Mechanism for wave absorption or amplification in stochastic acceleration of particles

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A particle accelerated by stochastic resonant fields can either acquire energy from or transfer energy to nonresonant waves and fields. This effect underlies a nonlinear mechanism for wave amplification or absorption which may lead to an efficient transfer of electromagnetic energy upward in frequency.

§1. INTRODUCTION

Nonlinear amplification (or an absorption) of waves with several unusual properties was studied in a nonlinear theory as early as 1970 (Refs. 1 and 2; see also Refs. 3 and 4). In the simplest case, the occurrence of this effect requires (1) an intense random resonant field (well above the thermal level), e.g., a radiation field satisfying the condition for a Čerenkov or cyclotron resonance, (2) a steady-state distribution of particles, which may be resonant with the random field, and (3) a probing wave which does not satisfy the resonance conditions and which may be amplified or absorbed in the system. The absorption (or amplification) which occurs makes possible a significant change in frequency when no energy inversion of the particle populations is present but the resonant radiation is anisotropic. The damping rate (or growth rate) decreases slowly with the frequency, in proportion to $1/\omega^2$; i.e., there is a possibility, for example, of exciting electromagnetic waves of extremely high frequency. This absorption (or growth) cannot be an induced process with respect to some spontaneous emission.

In the motion of particles in a low-frequency resonant field there are no high-frequency harmonics, and such harmonics cannot be emitted spontaneously at high frequencies. It was shown in Ref. 4 that spontaneous emission does not occur in that order (q^4) in the charge q or that order ($|E_0|^2$) in the resonant field E_0 found from the relationship between absorption and spontaneous emission which is ordinarily used.

It was shown in Ref. 4 that the result of Ref. 1 for the nonlinear absorption (damping) under the assumption of a steady-state distribution of resonant particles can be derived rigorously by the Landau method[5] and therefore has the same theoretical foundation as the well-known linear Landau damping.[6,7]

The existence of absorption in a system which is far from equilibrium when spontaneous emission is absent does not rule out the possibility of a transition of the system to thermal equilibrium.

It was shown in Ref. 4 that the nonlinear absorption coefficient is proportional to the square amplitude of the resonant field, $|E_0|^2$, only under conditions such that the energy of the resonant field is much higher than the energy of the thermal fluctuations of this field, $|E_{0T}|^2$. In general, on the other hand, the nonlinear absorption coefficient is proportional to $|E_0|^2 - |E_{0T}|^2$, and it vanishes as the energy of the resonant field approaches the thermal noise level.

Many recent papers have developed certain aspects of Refs. 8 and 9, have extended the list of wave types for which this interaction has been studied,[4,10,11] and have taken into account the effects of an iteration of currents of second order in the field in the third-order nonlinear responses[12–14] (an effect which vanishes for high-frequency waves[1,4]). The physical interpretation of the effect in many of these papers has been, in our opinion, inexact or, in several cases,[12–14] simply wrong.

An error was made in Ref. 15 in connection with the symmetry properties of the real parts of the nonlinear responses. The interpretation and the diagram technique used in Ref. 15 are also inexact (see also Refs. 16–18). Our purpose in the present paper is to described the physical mechanism which is responsible for the exchange of energy between particles and a nonresonant high-frequency field when intense resonant fields are present.

§2. PHYSICAL MODEL FOR THE NONLINEAR WAVE ABSORPTION OR AMPLIFICATION

It was pointed out back in Ref. 2 that this nonlinear mechanism is closely related to the processes by which particles are accelerated by random fields or oscillations. The absorption coefficient in the isotropic case, for example, is proportional to the diffusion coefficient in momentum space describing the acceleration process.

What distinguishes Fermi acceleration by waves and oscillations[14] is whether the particle passes through a "cloud" (this is the case of acceleration by "waves") or is reflected from it (Fermi acceleration). The cloud might be a wave packet or a soliton (for magnetic clouds, it would be a packet of MHD waves). The condition that the time required for the particle to traverse the soliton (or cloud), $t = l/v_0$, be smaller than the characteristic period $2\pi/\omega$ (ω is the wave frequency, v_0 is the velocity of the particle, and l is the size of the soliton or cloud) is the same as the Čerenkov condition $v_0 > \omega l/2\pi = \omega/k$. Under this condition, a particle in a soliton is acted upon by a static field E_0, so that the simplest model for stochastic acceleration would be the following.

