Exchange scattering of quasiparticles by a positive ion in superfluid ^{3}He-*B*

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(Submitted 27 March 1985)
Zh. Eksp. Teor. Fiz. **89**, 894–904 (September 1985)

The difference between the temperature dependences of the mobilities of positive and negative ions in superfluid ^{3}He-*B* is related to exchange interaction between quasiparticles and helium atoms from ice-like shells surrounding positive ions. Exact wave functions are obtained for a quasiparticle in the field of a spherical potential barrier of large but finite height. These wave functions are used to calculate the exchange scattering by an ion, and it is shown that the superfluid transition influences exchange scattering less than potential scattering. The results agree approximately with experiment.

I. INTRODUCTION

According to prevailing notions concerning ion structure, an electron placed in helium becomes surrounded by a spherical cavity of microscopic size, whereas a positive ion, in view of the electric field, is contained in a region in which the pressure is high enough to crystallize a small region around the ion.

Study of ion motion in ^{3}He at low temperatures[1–6] has revealed a large difference between ions of opposite sign. This become particularly pronounced in experiments performed at high pressures, $p \approx 30$ atm. It was estimated[6] that under these conditions the radius a_- of the bubble around the electron and the radius a_+ of the solidified helium[8] should become practically equal ($a_+ \approx a_- \approx 10$ Å), so that their mobilities should have a similar behavior. It was observed, on the contrary, that in the normal phase the mobility $\mu_-(T)$ of negative ions at $T_c \leqslant T \leqslant 50$ mK is almost constant, but the mobility of positive ions increases logarithmically with rising temperature. The difference is preserved also when superfluidity sets in, and $\mu_+(T)$ increases with decreasing temperature more slowly than $\mu_-(T)$. The difference in the temperature dependence cannot be attributed to the small difference between the ion radii, and points to a qualitative difference between the mechanisms whereby the quasiparticles interact with ions of opposite sign.

One of the possible scattering mechanisms that distinguishes different ions may be the exchange interaction. In the case of a negative ion the quasiparticle scattering is due to repulsion by the electron contained in the bubble, whereas in the case of a positive ion the quasiparticle interacts with a solid-helium iceberg. The potential scattering can therefore be supplemented by exchange scattering from the helium-atom iceberg with frozen surface. It was shown in Ref. 9 that allowance for exchange scattering in the normal phase can explain the logarithmic increase of $\mu_+(T)$ with decreasing temperature. In the approximation in which the quasiparticles collide elastically with the ion, exchange scattering is similar to electron scattering in a metal by a paramagnetic impurity i.e., the Kondo effect.[11,12] If the sign of the exchange constant is that of a ferromagnet, i.e., the same as between quasiparticles and a liquid, the interaction with the ion weakens logarithmically with decreasing temperature, and the mobility increases.

It follows from experiment[5] that at $T < T_c$ the relative mobility $\mu_+(T)/\mu_+(T_c)$ of positive ions increases with decreasing temperature noticeably more slowly than $\mu_-(T)/\mu_-(T_c)$ for a bubble. The value of $\mu_-(T)$ was calculated in Ref. 12 and provides a good quantitative description of the results of experiments near T_c. It follows from the results of Ref. 12 that $\mu(T)/\mu(T_c)$ is independent of the ion radius and is determined only by the modulus of the order parameter $\Delta(T)$ and by the character of the interaction between the quasiparticle and the ion. In the present paper the cause of this difference is taken to be the exchange interaction of the quasiparticles with the iceberg.

