Meson-current correlation functions in instanton vacuum

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The correlation functions (polarization operators) of colorless currents are calculated over the entire range of (Euclidean) momenta in the instanton vacuum of quantum chromodynamics. The appearance of a pion pole due to the spontaneous breaking of chiral invariance is studied explicitly. The mass and the axial constant of the π meson, f_π, are calculated in terms of the parameters found previously for the instanton medium.

1. INTRODUCTION

We showed in Ref. 1 that spontaneous breaking of the chiral symmetry of strong interactions necessarily occurs in the instanton vacuum of quantum chromodynamics (QCD). We calculated the Green's function of a quark in an instanton medium and, in particular, the chiral condensate $\langle\bar{\psi}\psi\rangle$ in terms of the basic characteristics of the instanton medium: the expectation value of the size of the instantons, $\bar{\rho}$, and the expectation value of the distance between instantons, $\bar{R}$. These expectation values can in turn be expressed in terms of the fundamental parameter Λ of QCD by means of a variational principle.[2]

An unavoidable consequence of the spontaneous breaking of chiral invariance in strong interactions is the existence of Goldstone particles, an octet of pseudoscalar mesons.

In the present paper we explicitly demonstrate the appearance of a massless pion pole in the correlation functions (polarization operators) of colorless currents which have a nonvanishing coupling constant with the π meson. Using the methods and approximations of Ref. 1, we calculate the two-current correlation function

$$\Pi^\Gamma(p)=\int d^4x e^{-i(px)}\langle J^\Gamma(x),J^\Gamma(0)\rangle,\quad J^\Gamma=\psi^+\hat{\Gamma}\psi, \tag{1}$$

in an instanton medium over the entire Euclidean range of the momentum p for the five Fermi types of interaction ($\Gamma=1,i\gamma_5,\gamma_\mu\,\gamma_\mu\gamma_5,\sigma_{\mu\nu}$). Here Γ is understood as a unit matrix in terms of color. We will see that an important difference between channels arises in a natural way in an instanton vacuum. We will find the residue at the pole in the pseudoscalar and axial channels, where a massless pion pole arises. In this manner we calculate the coupling constant of the pion with the axial current, f_π, in terms of the parameters $\bar{R}$ and $\bar{\rho}$ of the medium. We recall that the expectation value of the distance between pseudoparticles, $\bar{R}$ is determined by the density of the medium [$\bar{R}=(N/V)^{1/4}$], and the latter is directly related to the gluon condensate[2,3]: $N/V=\langle F_{\mu\nu}{}^2/32\pi^2\rangle\approx(200\text{ MeV})$.[4] As for the expectation value of the size of the pseudoparticles, $\bar{\rho}$ we note that we have the ratio $\bar{\rho}/\bar{R}\approx 1/3$ from a variational principle.[2] This ratio was derived previously from phenomenological analysis.[4] Using these values of $\bar{R}$ and $\bar{\rho}$, we find good agreement between the calculated value of f_π and the experimental value $f_\pi\approx 132$ MeV.

When (small) current quark masses are introduced, the mass of the pion becomes nonzero. In this case we calculate the shift of the pole from the point $p^2=0$, and we find the mass of the pion.

The representation of the QCD vacuum as an instanton medium makes it possible to calculate the characteristics of the hadronic world by methods analogous to those used in the theory of disordered systems.[5] In fact, we examine the propagation of quarks in a random external field. This field is specified by the collective coordinates of the pseudoparticles: the set of their positions z_I, their sizes ρ_I, and their orientations U_I in color space. The result should be averaged over all the parameters which determine the field, by expressing the field in terms of the expectation values of the characteristics of the medium. As in Ref. 1, we ignore correlations between different pseudoparticles, since they are small, on the order of the density.

The analogy between problems of the theory of disordered systems (e.g., the problem of the behavior of an electron in a metal with random impurities[6]) and the QCD instanton vacuum with which we are concerned here is rather far-reaching. The acquisition of an effective mass by a quark, for example, is completely analogous to the appearance in the Green's function of an electron in a metal of a finite relaxation time (but in the present case, this time depends on the momentum of the particle), and the appearance in the pseudoscalar channel of a massless pole corresponding to a π meson can be associated with the formation of a diffusion in the density-density correlation function.

A standard approximation in the theory of disordered systems uses planar diagrams (this approximation corresponds to the classical equation). The use of this approximation is ordinarily justified by assuming that the electron wavelength is short in comparison with the mean free path. We will also use this approximation here, but now the parameter is the number of colors, $N_c\to\infty$. The corrections in N_c correspond to "intersecting" diagrams and can be dealt with systematically by perturbation theory.

As in our preceding study, we ignore the vacuum loops of quarks. Incorporating these loops theoretically is equivalent to making a correction $\sim 1/N_c$, and they are apparently unimportant in a study of currents which are not flavor-singlets. (Incorporating the vacuum loops becomes of fundamental importance only in a study of a singlet current associated with η' meson; the solution of the U_1 problem in an instanton medium requires a special study.) We will accordingly treat the correlation functions (1) in a theory with a single flavor, and we will ignore the vacuum loops with the

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understanding that we are calculating the correlation functions of nonsinglet currents.

2. DIAGRAMS FOR THE POLARIZATION OPERATOR

We rewrite the two-current correlation function (1) in the external field of the pseudoparticles in the form

$$\Pi^\Gamma(p) = -\int d^4x e^{-i(px)} \operatorname{Tr}\overline{\{S(x,0)\Gamma S(0,x)\Gamma\}}. \qquad (2)$$

Here $S(x,y)$ is the quark propagator in the instanton medium, Tr includes a summation over the Lorentz and color indices, and the superior bar denotes the expectation value over the statistical ensemble of the instanton medium.

