sim Z_0^4(Z_0^{2^{6/5}})$ original Ising (vector) spins], so that physical phenomena associated with the existence of many levels of the hierarchy could hardly be observed in numerical simulations. For the vector model, we might note, this increase is more rapid, possibly explaining the absence of freezing from the numerical simulations of Refs. 20 and 21 with Heisenberg spins.

Our results do not contradict the conclusion that no phase transition occurs in a space of dimensionality $d<4$, which was reached in Ref. 12 through an analysis of high-temperature series, since that analysis tells us only that there is no phase transition at temperatures where all the terms of

the series are comparable in magnitude ($|\tau| \sim Z^{-2/3}$ in the case $d = 3$). As we have shown above, it is quite true that no phase transition occurs in this region, but one does occur at a lower temperature ($-\tau \sim Z_0^{-1/3}$). A characteristic feature of this phase transition is the absence of continuous scaling, a direct consequence of the discrete hierarchical structure of the phase space of the system.

2. The slowing of the relaxation as the freezing point is approached from above stems from a broadening of the spectrum of relaxation times $g(t)$ toward longer times because of an increase in the number of levels of the hierarchy. The relaxation times of the intermediate levels change only slightly. At $T < T_f$, with $(T_f - T)/T_f = \varepsilon > 0$, an infinite number of levels of the hierarchy [beginning at the index $N(\varepsilon)$ determined by expression (3.30), with $\ln(1/\varepsilon)$ as its left side] turn out to be close to zero at their effective temperatures. This result means that at $T \leqslant T_f$ the spectrum of times becomes an unbounded spectrum which decays very slowly; this is the primary distinction between the spin-glass phase and the low-temperature phases of ferromagnets or antiferromagnets. In ordinary magnets the spectrum of relaxation times can be broken up into two distinct regions; the times in one region do not depend on the total number of spins, $\mathcal{N}$, while those in the other increase with increasing $\mathcal{N}$. From this picture we can find the limit of equilibrium thermodynamics, by first letting $\mathcal{N} \to \infty$ and then letting $t \to \infty$. The situation is completely different in a spin glass, where the spectrum of relaxation times stretches continuously from microscopic times up to the very longest times, which increase exponentially with the size of the system, $\mathcal{N}$. Consequently, a thermodynamic equilibrium is never reached in such a system. Generally speaking, the ergodic behavior is disrupted even at an ordinary phase transition, since in its low-temperature phase the system sweeps over only some hypersurface in phase space. However, the codimensionality of this hypersurface is finite and determined by the number of order parameters. In our case, the codimensionality (at $\mathcal{N} = \infty$) is infinite at any finite t, since there are infinitely many levels of the hierarchy with relaxation times greater than t.

We can use this hierarchical picture to find a simple qualitative explanation of the validity of the de Almeida-Thouless law for the magnetic-field dependence at which the irreversible response appears. At temperatures below the point of the phase transition, the relaxation times of the levels of the hierarchy increase very rapidly with increasing level index [at any $T < T_f$, the effective temperatures $T_{\text{eff}}^{(N)}$ of all levels with $N > N(T)$ are very small, and we have $T_{\text{eff}}^{(N)} \to 0$ in the limit $N \to \infty$], so that in a real experiment (with a scale time t) the point at which the irreversible response appears is related to the "quasiphase transition" which occurs in the highest thermalized level (with $\Gamma_N^{-1} \sim t$). The corresponding number of nearest neighbors, Z_N, is very large, so that we can use the mean field theory, which predicts a de Almeida-Thouless line, but the effective magnetic field acting on a "spin" of the Nth level is proportional (with a large factor) to the physical field h. As a result, we find $A \gg 1$ in this functional dependence (see the Introduction).