We assume that the ion moves at constant velocity, meaning that recoil in collisions is disregarded. This assumption is based on experiments with bubbles at high pressures,[1] where, in accordance with this assumption, the mobility is independent of temperature at any $T > T_c$. This circumstance seems to indicate that the use of hydrodynamic concepts to estimate the effective masses of ions in a degenerate Fermi liquid is not fully justified. There is at present no complete solution to the problem of the ion recoil energy.[13,14]

2. SCATTERING BY A SPHERICAL POTENTIAL

The exchange interaction between an atom of a liquid and a surface atom located at a point **R** is of the form

$$V_{\alpha i|\beta j} = {}^1/_2\Gamma(\mathbf{r}-\mathbf{R})\,\sigma_{\alpha\beta}\sigma_{ij}, \tag{1}$$

where the Greek subscripts label the spin of the liquid atom, and the Latin ones refer to the surface atom. The function Γ is concentrated in a region having a linear dimension of the order of the diameter d of the ^{3}He atom and its value V_0 is of the order of the repulsion between two atoms separated by the same distance, i.e., $V_0 \leqslant U_0$, where U_0 is the potential barrier produced by the entire ion. We shall assume that the exchange interaction of the quasiparticle with the surface has the same form (1). The scattering amplitude is determined by the probability of finding the particle at a point where the exchange potential differs substantially from zero,

 0038-5646/85/090514-06$04.00

i.e., under the barrier of height V_0, multiplied by the value of this potential. Although this probability is low at large V_0, this product tends to a finite limit. To take better account of this circumstance and to simplify the calculations, we make the substitution

$$\Gamma(\mathbf{r}-\mathbf{R}) \to (4\pi d^3/3) V_0 \delta(\mathbf{r}-\mathbf{R}), \tag{2}$$

and assume $\mathbf{R}$ to be arbitrarily located on the surface of a sphere of radius a. To obtain the cross section for exchange scattering by the entire ion, we multiply the cross section for scattering by the potential (2), averaged over the directions of $\mathbf{R}$, by the total number $N_s = 4\pi a^2 d n_s$, of the surface ions, where n_s is the density of the solid ^{3}He.

To calculate the Born corrections for V_0 we need the exact wave functions in the potential U_0. This is needed more in a superfluid liquid than in a normal one, for otherwise the singularities in the density of states lead in the calculations to integrals that diverge with respect to energy. The scattering states are obtained from the Bogolyubov equation[15,16]

$$H_0 \Psi_{\mathbf{k},\lambda} = E_{\mathbf{k}} \Psi_{\mathbf{k},\lambda},$$

$$H_0 = \begin{pmatrix} \xi(\mathbf{p}) + U(\mathbf{r}) & \hat{\Delta} \\ \hat{\Delta}^+ & -\xi(\mathbf{p}) - U(\mathbf{r}) \end{pmatrix}, \tag{3}$$

$$U(\mathbf{r}) = U_0 \theta(a-r), \quad \hat{\Delta} = \Delta \sigma \sigma_2 \mathbf{p}/k_F, \quad \mathbf{p} = -i\nabla,$$

where the subscript λ numbers two different spin states. We assume the order parameter in (3) to be equal to its value in a homogeneous medium, so that Eqs. (3) become linear. It is shown in Ref. 17 that the deviation of the order parameter from the equilibrium value at distances on the order of the ion radius is small in the parameter $ak_F\Delta/\varepsilon_F$. In a layer of thickness $\sim k_F^{-1}$ around the sphere the deviations are substantial, but it can be shown, by estimating the influence of $\Delta(R) - \Delta$ by perturbation theory, that the correction to the resulting wave functions will be of higher order in this parameter. The assumption that (3) is valid in the effective region of the potential is not supplementary. As $U_0 \to \infty$ the wave functions penetrate into the sphere to a distance $\sim (mU_0)^{-1/2}$. The pairing interaction has a nonlocality radius $r_0 > k_F^{-1}$. At $U_0 \gg \varepsilon_F$ we can therefore neglect the change of Δ inside the ion when the wave functions are calculated. As $U_0 \to \infty$ it is necessary that (3) be applicable only in the exterior and on the boundary of the ion.