A system contains randomly distributed regions with a static electric field E_0 (capacitors)[1)] the size of a region is l, the average distance between regions is L, and there are equal probabilities that a particle will encounter fields with

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E_0 and $-E_0$, i.e., with opposite signs, along its path. After it has encountered a field E_0 directed along its velocity, a particle acquires an energy qE_0l, where q is the charge of the particle. When it encounters a field $-E_0$, it loses an energy of the same magnitude (for simplicity we assume that the soliton capacitors are at rest), but when it acquires energy the particle spends less time in traversing the distance L to the next region with a field. Consequently, the particle on the average acquires an energy

$$\frac{d}{dt}\langle\varepsilon\rangle=\frac{qE_0l}{t}-\frac{qE_0l}{t'}=\frac{qE_0l}{L}\left(v_0+\frac{qE_0l}{mv_0}\right)-\frac{qE_0l}{L}\left(v_0-\frac{qE_0l}{mv_0}\right)=\frac{2q^2E_0{}^2l^2}{mv_0L}=\frac{2q^2l}{mv_0}\langle E_0{}^2\rangle, \qquad (1)$$

where $\langle E_0{}^2\rangle = E_0{}^2l/L$ is the average square field.

This result agrees (to within a coefficient of order unity) with the result first derived in Ref. 19 (see Ref. 20) and also with a result derived from a quasilinear equation[21] (more on this below).

We now consider a nonresonant field E which causes particles to oscillate with a frequency Ω different from the field frequency ω because of the Doppler effect, i.e., $\Omega=\omega-\mathbf{kv}$. This event occurs in the absence of a resonant field E_0. It is customary to assume that the nonresonant field on the average performs no work; this assumption is of course correct if we are talking about the average work over the period.

When there is a resonant field E_0, the average must be calculated slightly more accurately. Let us assume, for example, that the size l of a region with $E_0=0$ is equal to an integer number of oscillation periods of the particle in the field E, so that after traveling a distance l a particle has the same velocity it had initially. If, on the other hand, a resonant field E_0 points in the same direction over a distance l, the average velocity of the particle (and its energy) will increase, so that the period $2\pi/\Omega$ increases, by virtue of the Doppler effect. The distance l is now not equal to an integer number of periods. Let us assume that E and E_0 are small, so that the deviations from a whole number of periods are small. At the exit from the slab the energy acquired is then slightly smaller than qE_0l, by an amount which depends on E. We set

$$\Delta\varepsilon=qE_0l(1-\alpha E),$$

where α is some coefficient.

If a particle meets an oppositely directed field E_0, its velocity will decrease, the period $2\pi/\Omega$ will also decrease, and the particle will lose slightly more energy,

$$\Delta\varepsilon=-qE_0l(1+\alpha E).$$

As a result, the average change in the energy of the particle is

$$\frac{d}{dt}\langle\varepsilon\rangle=\frac{qE_0l(1-\alpha E)}{L}\left(v_0+\frac{qE_0l(1-\alpha E)}{mv_0}\right)-\frac{qE_0l(1+\alpha E)}{L}\left(v_0-\frac{qE_0l(1+\alpha E)}{mv_0}\right)=\frac{q^2E_0{}^2l^2}{mv_0L}(2+\alpha^2E^2). \qquad (2)$$

We actually find in the energy acquired by the particle an additional term which depends on the amplitude of the nonresonant field; i.e., the field E performs work on the particle.

Obviously, this model is extremely crude. Among the many factors which it neglects are the change in the synchronization of the particles with the resonant fields due to the effect of the resonant fields and the result change in the rate of acceleration by the resonant fields. We thus turn to a rigorous, quantitative analysis of the question.

§3. ENERGY TRANSFERRED TO PARTICLES

For simplicity we consider the most elementary case, of nonrelativistic particles and longitudinal fields, both resonant and nonresonant. The resonant fields $\mathbf{E}_0$ are described by the Fourier components $\mathbf{E}_{k0}$,

$$\mathbf{E}_0(\mathbf{r},t)=\int\mathbf{E}_{k0}\exp(-i\omega_0t+i\mathbf{k}_0\mathbf{r})\,dk_0,\quad k_0=\{\mathbf{k}_0,\omega_0\};\quad dk_0=d\mathbf{k}_0d\omega_0, \qquad (3)$$

while the nonresonant fields $\mathbf{E}$ are described by the Fourier components $\mathbf{E}_k$,

$$\mathbf{E}(\mathbf{r},t)=\int\mathbf{E}_k\exp(-i\omega t+i\mathbf{kr})\,dk,\quad k=\{\mathbf{k},\omega\};\quad dk=d\mathbf{k}\,d\omega. \qquad (4)$$

The field $\mathbf{E}_0$ is random

$$\langle E_{k_0,i}E_{k_0',j}\rangle=(k_{0,i}k_{0,j}/k_0{}^2)\,|E_{k_0}|^2\delta(k_0+k_0'). \qquad (5)$$