3. We have shown that two universality classes exist in this model: one corresponding to Ising spins (the number of components is $n = 1$), and another corresponding to vector spins ($n \geqslant 2$) with an isotropic interaction. It can be shown that the incorporation in the vector model of weak interactions of the uni axial-anisotropy type,

$$\delta H = \Lambda \sum_i (\sigma_i^x)^2 ,$$

or a randomly anisotropic interaction,[35]

$$\delta H = \sum_{ik} G_{ik}^{\alpha\beta} \sigma_i^\alpha \sigma_k^\beta , \tag{6.1}$$

$$\overline{G_{ik}^{\alpha\beta} G_{lm}^{\gamma\delta}} = \frac{G^2}{4} (\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) ,$$

leads to a rapid transition (crossover) as $T \to T_f$ to an Ising asymptotic behavior, since the relative values of Λ and G increase:

$$\Lambda_N \propto \Lambda_0 \left(\frac{T_N - T_f}{T_f} \right)^{-\nu/2} , \quad G_N \propto G_0 \left(\frac{T_N - T_f}{T_f} \right)^{-\nu/8} . \tag{6.2}$$

The fact that interaction (6.1) leads to an Ising asymptotic behavior follows from the nondegeneracy of the eigenstates of the exchange matrix $J_{ij}\delta^{\alpha\beta} + G_{ij}^{\alpha\beta}$, because of which the eigenfunctions $\psi_\lambda^\alpha(i)$ carry the vector exponent α, while the amplitudes a_λ do not (as in the Ising case). The appearance of new universality classes is possible in an analysis of models with a constant anisotropy of higher order, but we will not take up that question here.

4. This entire analysis leans heavily on the assumption of a large initial number of nearest neighbors (i.e., of spins within the interaction radius): $Z_0 \gg 1$. When we make the transition from the original spins to the "block spins" of the first level, the parameters Z_0 becomes $Z_1 = \mu Z_0^q$, where $q > 1$, and μ is an unknown numerical factor. At $Z_0 > Z_{\text{crit}} = \mu^{1-q}$ we find the hierarchical construction described above, with increasing Z as we go to a higher level: $Z_2 < Z_1 < Z_2 \ldots$. In classical spin glasses with an RKKY interaction, $V_0 r^{-3} \cos \mathbf{p}_0 \cdot \mathbf{r}$, the effective number of nearest neighbors increases with decreasing temperature[36]: $Z_{\text{eff}} \propto cV_0/T$. A phase transition usually occurs at $T \sim cV_0$; i.e., we have $Z_{\text{eff}} \sim 1$. We thus have two possibilities: Either the scheme studied above begins to take form with increasing Z_N when Z_{eff} reaches the critical value Z_{crit}, or there exists another scenario of the freezing phase transition, which holds at $Z \sim 1$ and which corresponds to a fixed point along the scale of the parameter Z_N. Just how this second possibility may unfold is not clear at this point, but it is obvious that in this case the appearance of a nonlinear susceptibility would be described by a single power function, not by two power-law envelopes, as in the case studied here.

We do not rule out the possibility that it is this circumstance which explains the discrepancy between the asymptotic results which we derived for $\tilde{\chi}(T)$ and the experimental results of Refs. 8 and 9 for RKKY alloys. The same comment applies to the dependence $t_{\max}(T)$, although experimentally it is not known as accurately; in particular, the

high-frequency experiments are described better by power-law formulas of the type $t_{\max} \sim (T - T_f)^{-\Delta}$, while quasistatic experiments are described better by a Vogel-Fulcher law $\ln t_{\max} \sim T(T - T_f)^{-1}$. There is the possibility that a description of the experimental data over the entire range of times (10^{-10}–10^4 s) will require the use of an expression of the type

$$t_{max} \propto (T-T_f)^{-\Delta} \exp\{C(T-T_f)^{-\omega}\}.$$

Furthermore, as the observation time is increased, there is the real possibility of a transition from a "vector" regime, (5.12), to an "Ising" regime, (5.13).