Equation (3) is similar in many respect to the Dirac equation; we can seek accordingly stationary states with definite angular momentum, with projection n of this momentum on the z axis, and with parity in the form

$$\Psi_{k,jln}(\mathbf{r}) = i^l \begin{pmatrix} u(r) & \Omega_{jln}(\hat{\mathbf{r}}) \\ iv(r) & \sigma_2 \Omega_{jl'n}(\hat{\mathbf{r}}) \end{pmatrix}, \tag{4}$$

where $l = j \pm 1/2$, $l' = 2j - 2$, and Ω_{jln} is a spherical spinor.[18] Using the properties of spherical spinors (Ref. 18, Chap. II, §10), we obtain for the radial functions the system of equations

$$[\hat{T}_l - \varepsilon_F + U(r)] u(r) - \frac{\Delta}{k_F}\left(\frac{d}{dr} + \frac{1-\varkappa_{jl}}{r}\right) v(r) = Eu(r),$$

$$\frac{\Delta}{k_F}\left(\frac{d}{dr} + \frac{1+\varkappa_{jl}}{r}\right) u(r) - [\hat{T}_{l'} - \varepsilon_F + U(r)] v(r) = Ev(r), \tag{5}$$

where

$$\varkappa_{jl} = -(j+{}^1/_2) = -(l+1), \quad j = l + {}^1/_2,$$

$$\varkappa_{jl} = j + {}^1/_2 = l, \quad j = l - {}^1/_2,$$

$$\hat{T}_l = \frac{1}{2m}\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + \frac{l(l+1)}{r^2}\right].$$

Consider the region $r > a$. We introduce the notation

$$\vec{\mathcal{F}}(r) = i^l \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = i^l \frac{\mathbf{f}(r)}{r^{1/2}}. \tag{6}$$

It is easily verified that there are two vector functions (regular and singular at the origin) corresponding to a given momentum $\mathbf{k}$ and to an energy $E_{\mathbf{k}} = (\xi_{\mathbf{k}}^2 + \Delta^2)^{1/2}$:

$$\mathbf{f}_r(k) = (2\pi k)^{1/2} \begin{pmatrix} u(k) J_{l+1/2}(kr) \\ \operatorname{sgn} \varkappa v(k) J_{l'+1/2}(kr) \end{pmatrix},$$
$$\mathbf{f}_s(k) = (2\pi k)^{1/2} \begin{pmatrix} u(k) N_{l+1/2}(kr) \\ \operatorname{sgn} \varkappa v(k) N_{l'+1/2}(kr) \end{pmatrix}, \tag{7}$$

where

$$u(k) = \left[\frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)\right]^{1/2},$$

$$v(k) = \left[\frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)\right]^{1/2}, \quad \xi_k = \frac{k^2}{2m} - \varepsilon_F,$$

and the notation for all the special functions is that used in Ref. 19. The significant difference between (3) and the Dirac equation is the additional double degeneracy with respect to a transition from a state with momentum $\mathbf{k}$ to a state with a dual momentum $\bar{\mathbf{k}}$ such that $\hat{\mathbf{k}} = \bar{\mathbf{k}}$, $\xi_{\mathbf{k}} = -\xi_{\hat{\mathbf{k}}}$. A quasiparticle with $\xi_k > 0$ will henceforth be called a particle (the p-branch) and one with $\xi_k < 0$ a hole (the h-branch). In the absence of a potential, the solutions of (3) can be chosen in the form of plane waves with definite momentum $\mathbf{k}$ and polarization λ:

$$\Phi_{\mathbf{k},\lambda}(\mathbf{r}) = \begin{pmatrix} u(k) \\ v(k)\sigma_2(\sigma\hat{\mathbf{k}}) \end{pmatrix} \frac{e^{i\mathbf{kr}}}{(2\pi)^{3/2}} |\eta^{(\lambda)}\rangle, \tag{8}$$

where $|\eta^{(\lambda)}\rangle$, $\lambda = 1, 2$ is an arbitrary complete set of spin functions. The scattering state is specified by the requirement that as $r \to \infty$ the wave functions each be a superposition of a plane wave and a spherically diverging wave:

$$\langle \mathbf{r} | \Psi_{\mathbf{k},\lambda} \rangle^{out} \approx \Phi_{\mathbf{k},\lambda}(\mathbf{r}) + G_p^{out}(\hat{\mathbf{r}}) \frac{e^{ik_p r}}{r} + G_h^{out}(\hat{\mathbf{r}}) \frac{e^{-ik_h r}}{r}, \tag{9}$$

where

$$k_p = (k_F^2 + 2m|\xi_k|)^{1/2} > k_F, \quad k_h = (k_F^2 - 2m|\xi_k|)^{1/2} < k_F.$$