We consider two cases regarding the nonresonant field: (1) a regular sinusoidal field

$$E_k=E^{(0)}\delta(k-k^{(0)})+E^{(0)*}\delta(k+k^{(0)}), \qquad (6)$$

(2) a random field

$$\langle E_{k,i}E_{k',j}\rangle=(k_ik_j/k^2)\,|E_k|^2\delta(k+k). \qquad (7)$$

The results for the regular field, (6), can be derived from the results for the random field, (7), by making the replacement

$$|E_k|^2\to|E^{(0)}|^2\delta(k-k^{(0)}). \qquad (8)$$

This circumstance is understandable since the result depends on the square of the amplitude of the regular field, and for a random field the only important consideration is the random nature of the relative phases of the different harmonics. When a single harmonic is left [as in (8)], its phase does not appear.

For a particle which is being accelerated, we can formulate an initial-value problem; we assume that the velocity of the particle at $t=0$ is $\mathbf{v}_0$. The energy acquired from the resonant field is determined by $(E=0)$

$$\frac{d}{dt}\langle\varepsilon\rangle=q\int\langle\mathbf{E}_{k_0}\mathbf{v}(t)\exp(-i\omega_0t+i\mathbf{k}_0\mathbf{r}(t))\rangle dk_0, \qquad (9)$$

$$m\frac{d\mathbf{v}(t)}{dt}=q\int\mathbf{E}_{k_0}\exp(-i\omega_0t+i\mathbf{k}_0\mathbf{r}(t))\,dk_0. \qquad (10)$$

We write $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_{E_0}$, $\mathbf{r}=\mathbf{v}_0t+\mathbf{r}_{E_0}$, where $\mathbf{v}_{E_0}$ and $\mathbf{r}_{E_0}$ are perturbations of the motion of the particles linear in the field $\mathbf{E}_0$. Retaining in (9) the terms quadratic in $\mathbf{E}_0$, we find

$$\frac{d}{dt}\langle\varepsilon\rangle=q\int\langle[(\mathbf{E}_{k_0}\mathbf{v})i(\mathbf{k}_0\mathbf{r}_{E_0})+(\mathbf{E}_{k_0}\mathbf{v}_{E_0})]\rangle\exp(-i\Omega_0t)\,dk_0, \qquad (11)$$

$$\mathbf{v}_{E_0}=\frac{q}{m}\int\frac{\mathbf{E}_{k_0}(\exp(-i\Omega_0t)-1)}{-i\Omega_0}dk_0,$$

$$\mathbf{r}_{E_0}=\frac{q}{m}\int\frac{\mathbf{E}_{k_0}(\exp(-i\Omega_0t)-1+i\Omega_0t)}{-\Omega_0{}^2}dk_0. \qquad (12)$$

Examining the asymptotic behavior at $t\to\infty$, and using

$$\sin\Omega_0 t/\Omega_0\to\pi\delta(\Omega_0),\quad \Omega_0=\omega_0-\mathbf{k}_0\mathbf{v},\tag{13}$$

we find

$$\frac{d}{dt}\langle\varepsilon\rangle_{E_0^2}=\frac{q^2}{m}\int|E_{k_0}|^2(\delta(\Omega_0)-\mathbf{k}_0\mathbf{v}_0\delta'(\Omega_0))\,dk_0.\tag{14}$$

The subscript on the left side of (14) specifies the power of the resonant field which is being taken into consideration.

Expression (14) describes the well-known resonant acceleration,[19] which can also be found from the quasilinear equation[21] ($\Phi_\mathbf{p}$ is the average distribution function)

$$\left(\frac{d\Phi_\mathbf{p}}{dt}\right)_{E_0^2}=\frac{q^2}{m^2}\int\frac{|E_{k_0}|^2}{k_0^2}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\Phi_\mathbf{p}\,dk_0.\tag{15}$$

We wish to stress that since we are dealing with the initial rate at which a particle acquires energy in (14), it is sufficient to substitute the initial particle distribution function $\Phi_\mathbf{p}^{(0)}$ into the right side of (15). We find

$$\frac{d}{dt}\int\varepsilon_p\frac{\Phi_\mathbf{p}d\mathbf{p}}{(2\pi)^3}=\frac{q^2}{m}\int|E_{k_0}|^2(\delta(\Omega_0)-(\mathbf{k}_0\mathbf{v})\delta'(\Omega_0))\frac{\Phi_\mathbf{p}\,d\mathbf{p}}{(2\pi)^3}dk_0.\tag{16}$$

The increase in the average energy of the particles is therefore determined by the rate at which an individual particle acquires energy, averaged over the initial distribution.