Diagonalizing the zero-fermion modes of the individual pseudoparticles, we derived the following expression for the propagator $S(x,y)$ in Ref. 1 [1, (40)][1)]:

$$S(x,y) = S_0(x,y) + \sum_{I,J} \psi_I(x) \left(\frac{1}{im - T}\right)_{I,J} \psi_J^+(y), \qquad (3)$$

where $S_0(x,y)$ is the free propagator, $\psi_I(x)$ is a zero mode in the field of pseudoparticle I (an instanton or anti-instanton), and the matrix T is constructed from the overlap integrals of the zero modes,

$$T_{IJ}(z_I+z_J, U_I^+, U_J) = \int d^4x \psi_I^+(x-z_I, U_I) i\hat{\partial}_x \psi_J(x-z_J, U_J). \qquad (4)$$

In practice, an element T_{IJ} is nonvanishing only for unlike pseudoparticles; an expression for it is derived in the Appendix of Ref. 1.

Expanding (2) and (3) in the reciprocal of the quark mass m (we will take the chiral limit $m \to 0$ in the final result), we can write the polarization operator as a sum of diagrams as in Fig. 1, where each circle is associated with a factor of $1/im$, while the line connecting circles I and K is associated with the value of T_{IK} (Ref. 1). The crosses represent external currents. Each line going into a cross is associated with a zero-mode ψ_I, while each outgoing line is associated with ψ_I^+. The averaging includes a summation over all pseudoparticles (and gives rise to a factor of $N/2$ for each species) and also an integral over all possible positions of the pseudoparticles in 4-space (divided by $V^{(4)}$) and their orientations in color space. In taking this average, one should allow for situations in which the same pseudoparticle is encountered several times in a diagram. As in Ref. 1, we use a dashed line to connect pseudoparticles which are encountered repeatedly.

We showed in Ref. 1 that a correlation which arises because a quark may be "scattered" several times by the same pseudoparticle plays a fundamental role in the mechanism for the spontaneous breaking of chiral invariance. It is ultimately this effect which is responsible for the appearance of the quark condensate. The contributions of leading order in N_c to both the Green's function of the quark and the polarization operator come from the planar diagrams in which the dashed lines intersect nowhere.

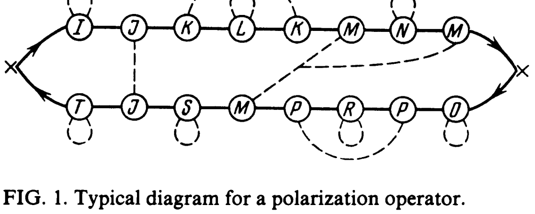

FIG. 1. Typical diagram for a polarization operator.

The diagrams for the polarization operator are conveniently classified in three groups (Figs. 2a, 2b, and 2c): a) diagrams in which the same pseudoparticle is never encountered in the upper line and the lower line; b) diagrams in which there is only one common pseudoparticle in the upper line and the lower line (this pseudoparticle may be encountered an arbitrary number of times); c) diagrams in which there are two or more common pseudoparticles in the upper line and the lower line. We denote by $\Pi_{0,1,2}(p)$ the corresponding contributions to the polarization operator. The heavy lines in Fig. 2 are the exact Green's functions of the quark in the "pseudoparticle representation," D_{IJ} and F_{IJ} (Ref. 1).

The diagrams in Fig. 2a correspond to a splitting of correlation function (2) into a product of expectation values. Incorporating the Green's function from (3) in $\Pi_0(p)$, we find

$$\Pi_0(p) = -\int d^4x e^{-i(px)} \operatorname{Tr}\{\Gamma\overline{S(x,0)}\Gamma\overline{S(0,x)}\}$$
$$= -\int (dp_1\, dp_2) \operatorname{Tr}\{\Gamma\overline{S(p_1)}\Gamma\overline{S(p_2)}\}. \qquad (5)$$

For brevity we have introduced the notation

$$\int (dp_1\, dp_2) \equiv \int \frac{d^4p_1\, d^4p_2}{(2\pi)^4} \delta^{(4)}(p_1 - p_2 - p); \qquad (6)$$

$\overline{S}(p)$ is the quark propagator in the instanton vacuum. In the planar approximation (to which we restrict the present paper), $\overline{S}(p)$ is [1, (54)]

$$\overline{S}(p) = \frac{\hat{p} + iM(p)}{p^2 + M^2(p)} 1\,(\text{color}), \qquad (7)$$

where $M(p)$ is the effective mass of the quark, expressed in terms of the Fourier transform of a zero mode [1, (50)]. Consequently, $\Pi_0(p)$ is the noncoupling part of the polarization operator made up of the exact propagators of the quark in the instanton medium.

Let us examine the "Born" part, $\Pi_1(p)$, represented by the diagrams in Fig. 2b. We first note that the interior part of these diagrams, including the dashed lines connecting a pair of pseudoparticles I, three Is, four Is, etc., can easily be summed. We denote the sum by a double dashed line (Fig. 3). The heavy line connecting two identical pseudoparticles in the diagrams in Fig. 3 is by definition[1] the Green's function in the pseudoparticle representation, $D_{IK}(z_K - z_I, U_I, U_K)$, with $I = K$, i.e., the quantity γ [1, (44)]. The summation represented in Fig. 3 corresponds to a double geometric progression (over the upper and lower lines) and gives us

$$\left(\frac{1}{im}\right)^2 (1 + im\gamma + \ldots)(1 + im\gamma + \ldots) = \left[\frac{1}{im(1 - im\gamma)}\right]^2 = -\varepsilon^2 \qquad (8)$$

(see definition [1, (49)]).