Since the group velocity of the particles is directed along $\hat{\mathbf{k}}$, and that of the holes oppositely, it can be seen that both spherical waves describe particles and holes that move away from the center. The superposition of a plane and spherically convergent wave is similar in form

$$\langle \mathbf{r}|\Psi_{\mathbf{k},\lambda}\rangle^{in}\approx\Phi_{\mathbf{k},\lambda}(r)+G_p{}^{in}(\hat{\mathbf{r}})\frac{e^{-ik_pr}}{r}+G_h{}^{in}(\hat{\mathbf{r}})\frac{e^{ik_hr}}{r}. \tag{10}$$

Such solutions can be sought for in the form of a superposition of waves of type (4) with a radial part

$$\vec{\mathscr{F}}_{\mathbf{k}}(r)=A\vec{\mathscr{F}}_r(k)+B\vec{\mathscr{F}}_r(\bar{k})+C\vec{\mathscr{F}}_s(k)+D\vec{\mathscr{F}}_s(\bar{k}). \tag{11}$$

Taking into account the formula for the expansion of a plane wave in spherical waves

$$\Phi_{\mathbf{k},\lambda}(\mathbf{r})=(kr)^{-1/2}\sum_{jln}\langle\Omega_{jln}(\hat{\mathbf{k}})|\eta^{(\lambda)}\rangle \times\begin{pmatrix} u(k)i^lJ_{l+1/2}(kr)\Omega_{jln}(\hat{\mathbf{r}})\\ -v(k)i^{l'}J_{l'+1/2}(kr)\sigma_2\Omega_{jl'n}(\hat{\mathbf{r}})\end{pmatrix} \tag{12}$$

and the equality $i^{l'}=-\operatorname{sgn}\varkappa_{jl}i^{l+1}$, the requirement that the asymptotic conditions (9) or (10) be satisfied reduces the number of constants to two:

$$\vec{\mathscr{F}}_{\mathbf{k}}^{out}(r)=\gamma\Big[i\Big(\frac{2\pi k}{r}\Big)^{1/2}\begin{pmatrix} i^lu(k)J_{l+1/2}(kr)\\ -i^{l'}v(k)J_{l'+1/2}(kr)\end{pmatrix}+a^{out}(k)\Big(\frac{2\pi k_p}{r}\Big)^{1/2}\begin{pmatrix} i^lu(k_p)H^{(1)}_{l+1/2}(k_pr)\\ -i^{l'}v(k_p)H^{(1)}_{l'+1/2}(k_pr)\end{pmatrix}+b^{out}(k)\Big(\frac{2\pi k_h}{r}\Big)^{1/2}\begin{pmatrix} i^lu(k_h)H^{(2)}_{l+1/2}(k_hr)\\ -i^{l'}v(k_h)H^{(2)}_{l'+1/2}(k_hr)\end{pmatrix}\Big]$$

$$\gamma=\langle\Omega_{jln}(k)|\eta^{(\lambda)}\rangle/ik(2\pi)^{1/2}. \tag{13}$$

In the case of an impermeable sphere, the condition that the functions vanish on the surface $r=a$ determines a_0 and b_0. If the momenta are replaced by Fermi momenta wherever possible with respect to the parameter $k_Fa\Delta/\varepsilon_F$, we obtain[1)] at $k>k_F$

$$a_0^{out}(k)=\frac{1}{Z_{ll'}(k)}\Big[-u^2(k)t_l+v^2(k)t_{l'}-it_lt_{l'}\Big|\frac{\xi_k}{E_k}\Big|\Big],$$

$$t_l=\frac{J_{l+1/2}(k_Fa)}{N_{l+1/2}(k_Fa)},$$

$$b_0^{out}(k)=\frac{\Delta}{2E_k}\frac{t_l-t_{l'}}{Z_{ll'}(k)},\quad Z_{ll'}=-it_l+it_{l'}+(1+t_lt_{l'})\Big|\frac{\xi_k}{E_k}\Big|, \tag{14}$$

and at $k<k_F$

$$a_0^{out}(k)=b_0^{out}(\bar{k}),\quad b_0^{out}(k)=a_0^{out}(\bar{k}). \tag{15}$$