We now consider an additional effect in the transfer of energy to particles; this effect stems from the presence of a nonresonant field. The term linear in E (for a regular field) oscillates markedly, and in the limit $t\to\infty$ it vanishes. For a random nonresonant field its average value is zero. The term quadratic in E is determined by the expansion

$$\mathbf{v}(t)=\mathbf{v}_0+\mathbf{v}_{E_0}+\mathbf{v}_E+\mathbf{v}_{E_0^2}+\mathbf{v}_{E^2}+\mathbf{v}_{E_0,E}+\mathbf{v}_{E_0^2,E}+\mathbf{v}_{E_0,E^2}+\dots\tag{17}$$

and by an analogous expansion for $\mathbf{r}(t)$, where the subscript specifies the power of the field which is being taken into account. The quantities $\mathbf{v}_E$ and $\mathbf{r}_E$ are determined by (12) with $E_{k_0}\to E_k$ and $k_0\to k$; the other quantities are determined from the equations

$$\frac{d\mathbf{v}_{E_0,E}}{dt}=\frac{q}{m}\int\mathbf{E}_{k_0}i(\mathbf{k}_0\mathbf{r}_E)e^{-i\Omega_0t}dk_0+\frac{q}{m}\int\mathbf{E}_ki(\mathbf{k}\mathbf{r}_{E_0})e^{-i\Omega t}dk,$$

$$\frac{d\mathbf{v}_{E_0^2}}{dt}=\frac{q}{m}\int\mathbf{E}_{k_0}i(\mathbf{k}_0\mathbf{r}_{E_0})e^{-i\Omega_0t}dk_0,\quad\frac{d\mathbf{v}_{E_0^2,E}}{dt}\tag{18}$$

$$=\frac{q}{m}\int\mathbf{E}_{k_0}dk_0\exp(-i\Omega_0t)\left[i(\mathbf{k}_0\mathbf{r}_{E_0})i(\mathbf{k}_0\mathbf{r}_E)+i(\mathbf{k}_0\mathbf{r}_{E_0,E})\right]$$

$$+\frac{q}{m}\int\mathbf{E}_k\,dke^{-i\Omega t}\left[-\frac{1}{2}(\mathbf{k}\mathbf{r}_{E_0})^2+i(\mathbf{k}\mathbf{r}_{E_0^2})\right]$$

with analogous equations for $\mathbf{v}_{E_0,E^2}$ and $\mathbf{v}_{E^2}$, found by making the substitutions $\mathbf{E}_0\to\mathbf{E}$ and $k_0\to k$.

The energy change which is quadratic in E and E_0, i.e., a term which arises in addition to the quasilinear acceleration in (15), is determined by

$$\delta\frac{d}{dt}\langle\varepsilon\rangle=\frac{d}{dt}\langle\varepsilon\rangle_{E_0^2,E^2}+\frac{d}{dt}\langle\varepsilon\rangle_{E^2,E_0^2}.\tag{19}$$

The second term in (19) can be found from the first term in (19) by making the interchanges $\mathbf{E}_0\leftrightarrow\mathbf{E}$ and $k_0\leftrightarrow k$:

$$\frac{d}{dt}\langle\varepsilon\rangle_{E_0^2,E^2}=q\Big\langle\int\Big\{(\mathbf{E}_k\mathbf{v}_0)\Big[i(\mathbf{k}\mathbf{r}_E)\Big(-\frac{1}{2}(\mathbf{k}\mathbf{r}_{E_0})^2+i(\mathbf{k}\mathbf{r}_{E_0^2})\Big)$$
$$-(\mathbf{k}\mathbf{r}_{E_0})(\mathbf{k}\mathbf{r}_{E_0,E})-(\mathbf{k}\mathbf{r}_E)(\mathbf{k}\mathbf{r}_{E_0^2})+i(\mathbf{k}\mathbf{r}_{E,E_0^2})\Big]$$
$$+(\mathbf{E}_k\mathbf{v}_{E_0})\left[-(\mathbf{k}\mathbf{r}_{E_0})(\mathbf{k}\mathbf{r}_E)+i(\mathbf{k}\mathbf{r}_{E_0,E})\right]$$
$$+(\mathbf{E}_k\mathbf{v}_E)\left[-\frac{1}{2}(\mathbf{k}\mathbf{r}_{E_0})^2+i(\mathbf{k}\mathbf{r}_{E_0^2})\right]\tag{20}$$
$$+(\mathbf{E}_k\mathbf{v}_{E_0,E})i(\mathbf{k}\mathbf{r}_{E_0})+(\mathbf{E}_k\mathbf{v}_{E_0^2})i(\mathbf{k}\mathbf{v}_E)+(\mathbf{E}_k\mathbf{v}_{E,E_0^2})\Big\}$$
$$\times\exp(-i\Theta t)\,dk\Big\rangle.$$