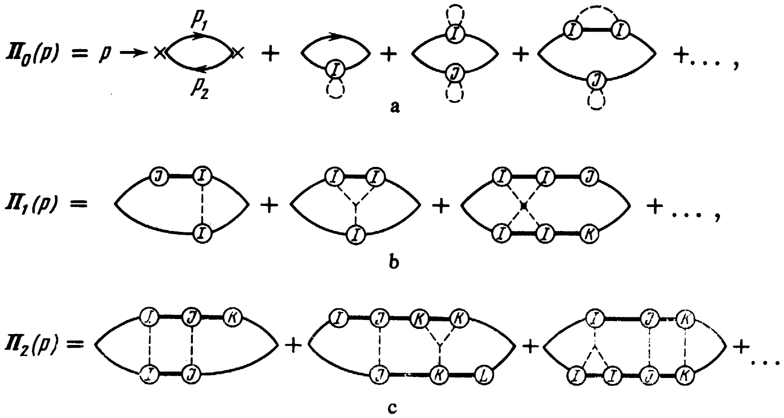

FIG. 2. Three groups of planar diagrams for $\Pi(p)$. a—Uncoupled; b—with one common pseudoparticle; c—with two or more common pseudoparticles in the upper and lower lines.

To calculate $\Pi_1(p)$ we must now take into account the fact that the exact quark propagators $D(F)$ can be (and must be) inserted in the upper and lower lines both before and after the double dashed line. It is natural to relate the corresponding diagrams to the definition of the vertices of the polarization operator, which we denote by $\gamma_I(p)$ (Fig. 4). Analytically, the vertex function $\gamma_I(p)$ is

$$\gamma_I(p)=\int (dp_1\,dp_2)\left[\psi_{Ii}{}^+(p_1)+\frac{N}{2V}\frac{im}{1-im\gamma}\left(\int dU_K D_{IK}(p_1)\psi_{Ki}{}^+(p_1)+\int dU_{\bar K}F_{I\bar K}(p_1)\psi_{\bar K i}{}^+(p_1)\right)\right]\Gamma_{ij}\left[\psi_{Ij}(p_2)+\frac{N}{2V}\frac{im}{1-im\gamma}\left(\int dU_L\psi_{Lj}(p_2)D_{LI}(p_2)+\int dU_{\bar L}\psi_{\bar L j}(p_2)F_{\bar L I}(p_2)\right)\right]. \quad (9)$$

Since the diagrams in Fig. 4 end with the same pseudoparticle I, $\gamma_I(p)$ in the momentum representation is an integral over the momentum in a closed loop, which we have written in the notation of (6). The factors $N/2$ in (9) stem from the summation over the intermediate pseudoparticles K and L, $1/V$ stems from the average over the positions of these pseudoparticles, and the factors $im/(1-im\gamma)\equiv im^2\varepsilon - c$ stem from the summation of the geometric progression which arises because pseudoparticles K and L may repeat an arbitrary number of times. The matrix Γ_{ij} is one of the five Fermi types, and it is a unit matrix in terms of color.

The integrals over orientation in (9) can be evaluated easily by making use of the explicit expressions for the Green's functions D_{IK} and F_{IK} [1, (46),(51)] and for the density matrix of the zero modes (see the Appendix in Ref. 1). We find

$$\gamma_I(p)=\frac{2VN_c}{N\varepsilon}\int (dp_1\,dp_2)\frac{(M_1M_2)^{1/2}}{(p_1{}^2+M_1{}^2)(p_2{}^2+M_2{}^2)}\times\mathrm{Tr}\left\{\Gamma(\hat p_1+iM_1)\frac{1+\gamma_5}{2}\times(\hat p_2+iM_2)\right\}\equiv\frac{2VN_c}{N\varepsilon}\Gamma_I(p),\quad M_{1,2}\equiv M(p_{1,2}). \quad (10)$$

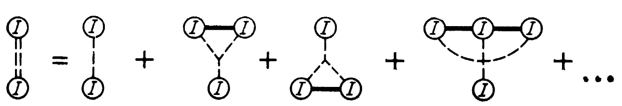

FIG. 3. Definition of a double dashed line.

Here we have determined the reduced vertex function $\Gamma_I(p)$, divided by a factor $2VN_c/N\varepsilon$. The factor $VN_c/N\varepsilon$ arises in (10) because we have preferred to express the result in terms of the effective mass $M(p)$ [1, (50)] rather than directly in terms of the Fourier transforms of the zero modes. The coefficient 2 results from taking the trace over color of the density matrices of the zero modes.

We have derived an expression for the left-hand vertex of the polarization operator in the case in which the pseudoparticle I is an instanton. For the right-hand vertex we should make the replacement $p\to -p$, and in the case in which I is an anti-instanton we must use the replacement $\gamma_5\to-\gamma_5$. We note that $\Gamma_I(p)$ does not depend on the orientation U_I of pseudoparticle I, because the currents under consideration here are colorless.