It is important to note that the degenerate states we have constructed are orthogonal:

$$\int\vec{\mathscr{F}}_{\mathbf{k}}(r)\vec{\mathscr{F}}_{\mathbf{k}}(r)r^2\,dr=0. \tag{16}$$

The equations for waves that satisfy (10) differ from (13) by the substitutions

$$a^{in}=-a^{*\,out},\quad b^{in}=-b^{*\,out}. \tag{17}$$

It follows from (13) and (14) that as $r\to\infty$ and at $k>k_F$ the scattered wave takes the form

$$\begin{pmatrix} u(k)\\ -v(k)\sigma_2\sigma\hat{\mathbf{r}}\end{pmatrix}F_p(\hat{\mathbf{r}})\frac{e^{ikr}}{r}+\begin{pmatrix} v(k)\\ u(k)\sigma_2\sigma\hat{\mathbf{r}}\end{pmatrix}F_h(\hat{\mathbf{r}})\frac{e^{-i\bar{k}r}}{r}, \tag{18}$$

$$F_p(\hat{\mathbf{k}},\hat{\mathbf{r}})=\sum_{jln}-2i\gamma a_0^{out}\Omega_{jln}(\hat{\mathbf{r}}),\quad F_h(\hat{\mathbf{k}},\hat{\mathbf{r}})=\sum_{jln}2i\gamma b_0^{out}(-1)^l\Omega_{jln}(\hat{\mathbf{r}}).$$

Since the quasiparticle velocity $\mathbf{v}_{\mathbf{k}}=\partial E_{\mathbf{k}}/\partial\mathbf{k}$ is equal to $\hat{\mathbf{k}}v_F\xi_k/E_k$, the ratio of the flux density of the quasiparticles scattered in the $\hat{\mathbf{r}}$ direction to the density of the incident flux yields an equation for the differential cross section

$$d\sigma/d\Omega=(2\pi)^3(|F_p|^2+|F_h|^2). \tag{19}$$

The quantities F_p and F_h are the amplitudes for scattering a particle with momentum $\mathbf{k}$ in the direction of $\hat{\mathbf{r}}$ in the cases without and with a change of the branch, respectively (Andreev scattering[20]). To calculate the transport properties we must know the probability of the transition for which the momentum, and not the velocity of the final state, is directed along $\hat{\mathbf{r}}$. The momentum of the scattered wave (18) in the "particle" channel $\exp(ikr)$ is equal to $k\hat{\mathbf{r}}$, i.e., it has the required direction, but the momentum of the wave in the "hole" channel $\exp(-i\bar{k}r)$ is equal to $-\bar{k}\hat{\mathbf{r}}$, i.e., it is directed counter to $\hat{\mathbf{r}}$. Therefore the cross section of the $\mathbf{k}\to\mathbf{k}'$ transition (the "impulse" cross section) is given by

$$d\sigma^{imp}/d\Omega=(2\pi)^3[|F_p(\hat{\mathbf{k}},\hat{\mathbf{k}}')|^2+|F_h(\hat{\mathbf{k}},-\hat{\mathbf{k}}')|^2]. \tag{20}$$

Let us simplify this expression. We introduce the operator

$$\Lambda_{jl}(\hat{\mathbf{x}},\hat{\mathbf{k}})=\sum_n|\Omega_{jln}(\mathbf{x})\rangle\langle\Omega_{jln}(\hat{\mathbf{k}})|. \tag{21}$$