After some lengthy but straightforward calculations involving the solution of (16), finding the corresponding $\mathbf{r}_E,\dots$, taking an average with the help of (5) and (7), and calculating the asymptotic behavior in the limit $t\to\infty$ (making use of the fact that only the resonance $\Omega_0=0$ is possible; both the direct resonance of the fields $\mathbf{E}$, i.e., $\Omega=0$, and the resonance $\Omega-\Omega_0=0$ are impossible), we find the final results:

$$\frac{d}{dt}\langle\varepsilon\rangle_{E_0^2,E^2}+\frac{d}{dt}\langle\varepsilon\rangle_{E^2,E_0^2}=\frac{q^4}{m^3}\int|E_k|^2|E_{k_0}|^2dk\,dk_0$$
$$\times\Big\{\Big[9\frac{(\mathbf{kk}_0)^2}{k_0^2\Omega^4}+12\frac{(\mathbf{kk}_0)^2(\mathbf{kv})}{k_0^2\Omega^5}-\frac{4(\mathbf{kk}_0)^3(\mathbf{k}_0\mathbf{v})}{k^2k_0^2\Omega^5}-\frac{(\mathbf{kk}_0)^2}{k^2\Omega^4}\Big]\delta(\Omega_0)$$
$$+\Big[-5\frac{(\mathbf{kk}_0)}{\Omega^3}-\frac{3(\mathbf{kk}_0)(\mathbf{kv})}{\Omega^4}+\frac{(\mathbf{kk}_0)^2(\mathbf{k}_0\mathbf{v})}{k^2\Omega^4}-\frac{3(\mathbf{kk}_0)^2(\mathbf{kv})}{k_0^2\Omega^4}\Big]$$
$$\times\delta'(\Omega_0)$$
$$+\Big[\frac{3(\mathbf{kk}_0)^2}{2k^2\Omega^2}+2\frac{(\mathbf{kk}_0)(\mathbf{k}_0\mathbf{v})}{\Omega^3}\Big]\delta''(\Omega_0)+\frac{(\mathbf{kk}_0)^2(\mathbf{k}_0\mathbf{v})}{2k^2\Omega^2}\delta'''(\Omega_0)\Big\}.\tag{21}$$

The primes on the δ-functions indicate derivatives with respect to their arguments. The result in (19) contains no secular terms of any sort, proving that the expansion is legitimate and also showing that a particle undergoes a change in energy under the influence of the nonresonant field. It is also obvious that this change is due not only to the absorption (or amplification) by the nonresonant fields but also to a change which they bring about in the resonant acceleration (14). The most convenient way to separate the effects is to turn to a kinetic description of the process.

§4. KINETIC DESCRIPTION

We now consider the problem from the standpoint of a kinetic description. Introducing the average part $\Phi_\mathbf{p}$ and the fluctuational part $\delta f_\mathbf{p}$ of the particle distribution, $f_\mathbf{p}=\Phi_\mathbf{p}+\delta f_\mathbf{p}$, we find it convenient to consider the initial-value problem in which the fields E and E_0 are turned on at $t=0$. We can then write

$$\frac{d\Phi_\mathbf{p}}{dt}=-\frac{q}{m}\int\Big\langle\Big(\mathbf{E}_{k'}\frac{\partial}{\partial\mathbf{v}}\Big)\delta f_\mathbf{k}(t)\Big\rangle$$
$$\times\exp(-i\omega't+i(\mathbf{k}+\mathbf{k}')\mathbf{r})\,d\mathbf{k}\,dk'$$
$$-\frac{q}{m}\int\Big\langle\Big(\mathbf{E}_{k_0'}\frac{\partial}{\partial\mathbf{v}}\Big)\delta f_{\mathbf{k}_0}(t)\Big\rangle\exp(-i\omega_0't+i(\mathbf{k}_0+\mathbf{k}_0')\mathbf{r})\,dk_0\,dk_0',\tag{22}$$

$$\frac{\partial}{\partial t}\delta f_{\mathbf{k}}(t)+i(\mathbf{kv})\delta f_{\mathbf{k}}(t)=-\frac{q}{m}\left(\mathbf{E}_{\mathbf{k}}(t)\frac{\partial}{\partial\mathbf{v}}\right)\Phi_{\mathbf{p}}(t)$$
$$-\frac{q}{m}\int d\mathbf{k}_1\left\{\left(\mathbf{E}_{\mathbf{k}_1}(t)\frac{\partial}{\partial\mathbf{v}}\right)\delta f_{\mathbf{k}-\mathbf{k}_1}(t)\right.$$
$$\left.-\left\langle\left(\mathbf{E}_{\mathbf{k}_1}(t)\frac{\partial}{\partial\mathbf{v}}\right)\delta f_{\mathbf{k}-\mathbf{k}_1}(t)\right\rangle\right\}d\mathbf{k}_1$$
$$+\{\mathbf{E}\to\mathbf{E}_0,\ \mathbf{k}\to\mathbf{k}_0\}. \tag{23}$$