Summing over the common pseudoparticle I, taking an average over its position and orientation, and recalling the expression for the double dashed line [see (8)], we find the following expression for the sum of the diagrams in Fig. 2b:

$$\Pi_1(p)=N_c\frac{VN_c}{N}2[\Gamma_I(p)\Gamma_I(-p)+\Gamma_{\bar I}(p)\Gamma_{\bar I}(-p)]$$
$$=N_c\frac{VN_c}{N}\{[\Gamma_I(p)+\Gamma_{\bar I}(p)][\Gamma_I(-p)+\Gamma_{\bar I}(-p)]+[\Gamma_I(p)-\Gamma_{\bar I}(p)][\Gamma_I(-p)-\Gamma_{\bar I}(-p)]\}. \quad (11)$$

Finally, we consider the diagrams which have two or more common pseudoparticles in the upper and lower lines; we denote their sum by $\Pi_2(p)$ (Fig. 2c). To calculate $\Pi_2(p)$ we must first determine the sum of the ladder diagrams in Fig. 5, where a double dashed line appears as a rung (Fig. 3). Nonladder diagrams and also ladder diagrams with "renormalized" vertices (Fig. 6) are not planar. It is not difficult to see that they contain an extra factor of $1/N_c$ (from the more complicated average over orientations) in comparison with simple ladder diagrams (Fig. 5); they also have an additional numerically small factor which stems from the integration over the angles of the relative positions of the pseudoparticles. We are systematically ignoring nonplanar diagrams.

We denote by $S_{IK}(p)$ the sum of the s-channel ladder

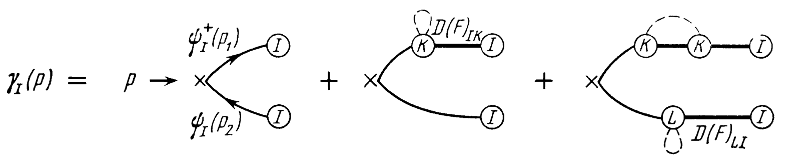

FIG. 4. Diagrams for the vertex function of a polarization operator.

diagrams in Fig. 5. Using the vertex functions $\gamma_I(p)$ calculated above, we can then write the contribution to the polarization operator from the diagrams in Fig. 2c as the sum of four terms, because pseudoparticles I,K can be both instantons and anti-instantons (we use a superior bar to denote the latter):

$$\Pi_2(p)=\left(\frac{N}{2V}\right)^2\left\{\int dU_I\gamma_I(p)\left[\int dU_K S_{IK}(p,U_I,U_K)\gamma_K(-p)\right.\right.$$
$$\left.+\int dU_{\bar K}S_{I\bar K}(p,U_I,U_{\bar K})\gamma_{\bar K}(-p)\right]$$
$$+\int dU_{\bar I}\gamma_{\bar I}(p)\left[\int dU_K S_{\bar I K}(p,U_{\bar I},U_K)\right.$$
$$\left.\left.\times\gamma_K(-p)+\int dU_{\bar K}S_{\bar I\bar K}(p,U_{\bar I},U_{\bar K})\gamma_{\bar K}(-p)\right]\right\}. \qquad (12)$$

Here $(n/2V)\int dU_I U_K$ arises, as always, from the summation and averaging over pseudoparticles I and K. The task which remains is to calculate S_{IK}.

3. THE s-CHANNEL FOUR-TAIL FUNCTION S_{IK}

The ladder diagrams in Fig. 5 are easily summed with the help of the Bethe-Salpeter equation shown in Fig. 7. The heavy lines in Fig. 7 represent the exact quark propagators in the pseudoparticle representation, $D(F)$, while the double dashed lines represent the sum of the diagrams in Fig. 3, equal to $-\varepsilon^2$ [see (8)]. Since the propagators $D(F)$ by definition[1] include the "end" factors $1/im$, both the seed term and the integral term in the Bethe-Salpeter equation should be multiplied by $(im)^4$ if we use the definition of the double dashed line in (8). The integral term in Fig. 7 essentially represents two diagrams, since the intermediate pseudoparticle L can be either an instanton or an anti-instanton.

After these comments, we write the Bethe-Salpeter equation in the momentum representation [we are using the notation in (6)]:

$$S_{IK}(p,U_I,U_K)=m^4\varepsilon^4\int(dk_1\,dk_2)D_{KI}(k_1,U_K,U_I)\,D_{IK}(k_2,U_I,U_K)$$
$$-m^4\varepsilon^2\frac{N}{2V}\int dU_L\int(dk_1\,dk_2)D_{KL}(k_1,U_K,U_L)S_{IL}(p,U_I,U_L)\,D_{LK}(k_2,U_L,U_K)$$
$$-m^4\varepsilon^2\frac{N}{2V}\int dU_{\bar L}\int(dk_1\,dk_2)F_{K\bar L}(k_1,U_K,U_{\bar L})S_{I\bar L}(p,U_I,U_{\bar L})\,F_{\bar L K}(k_2,U_{\bar L},U_K). \qquad (13)$$

This equation is written for the case in which both of the pseudoparticles I and K are instantons. It is easy to see that $S_{\bar I\bar K}=S_{IK}$ and $S_{\bar I K}=S_{I\bar K}$. Consequently, only two of the four quantities are independent. If I and K are distinct, the Bethe-Salpeter equation becomes

$$S_{I\bar K}=m^4\varepsilon^4\int(dk_1\,dk_2)F_{\bar K I}(k_1)F_{I\bar K}(k_2)-m^4\varepsilon^2\frac{N}{2V}$$
$$\times\int(dk_1\,dk_2)F_{\bar K L}(k_1)S_{IL}(p)F_{L\bar K}(k_2)-m^4\varepsilon^2\frac{N}{2V}$$
$$\times\int dU_{\bar L}\int(dk_1\,dk_2)D_{\bar K\bar L}(k_1)S_{I\bar L}(p)D_{\bar L\bar K}(k_2). \qquad (14)$$