It would apparently be possible to experimentally study spin glasses which correspond literally to the model which we have studied here: the rare earth alloys $Y_{1-x}R_x$ (R = Er, Dy, Tb, Gd), with small x (apparently on the order of 0.5–1 at. %). Since yttrium is close to the instability with respect to the formation of spin-density waves, the interaction of impurity magnetic moments is of the form[36,37]

$$V(r) \sim \frac{V_0}{r} e^{-r/r_0} \sin p_0 r$$

with an interaction radius $r_0 \gg a, p_0^{-1}$. The behavior of a system of this sort depends strongly on the value of the parameter $\gamma = p_0^2/4\pi c V_0$, where c is the dimensional concentration. At $\gamma \ll 1$, a spin-glass phase with a local helical order should form.[36] At $\gamma \gtrsim 1$, helical correlations are unimportant, and the problem reduces to the Edwards-Anderson model with $Z_0 \approx 4cr_0^3 \gg 1$, which we have studied here (the conditions $\gamma \gtrsim 1$ and $Z_0 \gg 1$ are compatible if $r_0 p_0 \gg 1$). Preliminary experiments[38–40] carried out with these alloys indicate a local helical order at $x \gtrsim 2$–3%. In addition to magnetic measurements, it would be possible to observe a transition from a "helical" spin glass to a completely random glass by means of neutron scattering and also from the change in the function $T_f(c)$: At $\gamma \ll 1$ we should find $T_f \propto c$, while in the opposite limit we should find $T_f \propto c^{1/2}$ (for classical spins; how quantum effects will influence this behavior is not known at this point).

The hierarchical structure of a real spin glass (with a finite Z) described above converts in a natural way into the hierarchical structure, discovered by Parisi,[4] of the Sherrington-Kirkpatrick model. The infinite set of relaxation times, which become infinite at the phase transition and which remain infinite below T_f (Ref. 5), acquires a clear physical meaning at a finite Z: These are the relaxation times which correspond to different levels of the hierarchy, each of which is now finite (but increases with increasing Z), while the maximum time becomes infinite at the phase transition.

Some slightly different possible hierarchical structures were recently proposed by Palmer *et al.*[41] on the basis of some phenomenological arguments. Our structure differs from those proposed in Ref. 41 in that the number of "spins" which are controlled by one "spin" of the next level of the hierarchy increases in our version but decreases in the scenarios of Ref. 41.

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APPENDIX

To calculate the functional $F\{\mathbf{m}_i\}$, we will determine its second variational derivative $\delta^2 F/\delta \mathbf{m}_i \delta \mathbf{m}_j$ and then reconstruct $F\{\mathbf{m}_i\}$ from it. We note that we have $\delta^2 F/\delta m_i^\alpha \delta m_j^\beta = (\chi^{-1})_{ij}^{\alpha\beta}$, where $\chi_{ij}^{\alpha\beta}$ is the matrix of the magnetic susceptibility, determined by the relation $\chi_{ij}^{\alpha\beta} = -\partial^2 \tilde{F}/\partial h_i^\alpha \partial h_j^\beta$ ($\tilde{F}$ is related to F by a functional Legendre transformation). In the approximation $Z = \infty$ (which is sufficient for deriving $F\{\mathbf{m}_i\}$), we can write $\chi_{ij}^{\alpha\beta}$ as[23]

$$\chi_{ij}{}^{\alpha\beta} = [\hat{J} + \hat{A}]^{-1}, \quad J_{ij}{}^{\alpha\beta} = J_{ij}\delta^{\alpha\beta}, \quad A_{ij}{}^{\alpha\beta} = \delta_{ij} A_i{}^{\alpha\beta}, \tag{A.1}$$

i.e.,

$$\frac{\partial^2 F}{\partial m_i^\alpha \partial m_j^\beta} = J_{ij}\delta_{\alpha\beta} + A_i{}^{\alpha\beta}\delta_{ij}. \tag{A.2}$$