Here we have used only a spatial Fourier expansion for $\delta f_{\mathbf{p}}$. Actually, the quasilinear equation (15) can be derived in the case in which there is only a resonant field ($\mathbf{E}_0\neq 0$, $\mathbf{E}=0$) by taking into account only the first term on the right side of (23), replacing $\mathbf{E}$ by $\mathbf{E}_0$, solving (23) with the initial condition $\delta f_{\mathbf{p}}=0$ at $t=0$, substituting the solution into the second term in (22), and taking the asymptotic limit $t\to\infty$ with the help of (13). In the ordinary derivation of (15) we find, instead of a δ-function,

$$-\frac{1}{\pi}\operatorname{Im}\frac{1}{(\Omega_0+i\Delta)}\Big|_{\Delta\to+0}.$$

The pole $\Delta\to+0$ is traversed by Landau's rule. All that we achieve through this derivation of the quasilinear equation is to prove Landau's rule. The same method will be used in calculating the additional terms which arise from the nonresonant fields, in which there are singularities of higher order ($1/\Omega_0^2$ and $1/\Omega_0^3$), so we need to rigorously prove our method for treating them.

The method is the same as was used above for a test particle: an expansion in the fields $\mathbf{E}$ and $\mathbf{E}_0$, with terms up to order $|E|^2$ and $|E_0|^2$ being retained. This calculation makes it possible to identify those effects which describe the changes in the absorption of the resonant fields and which do not result from the absorption (or amplification) of nonresonant fields.

Omitting the lengthy calculations, which are carried out by the method described above, which involve an expansion in the fields $\mathbf{E}$ and $\mathbf{E}_0$, and which are the assumption that only the resonance $\Omega_0=0$ occurs (the resonances $\Omega=0$ and $\Omega=\Omega_0$ do not occur), we write the final result:

$$\frac{d\Phi_{\mathbf{p}}}{dt}=\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2,E^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2,E_0^2}, \tag{24}$$

where $(d\Phi_{\mathbf{p}}/dt)_{E_0^2}$ is a quasilinear collision integral describes by (15), $(d\Phi_{\mathbf{p}}/dt)_{E^2}$ is a collision integral of nonresonant fields which is linear in $|E|^2$, $(d\Phi_{\mathbf{p}}/dt)_{E_0^2,E^2}$ is a collision integral describing the change in the absorption of the resonant fields due to the effect of the nonresonant fields, and $(d\Phi_{\mathbf{p}}/dt)_{E^2,E_0^2}$ is a collision integral describing the absorption (or amplification) of nonresonant fields due to the presence of the resonance fields (this is the effect which we wish to examine). Since the fields E are nonresonant, the quantity $(d\Phi_{\mathbf{p}}/dt)_{E^2}$ is determined exclusively by the time variation of the particle distribution:

$$\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2}=\frac{q^2}{m^2}\int\frac{|E_k|^2}{k^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{\partial\Phi_{\mathbf{p}}}{\partial t}dk \tag{25}$$

(for simplicity here we are considering a spatially homogeneous particle distribution). A corresponding term with $\partial\Phi_{\mathbf{p}}/\partial t$ is also present in $(d\Phi_{\mathbf{p}}/dt)_{E_0^2}$, but it can give rise only to effects of order E_0^4, in which we are not interested here. For the resonant field we thus restrict the discussion to the quasilinear expression (15). Further calculations lead to

$$\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2,E^2}=-\frac{q^4\pi}{m^4}\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)$$
$$\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\Phi_{\mathbf{p}}dk\,dk_0$$
$$+\frac{q^4\pi}{m^4}\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)$$
$$\times\left[-\frac{(\mathbf{kk}_0)^2}{\Omega^4}\delta(\Omega_0)-\frac{(\mathbf{kk}_0)^2}{2\Omega^2}\delta''(\Omega_0)-\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)(\mathbf{kk}_0)\delta'(\Omega_0)\right.$$
$$\left.+\delta(\Omega_0)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\ \mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\Big]\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\Phi_{\mathbf{p}}dk\,dk_0. \tag{26}$$

We stress that both terms in (26) are the same as those found in Ref. 4 [Eqs. (23b) and (23c) of Ref. 4; there is a misprinted sign in (23c); in (23a), $1/(\Omega-\Omega_0)$ can be replaced by $1/\Omega$ to the accuracy of this treatment] if we assume

$$\operatorname{Im}\frac{1}{\Omega_0}=-\pi\delta(\Omega_0),\qquad \operatorname{Im}\frac{1}{\Omega_0^2}=\pi\delta'(\Omega_0),$$
$$\operatorname{Im}\frac{1}{\Omega_0^3}=-\frac{\pi}{2}\delta''(\Omega).$$

This result corresponds to a generalized Landau rule for integrating around a pole in all the last relations: $1/\Omega_0\to 1/(\Omega_0+i\Delta)$, $\Delta\to+0$.