The general solution of Eqs. (13) and (14) is quite complicated because several different structures which depend on the orientation matrices U_I and U_K arise during the iteration of the seeds. However, in a study of the correlation functions of colorless currents the vertex functions $\gamma_{I,K}(p)$ do not depend on the orientation of the pseudoparticles U_I and U_K. For this reason, expression (12) includes the four-tail functions $S_{IK}(p,U_I,U_K)$ averaged over the orientations U_I and U_K, which we denote by $\bar S_{IK}(p)$. For $\bar S_{IK}(p)$ we can also write a closed system of equations, found from (13) and (14) by taking an average over U_I and U_K. The integration $\int dU_{I,K,L}$ over the Haar measure is easily carried out by making use of property [1, (23),(24)]. We find a system of algebraic equations for the quantities $\bar S_{IK}$:

$$\bar S_{IK}(p)=-\frac{2V\varepsilon^2}{N}G(p)+G(p)\bar S_{IK}(p)-H(p)\bar S_{I\bar K}(p),$$
$$\bar S_{I\bar K}(p)=\frac{2V\varepsilon^2}{N}H(p)-H(p)\bar S_{IK}(p)-G(p)\bar S_{I\bar K}(p), \qquad (15)$$

where G and $H(p)$ are dimensionless functions which are closed loops of the exact quark propagators D and F, given by

$$G(p)=\frac{4N_cV}{N}\int(dk_1\,dk_2)\frac{M_1^2M_2^2}{(k_1^2+M_1^2)(k_2^2+M_2^2)},$$
$$H(p)=\frac{4N_cV}{N}\int(dk_1\,dk_2)\cdot\frac{M_1M_2(k_1,k_2)}{(k_1^2+M_1^2)(k_2^2+M_2^2)},$$
$$M_{1,2}=M(k_{1,2}). \qquad (16)$$

We write a solution of system (15) in the form

$$\frac{\bar S_{IK}(p)}{2}\pm\frac{\bar S_{I\bar K}(p)}{2}=\frac{V\varepsilon^2}{N}\frac{-G(p)\pm H(p)}{1-[G(p)\mp H(p)]}. \qquad (17)$$

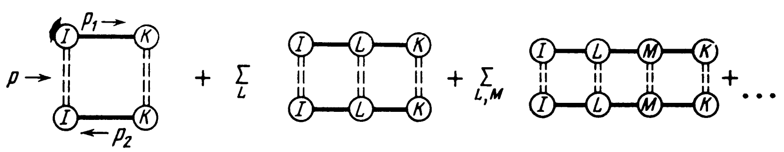

FIG. 5. Ladder diagrams for an s-channel four-tail function.

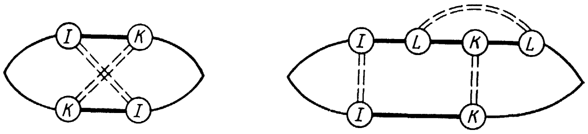

FIG. 6. Some nonplanar diagrams.

It remains to substitute these expressions and also the vertex functions $\gamma_I(p)$ found above [see (10)] into general expression (12) for $\Pi_2(p)$. Here it turns out to be convenient to combine $\Pi_2(p)$ with the Born contribution $\Pi_2(p)$ in (11). We call the sum of Π_1 and Π_2 the "coupled" part of the polarization operator $\Pi_{\rm con}(p)$, in contrast with the uncoupled part $\Pi_0(p)$ [see (5)]. The final expression for the polarization operator in (2) is

$$\Pi(p)=\Pi_0(p)+\Pi_{\rm con}(p), \tag{18}$$

$$\Pi_{\rm con}(p)=N_c\frac{VN_c}{N}\Big\{[\Gamma_I(p)+\Gamma_{\bar I}(p)]\times\frac{1}{R_+(p)}[\Gamma_I(-p)+\Gamma_{\bar I}(-p)]+[\Gamma_I(p)-\Gamma_{\bar I}(p)]\frac{1}{R_-(p)}[\Gamma_I(-p)-\Gamma_{\bar I}(-p)]\Big\}, \tag{19}$$

where the functions $\Gamma_{I,\bar I}(p)$ are the reduced vertex functions defined in (10), and $R_\pm(p)$ is, according to (16) and (17),

$$R_\pm(p)=1-\frac{4VN_c}{N}\int(dk_1\,dk_2)\frac{M_1M_2[M_1M_2\mp(k_1,k_2)]}{(k_1^2+M_1^2)(k_2^2+M_2^2)},$$

$$M_{1,2}=M(k_{1,2}). \tag{20}$$

Equations (18)–(20) and (10), along with the definition of the effective mass $M(k)$ [1, (50)], solve the problem of calculating the polarization operator of meson currents in the instanton vacuum. The expressions derived here are finite in the chiral limit $m\to 0$.

Remarkably, the coupled part of polarization operator (19) is of resonant form. The zeros of $R_\pm(p)$ determine the poles in some channel or other, while the reduced vertex functions $\Gamma(p)$ determine the residues at the poles.