By virtue of the isotropy, $F\{\mathbf{m}_i\}$ depends on only the scalars $\mathbf{m}_i^2$ and $(\mathbf{m}_i \mathbf{m}_j)$, so that we have restrictions on the form of :

$$A_i{}^{\alpha\beta} = f_i(\mathbf{m}_i^2)\delta^{\alpha\beta} + 2f_i'(\mathbf{m}_i^2) m_i^\alpha m_i^\beta. \tag{A.3}$$

The functional dependence $f_i(\mathbf{m}_i^2)$ can be determined most simply by making use of the form of the single-point susceptibility:

$$\sum_\alpha \chi_{ii}{}^{\alpha\alpha} = \frac{1}{T}\sum_\alpha (\langle \sigma_i^\alpha \sigma_i^\alpha \rangle - \langle \sigma_i^\alpha \rangle^2) = \frac{1}{T}(1 - \mathbf{m}_i^2). \tag{A.4}$$

Expanding the expression for $\chi_{ij}^{\alpha\beta}$, (A.1), in a series in $\hat{J}\hat{A}^{-1}$, and collecting the leading terms (in the limit $Z \to \infty$), we find a convenient equation which relates $\hat{\chi}$ and $\hat{A}$:

$$\sum_\beta A_i{}^{\alpha\beta}\chi_{ii}{}^{\beta\gamma} = \delta_{\alpha\gamma} + \frac{1}{Tn}\sum_j J_{ij}{}^2(1 - \mathbf{m}_j^2)\chi_{ii}{}^{\alpha\gamma}. \tag{A.5}$$

In deriving (A.5), the condition $Z \to \infty$ allowed us to replace

$$\sum_j J_{ij}{}^2 \chi_{jj}{}^{\alpha\beta} \quad \text{by} \quad \frac{1}{nT}\sum_j (1 - m_j^2)\delta_{\alpha\beta} J_{ij}{}^2.$$

Substituting (A.3) for $A_i^{\alpha\beta}$ and $\chi_{ii}^{\alpha\beta}$ in the form $\chi_i^{(0)}\delta_{\alpha\beta} + \chi_i^{(1)} m_i^\alpha m_i^\beta \mathbf{m}_i^2$, into (A.5), and using condition (A.4), i.e., $n\chi_i^{(0)} + \chi_i^{(1)} = (1 - \mathbf{m}_i^2)/T$, we find an equation for

$$\begin{aligned} &\tilde{f}(\mathbf{m}_i^2) = f(\mathbf{m}_i^2) - \frac{1}{nT}\sum_j J_{ij}{}^2(1 - \mathbf{m}_j^2): \\ &\tilde{f} + 2t\frac{d\tilde{f}}{dt}[1 - (n-1)/\tilde{f}(1-t)] = \frac{n}{1-t}, \end{aligned} \tag{A.6}$$

where $t = \mathbf{m}_i^2$. Solving this equation through a series expan-

sion in t, we find the functional dependence $F\{\mathbf{m}_i\}$:

$$F\{\mathbf{m}_i\} = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{m}_i\mathbf{m}_j - \frac{1}{4nT}\sum_{ij} J_{ij}^2(1-\mathbf{m}_i^2)(1-\mathbf{m}_j^2)$$
$$+\frac{nT}{2}\sum_i\left[\mathbf{m}_i^2 + \frac{n}{2(n+2)}(\mathbf{m}_i^2)^2 + \frac{n^2(n+8)}{3(n+4)(n+2)^2}(\mathbf{m}_i^2)^3\right]. \quad (A.7)$$

The first term in this expression corresponds to the direct interaction of $\mathbf{m}_i$; the second corresponds to the reaction of the remaining spins to the given $\mathbf{m}_i$; and the third corresponds to the free energy of one spin in the external field, expressed in terms of the expectation value of this spin.

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