Proof that it is in fact (26) which describes the change in the absorption of the resonant field can be found by comparing the rate of energy absorption found from (26) with that found from the relation

$$-\frac{1}{4\pi}\int|E_{k_0}|^2\omega_0(\operatorname{Im}\varepsilon_{k_0}^L+\operatorname{Im}\varepsilon_{k_0}^N)\,dk_0, \tag{27}$$

where $\varepsilon_{k_0}^L$ is the linear dielectric constant, and $\varepsilon_{k_0}^N$ is the nonlinear dielectric constant (proportional to $|E|^2$). The first term in (27) gives the quasilinear absorption found from (15), while the second term gives the absorption found from (26).

The corrections to the acceleration by the resonant fields found from (25) deserve a more detailed study. These corrections are important when $\Phi_{\mathbf{p}}$ must be expressed in terms of the initial distribution $\Phi_{\mathbf{p}}^{(0)}$, as is necessary for a comparison with the results of the preceding section. The accuracy of this treatment, we can find $\partial\Phi_{\mathbf{p}}/\partial t$ from the quasilinear term in this term. We then find an additional effect proportional to $|E_k|^2$ and $|E_{k_0}|^2$:

$$\frac{q^4}{m^4}\pi\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)$$
$$\times\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial\mathbf{v}}\right)dk\,dk_0. \tag{28}$$

Finally, if there were no resonant field, (25) would reduce to

$$\frac{\partial}{\partial t}\left[\Phi_{\mathbf{p}}-\frac{q^2}{m^2}\int\frac{|E_k|^2}{k^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\Phi_{\mathbf{p}}dk\right]=0, \qquad (29)$$

i.e.,

$$\Phi-\frac{e^2}{m^2}\int\frac{|E_k|^2}{k^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{v}}\right)dk=\text{const}=\Phi_{\mathbf{p}}^{(0)}, \qquad (30)$$

or, with an accuracy sufficient for our purposes,

$$\Phi_{\mathbf{p}}=\Phi_{\mathbf{p}}^{(0)}+\frac{q^2}{m^2}\int\frac{|E_k|^2}{k^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial\mathbf{p}}\right)dk. \qquad (31)$$

Substituting (31) into quasilinear term (15), we find an additional term proportional to $|E_{k_0}|^2$ and $|E_k|^2$:

$$\frac{q^4}{m^4}\pi\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial\mathbf{v}}\right)dk\,dk_0. \qquad (32)$$

In the two terms in (26) we can replace $\Phi_{\mathbf{p}}$ by $\Phi_{\mathbf{p}}^{(0)}$.

The change in the particle distribution can thus be expressed in terms of the initial distribution as follows:

$$\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2,E^2}$$
$$=\frac{q^2}{m^2}\pi\int\frac{|E_{k_0}|^2}{k_0^2}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial\mathbf{v}}\right)dk_0$$
$$+\frac{q^4}{m^4}\pi\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}$$
$$\times\Big\{-\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\left[\frac{(\mathbf{kk}_0)^2}{\Omega^4}\delta(\Omega_0)+\frac{(\mathbf{kk}_0)^2}{2\Omega^2}\delta''(\Omega_0)\right]\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)$$
$$+\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)$$
$$+\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\delta(\Omega_0)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\Big\}\Phi_{\mathbf{p}}^{(0)}dk_0\,dk. \qquad (33)$$

This relation can be used to find the average change in the energy of the particles which results from the change in the rate of resonant acceleration [we are considering all the terms in (33) except the first]:

$$\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2}+\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E_0^2,E^2}=\left(\frac{d\Phi_{\mathbf{p}}^{(0)}}{dt}\right)_{E_0^2}+\left(\frac{d\Phi_{\mathbf{p}}^{(0)}}{dt}\right)_{E_0^2,E^2}; \qquad (\text{r})$$