4. GOLDSTONE PARTICLES IN THE INSTANTON VACUUM

Using the explicit expression in (10) for the reduced vertex functions $\Gamma_{I(\bar I)}(p)$, we easily see that (1) in the vector and tensor channels ($\Gamma=\gamma_\mu,\sigma_{\mu\nu}$) the vertex functions $\Gamma_{I,\bar I}$ are zero [so that only the uncoupled part $\Pi_0(p)$ acts in these channels in our approximation], (2) in the scalar channel we have $\Gamma_I=\Gamma_{\bar I}$ [so that there is a pole associated with a zero of $R_+(p)$ in the scalar channel], and (3) we have $\Gamma_I=-\Gamma_{\bar I}$ in the axial and pseudoscalar channels, so that a pole associated with a zero of $R_-(p)$ appears in these channels.

We can show that $R_-(p)$ has, in the case of a zero quark mass, a zero at $p^2=0$ corresponding to a Goldstone particle, i.e., a π meson. For this purpose we recall the self-consistency condition [1, (52)] which determines the value of ε or, equivalently, the value of the effective mass $M(p=0)$:

$$\int\frac{d^4k}{(2\pi)^4}\frac{M^2(k)}{k^2+M^2(k)}=\frac{N}{4VN_c}. \tag{21}$$

Using this condition, we write the 1 in expression (20) for $R_\pm(p)$ as

$$1=\frac{4VN_c}{N}\int(dk_1\,dk_2)\left[\frac{1}{2}\frac{M_1^2}{k_1^2+M_1^2}+\frac{1}{2}\frac{M_2^2}{k_2^2+M_2^2}\right].$$

Substituting this expression into (20), and carrying out some simple identity transformations, we find

$$R_\pm(p)=\frac{2VN_c}{N}\int(dk_1\,dk_2)\frac{(M_1k_{2\mu}\pm M_2k_{1\mu})^2}{(k_1^2+M_1^2)(k_2^2+M_2^2)}. \tag{22}$$

Using the parametrization

$$k_{1,2\mu}=k_\mu\pm p_\mu/2,\qquad (dk_1dk_2)=d^4k/(2\pi)^4,$$

we find

$$\begin{aligned}(M_1k_{2\mu}-M_2k_{1\mu})^2&=\left[(M_1-M_2)\frac{k_{1\mu}+k_{2\mu}}{2}+(M_1+M_2)\frac{(k_{2\mu}-k_{1\mu})}{2}\right]^2\\&=\left[(M_1-M_2)k_\mu-\frac{M_1+M_2}{2}p_\mu\right]^2\\&\approx\left(-Mp_\mu+\frac{dM}{d|k|}\frac{(pk)}{|k|}k_\mu\right)^2.\end{aligned} \tag{23}$$

We thus see that $R_-(p)$ vanishes at $p=0$, and at small values of p we have

$$R_-(p)\approx p^2\frac{2VN_c}{N}\int\frac{d^4k}{(2\pi)^4}\left[M^2-\frac{1}{2}M\frac{dM}{d|k|}|k|+\frac{1}{4}\left(\frac{dM}{d|k|}|k|\right)^2\right](k^2+M^2)^{-2}\equiv\beta p^2. \tag{24}$$

We note that $R_+(p)$, which determines the spectrum of hadrons in the scalar channel, does not vanish at $p=0$. Substituting (24) into (19), we have a massless pole corresponding to a π meson in the coupled part of the correlation function $\Pi(p)$ in the pseudoscalar channel.

We thus see that in the approach developed here Goldstone's theorem holds by virtue of the same equation which determines the chiral condensate. This is as it must be, since the existence of a massless Goldstone particle cannot depend on the details of the mechanism by which the chiral invariance is broken [in particular, it cannot depend on the particular momentum dependence of the effective mass of the quark, $M(p)$]. We do wish to emphasize, however, that a massless particle appears only if diagrams of an identical

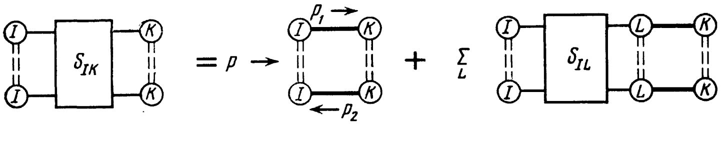

FIG. 7. Bethe-Salpeter equation for an s-channel four-tail function $S_{IK}(p)$.

type (in our case, these are planar diagrams) are taken into account for the Green's function and the coupled part of the correlation function $\Pi(p)$; there is a subtle equilibrium of these diagrams which maintains the Goldstone theorem. In particular, the corrections in $1/N_c$ must be taken into account simultaneously in the Green's function and in $\Pi(p)$.

We now calculate the correlation function of the pseudoscalar densities, $\Pi_5(p)$, for small momenta p^2, restricting the discussion to the coupled pole part of (19). The vertex in (10) takes the following form at small values of p $(\Gamma = \gamma_5)$:

$$\Gamma_I(0)=-\Gamma_{\bar{I}}(0)=-2\int\frac{d^4k}{(2\pi)^4}\frac{M(k)}{M^2(k)+k^2}=\frac{\langle\bar{\psi}\psi\rangle}{2N_c}. \qquad (25)$$

In other words, it is expressed in terms of the chiral condensate in the instanton vacuum (here we have used [1, (55)]).