$$\int\varepsilon_{\mathbf{p}}\left(\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial t}\right)_{E_0^2,E^2}\frac{d\mathbf{p}}{(2\pi)^3}=\int\frac{\Phi_{\mathbf{p}}^{(0)}d\mathbf{p}}{(2\pi)^3}|E_k|^2|E_{k_0}|^2dk\,dk_0$$
$$\times\frac{q^4\pi}{m^3}\Big\{\Big[\frac{21(\mathbf{kk}_0)^2}{k_0^2\Omega^4}+\frac{24(\mathbf{kk}_0)^2(\mathbf{kv})}{\Omega^5}-\frac{4(\mathbf{kk}_0)^3(\mathbf{k}_0\mathbf{v})}{k^2k_0^2\Omega^5}-\frac{(\mathbf{kk}_0)^2}{k^2\Omega^4}\Big]\delta(\Omega_0)$$
$$+\Big[-\frac{8(\mathbf{kk}_0)}{\Omega^3}-\frac{6(\mathbf{kk}_0)(\mathbf{kv})}{\Omega^4}+\frac{(\mathbf{kk}_0)^2(\mathbf{k}_0\mathbf{v})}{k^2\Omega^4}-\frac{3(\mathbf{kk}_0)^2(\mathbf{k}_0\mathbf{v})}{k_0^2\Omega^4}\Big]\delta'(\Omega_0)$$
$$+\Big[\frac{2(\mathbf{kk}_0)(\mathbf{k}_0\mathbf{v})}{\Omega^3}+\frac{3(\mathbf{kk}_0)^2}{2k^2\Omega^2}\Big]\delta''(\Omega_0)+\frac{(\mathbf{kk}_0)^2(\mathbf{k}_0\mathbf{v})}{2k^2\Omega^2}\delta'''(\Omega_0)\Big\}. \qquad (34)$$

As we mentioned earlier, this expression does not include the work associated with the absorption of the nonresonant field. These effects are described by the last term in (24). A calculation yields

$$\left(\frac{d\Phi_{\mathbf{p}}}{dt}\right)_{E^2,E_0^2}=-\frac{q^4\pi}{m^4}\int\frac{|E_k|^2|E_{k_0}|^2}{k^2k_0^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}_0\frac{\partial}{\partial\mathbf{v}}\right)\frac{1}{\Omega}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\times\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial\Phi_{\mathbf{p}}^{(0)}}{\partial\mathbf{v}}\right)dk\,dk_0, \qquad (35)$$

which is essentially the same as the result of Ref. 4. That it is this term which is associated with the absorption of the nonresonant field can be seen from the expression for the imaginary part of the nonlinear dielectric constant found in Ref. 4:

$$\operatorname{Im}\varepsilon_k^N=\frac{12\pi^2q^4}{m^3}\int\frac{|E_{k_0}|^2}{k_0^2}\frac{(\mathbf{kk}_0)}{\Omega^4}\delta(\Omega_0)\left(\mathbf{k}_0\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{v}}\right)\frac{d\mathbf{p}}{(2\pi)^3}dk_0. \qquad (36)$$

The rate of energy dissipation is determined by

$$-\frac{1}{4\pi}\int\operatorname{Im}\varepsilon_k^N\omega|E_k|^2dk, \qquad (37)$$

where the nonlinear dielectric constant ε_k^N is proportional to $|E_0|^2$. From (36) we find the dissipation rate corresponding to (35). The change in the energy of the particles associated with the absorption of the nonresonant field, (35), is

$$\int\varepsilon_{\mathbf{p}}\left(\frac{d\Phi_{\mathbf{p}}^{(0)}}{dt}\right)_{E^2,E_0^2}\frac{d\mathbf{p}}{(2\pi)^3}=\frac{q^4\pi}{m^3}\int\frac{\Phi_{\mathbf{p}}^{(0)}d\mathbf{p}}{(2\pi)^3}|E_k|^2\times|E_{k_0}|^2dk\,dk_0\Big\{-\Big[\frac{12(\mathbf{kk}_0)^2}{k_0^2\Omega^4}+\frac{12(\mathbf{kk}_0)^2(\mathbf{kv})}{k_0^2\Omega^5}\Big]\times\delta(\Omega_0)+\Big[\frac{3(\mathbf{kk}_0)}{\Omega^3}+\frac{3(\mathbf{kk}_0)(\mathbf{kv})}{\Omega^4}\Big]\delta'(\Omega_0)\Big\}. \qquad (38)$$

Adding (38) to (34), we find a result which agrees exactly with (21). This completes the proof of the assertions above.

These calculations provide a physical interpretation of the nonlinear absorption of nonresonant fields. In our case, it is sufficient to use equations in which all the terms in (24) are taken into account. However, the nonlinear absorption of nonresonant waves will be most obvious in the case in which the particle distribution is (at least in a first approximation) in a steady state for some reason or other.

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[1)]It is sufficient if the field is constant only over the time required for the particle to pass through the region with a field; on the average, on the other hand, the field can even be zero.

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