We thus have (we recall that we are working with a single quark species, but we are assuming a flavor-nonsinglet channel; see the Introduction)

$$\Pi_5(p)=N_c\frac{VN_c}{N}\left(\frac{\langle\bar{\psi}\psi\rangle}{N_c}\right)^2\frac{1}{\beta p^2}, \qquad p\to 0, \qquad (26)$$

where β is defined by (24). On the other hand, we can introduce an axial constant of the π meson in the standard way:

$$\langle 0|J_{\mu 5}|\pi\rangle=if_\pi p_\mu, \qquad J_{\mu 5}=\bar{\psi}\gamma_\mu\gamma_5\psi. \qquad (27)$$

Saturating the polarization operator $\Pi_5(p)$ by the contribution of the π-meson intermediate state as small values of p, we have the general relation

$$\Pi_5{}^M(p)=\int d^4xe^{ipx}i\langle T\bar{\psi}i\gamma_5\psi(x),\ \bar{\psi}i\gamma_5\psi(0)\rangle=-\frac{4\langle\bar{\psi}\psi\rangle^2}{f_\pi^2p^2}.$$

Transforming to a Euclidean space in accordance with

$$\begin{gathered}\int id^4x_M=\int d^4x_E, \qquad i\bar{\psi}_M=\psi_E{}^+, \qquad \gamma_{5M}=\gamma_{5E},\\ p_M{}^2=-p_E{}^2, \qquad \gamma_{0M}=\gamma_{4E}, \qquad \gamma_{iM}=i\gamma_{iE},\end{gathered} \qquad (28)$$

we find, for small values of p,

$$\Pi_5(p)=4\langle\bar{\psi}\psi\rangle^2/f_\pi^2p^2. \qquad (29)$$

Comparing (29) and (26), we find the constant f_π:

$$f_\pi^2=4\frac{N}{V}\beta=8N_c\int\frac{d^4k}{(2\pi)}\left(M^2-\frac{1}{2}MM'|k|+\frac{1}{4}M'^2k^2\right)\times(k^2+M^2)^{-2}. \qquad (30)$$

We recall[1] that the effective mass is parametrically small, on the order of the "packing parameter" of the instanton medium:

$$M(0)\sim(1/\bar{R})(\bar{\rho}/\bar{R}),$$

where $\bar{\rho}$ is the average size, and $\bar{R}$ is the average distance between instantons. The scale of the change in $M(k)$ is $1/\bar{\rho}$; dM/dk vanishes in the limit $|k|\to 0$. The integral in (30) is thus determined by the broad region of parametrically small momenta $1/\bar{R}\ll k\ll 1/\bar{\rho}$. Calculating it with logarithmic accuracy, we find

$$f_\pi^2\approx\frac{N_c}{\pi^2}M^2(0)\ln\frac{1}{M(0)\bar{\rho}}\sim\frac{1}{\bar{\rho}^2}\left(\frac{\bar{\rho}}{\bar{R}}\right)^4\ln\frac{\bar{R}}{\bar{\rho}}. \qquad (31)$$

We see that f_π is parametrically small in comparison with the hadron scale, which is determined in the instanton vacuum by the average size of the pseudoparticles, $1/\bar{\rho}\approx 600$ MeV. Furthermore, f_π is small even in comparison with the gluon condensate, $\langle F_{\mu\nu}{}^2/32\pi^2\rangle\approx\bar{N}/V\approx 1/\bar{R}^4\approx(200\text{ MeV})^4$:

$$f_\pi^4/\langle F_{\mu\nu}{}^2/32\pi^2\rangle\sim\bar{\rho}^4/\bar{R}^4.$$

Consequently, the small value of the constant f_π which is found experimentally (and which is frequently perplexing) finds a natural explanation in the instanton vacuum. Conversely, the small value of f_π may serve as an argument for a sparse instanton vacuum in QCD.

Numerically, for $1/\bar{R} = 200$ MeV and $1/\bar{\rho} = 600$ MeV whence[1] $M(0) = 345$ MeV, the value of f_π calculated from (30) is 138 MeV, in excellent with the experimental value $f_\pi\approx 132$ MeV [approximation (31) yields $f_\pi\approx 142$ MeV]. We also note that we find $f_\pi{}^2\propto N_c$, as is required by general considerations.

The pole corresponding to the π meson can be found even away from the chiral limit, when there is a nonvanishing quark mass m (we assume $m_u = m_d = m$). It is not sufficient to simply repeat the calculations above for the case $m\neq 0$; it is necessary to modify the overlap integral T_{IJ} in (4), replacing it by

$$T_{IJ}=\int d^4x\psi_I{}^+(x)\,(i\hat{\partial}_x+im)\,\psi_J(x), \qquad (32)$$

so that T_{IJ} becomes nonzero not only for unlike pseudoparticles but also for like pseudoparticles. We give without derivation the results of the corresponding calculation for the denominator $R_-(p)$ in the linear approximation in p at small values of p:

$$R_-(p)=-(Vm/N)\langle\bar{\psi}\psi\rangle+\beta p^2. \qquad (33)$$

From this expression we can determine the mass of the π meson:

$$m_\pi{}^2=-(m\langle\bar{\psi}\psi\rangle/\beta)(V/N). \qquad (34)$$

Combining this result with (30), we find the well-known relation

$$m_\pi{}^2=-4m\langle\bar{\psi}\psi\rangle/f_\pi{}^2, \qquad (35)$$

which follows from current algebra. Finally, using (33), we find the value of the correlation function $\Pi_5(p)$ with $p^2 = 0$ but $m\neq 0$:

$$\Pi_5(p^2)\,|_{p^2=0}=N_c\frac{VN_c}{N}\left(\frac{\langle\bar{\psi}\psi\rangle}{N_c}\right)^2\frac{N}{V}\frac{1}{-m\langle\bar{\psi}\psi\rangle}=-\frac{\langle\bar{\psi}\psi\rangle}{m}. \qquad (36)$$

This is the well-known Ward identity in the pseudoscalar channel.[7] Expressions (34)–(36) demonstrate the complete consistency of the description of the π meson as a (pseudo-) Goldstone excitation which arises in the instanton vacuum as a result of the spotaneous breaking of chiral symmetry.

This massless π pole must be manifested in not only the pseudoscalar channel but also the axial channel. The correlation function of the axial currents, $\Pi_{\mu\nu}^{(A)}(p)$, must have the following form for small momenta in the chiral limit:

$$\Pi_{\mu\nu}^{(A)}(p)=-(f_\pi{}^2/p^2)[p^2\delta_{\mu\nu}-p_\mu p_\nu]. \qquad (37)$$

This expression gives us a new and independent determination of the constant f_π. A test of expression (37) is made even more interesting by the fact that the first term here should arise from the uncoupled diagram $\Pi_0(p)$ in (5), while the second should arise from the sum of the latter diagrams for $\Pi_{\rm con}(p)$ in (19). There is no difficulty in calculating the correlation function $\Pi_{\mu\nu}^{(A)}$ from (5) and (19). The function $\Pi_{\mu\nu}^{(A)}$ diverges quadratically, of course, reflecting the divergence of the free quark loop. We can regularize it by subtracting the contribution of free quarks; we find that at small momenta we have

$$\Pi_{\mu\nu}^{(A)}=-f_1^2\delta_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}f_2^2, \tag{38}$$

$$f_1^2=8N_c\int\frac{d^4k}{(2\pi)^4}\frac{M^2(k)}{(k^2+M^2)^2}, \tag{39}$$

$$f_2^2=8N_c\int\frac{d^4k}{(2\pi)^4}\frac{M^2(k)-{}^1/_4|k|dM/d|k|}{(k^2+M^2)^2}. \tag{40}$$

Consequently, $\Pi_{\mu\nu}^{(A)}$ is nontransverse (this is also true of the correlation function in the vector channel), and its value at the origin is not equal to the constant f_π calculated for the pseudoscalar channel.

The origin of this deviation from consistency can be seen in the following way. In this paper we are calculating the polarization operator by making use of the interpolation model [1, (38)] for the Green's function of a quark in the field of a single instanton. Although this approach simplifies the calculations to a great extent, it does make all our equations inexact at intemediate momenta, $p\sim 1/\bar\rho$. The system of wave functions of the zero modes in the field of each instanton is incomplete, so in restricting the discussion to a rediagonalization of only the zero modes we are violating certain general relations underlying the transition from channel to channel.

The quantities in which momenta $p\sim 1/\bar\rho$ are important are determined by not only the collective effects associated with the zero modes but also the specific form of the Green's function in the field of a single instanton. Incidentally, we recall that the main contribution to f_π arises in the logarithmic approximation from momenta $1/\bar R \ll p \ll 1/\bar\rho$. In this approximation, all three expressions for f_π [(30), (39), and (40)] are the same, as would be expected. Numerically, the results are also in fairly good agreement: $f_\pi = 138$ MeV, $f_1 = 110$ MeV, and $f_2 = 100$ MeV.

5. DISCUSSION

We have shown that there is necessarily a spontaneous breaking of chiral invariance in the instanton vacuum, so that a pole corresponding to a (pseudo-) Goldstone excitation—a π meson—arises in the correlation functions of pseudoscalar densities and axial currents. The initial massless (or nearly massless) quarks disappear from the theory: The lowest state of the theory is a colorless hadron. In the other hadron channels, the correlation functions exhibit an exponential (not power-law) behavior at large distances.

This result does not, of course, mean that this theory demonstrates confinement. For example, in the uncoupled part of the polarization operator, $\Pi_0(p)$ in (5), there is undoubtedly a threshold corresponding to the production of a quark and an antiquark with effective masses $M(0)\approx 345$ MeV. Unfortunately, it is not clear at this point whether this result is a defect of the approximations used in our calculations or a fundamental flaw of the instanton vacuum.

We believe that the influence of confinement effects (if there are such effects and if they differ from instanton effects) on the properties of hadrons may turn out to be somewhat less dramatic. We believe that a second qualitative phenomenon which arises in QCD is far more important: the spontaneous breaking of chiral symmetry. In any case, we need a theory in which the breaking of chiral symmetry is reproduced automatically, so that confinement effects can be distinguished.

In analyzing the correlation functions of the meson currents we have found that the instanton vacuum leads to a qualitative (and, in many regards, quantitative) agreement with the phenomenological results in low-energy meson physics. The next step in this direction would apparently be to study the properties of baryons, in whose physics an exceedingly important role is undoubtedly also played by the spontaneous breaking of chiral invariance.

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[1)] [1 (n)] means Eq. (n) of Ref. 1.

[1] D. I. D'yakonov and V. Yu. Petrov, Zh. Eksp. Teor. Fiz. **89**, 361 (1985) [Sov. Phys. JETP **62**, 204 (1985)].
[2] D. I. Dyakonov and V. Yu. Petrov, Nucl. Phys. B **245**, 259 (1985).
[3] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385, 448, 519 (1979).
[4] E. V. Shuryak, Nucl. Phys. B **203**, 93, 116, 140 (1982).
[5] I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, Vvedenie v teoriyu neuporyadochennykh sistem (Introduction to the Theory of Disordered Systems), Nauka, Moscow, 1982.
[6] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Eng. transl. Prentice-Hall, Englewood Cliffs, NJ (1963), Fizmatgiz, Moscow (1962).
[7] S. Glashow and S. Weinberg, Phys. Rev. Lett. **20**, 224 (1968); R. Crewther, J. Phys. Lett. **70B**, 349 (1977).

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