After averaging over the polarization λ, replacing the right-hand side by an arithmetic mean with $\xi_k>0$ and $\xi_k<0$ (this is permissible in the calculation of the impulse cross section, for when k goes though the Fermi surface the momentum of the incident wave does not change direction), and substituting (14) and (15), we obtain

$$\Big\langle\frac{d\sigma^{imp}}{d\Omega}\Big\rangle=\Big(\frac{2\pi}{k_F}\Big)^2\sum_{Jl,Mm}\operatorname{Tr}(\Lambda_{Jl}{}^+(\hat{\mathbf{k}}',\hat{\mathbf{k}})\Lambda_{Mm}(\hat{\mathbf{k}}',\hat{\mathbf{k}}))Q(Jl;Mm),$$

$$Q=\frac{1}{Z^*_{ll'}Z_{mm'}}\Big[2\Big|\frac{\xi}{E}\Big|^2t_lt_{l'}t_mt_{m'}+t_lt_m+t_{l'}t_{m'}-\frac{\Delta^2}{E^2}(t_lt_{m'}+t_{l'}t_m)+i\Big|\frac{\xi}{E}\Big|t_lt_{l'}(t_{m'}-t_m)-i\Big|\frac{\xi}{E}\Big|t_mt_{m'}(t_{l'}-t_l)\Big], \tag{22}$$

From the properties of spherical spinors[18]

$$\Omega_{jl'n}(\hat{\mathbf{r}})=-\sigma\hat{\mathbf{r}}\Omega_{jln}(\hat{\mathbf{r}}) \tag{23}$$

it follows that

$$\operatorname{Tr}(\Lambda_{Jl}{}^+\Lambda_{Mm})=\operatorname{Tr}(\Lambda^+_{Jl'}\Lambda_{Mm'}). \tag{24}$$

Using this equality, the explicit form of the operators Λ (Ref. 12)

$$\Lambda_{jl}(\hat{\mathbf{r}},\hat{\mathbf{k}})=(1/4\pi)\,[|\varkappa_{jl}|P_l(\hat{\mathbf{r}}\hat{\mathbf{k}})+\operatorname{sgn}\varkappa_{jl}i(\sigma\mathbf{n})P_l^1(\hat{\mathbf{r}}\hat{\mathbf{k}})],$$
$$\mathbf{n}=[\hat{\mathbf{r}}\hat{\mathbf{k}}]/\sin\theta,\quad \cos\theta=\hat{\mathbf{k}}\hat{\mathbf{r}} \tag{25}$$

and the symmetry of the trace with respect to the interchange $\hat{\mathbf{k}}\leftrightarrow\hat{\mathbf{r}}$ we can derive the relation

$$\mathrm{Tr}(\Lambda_{Jl}^+\Lambda_{Mm})=\mathrm{Tr}(\Lambda_{Mm'}^+\Lambda_{Jl'}), \tag{26}$$

i.e., prove the invariance of the trace to the interchanges $J\leftrightarrow M, l\leftrightarrow m'$, $l'\leftrightarrow m$. This interchange does not alter the denominator in Q, so that it can be applied to individual terms of the numerator, and Q can be reduced to the form

$$Q=\frac{1}{Z_{ll'}^{*}Z_{mm'}}\left[2\left(t_l-i\left|\frac{\xi}{E}\right|t_lt_{l'}\right)\left(t_m+i\left|\frac{\xi}{E}\right|t_mt_{m'}\right)-\frac{\Delta^2}{E^2}(t_lt_{m'}+t_{l'}t_m)\right]. \tag{27}$$

For comparison with Ref. 12, we introduce the analogous notation

$$\alpha_{ll'}=-\frac{i}{Z_{ll'}}\left(t_l+i\left|\frac{\xi}{E}\right|t_lt_{l'}\right),\quad \beta_{ll'}=-\frac{\Delta t_l}{EZ_{ll'}},$$
$$\gamma_{ll'}=-\frac{\Delta t_{l'}}{EZ_{ll'}},\quad d_{ll'}=\left(1-i\left|\frac{E}{\xi}\right|t_l\right)\left(1+i\left|\frac{E}{\xi}\right|t_{l'}\right)-\frac{\Delta^2}{\xi^2}t_lt_{l'}=\left|\frac{E}{\xi}\right|Z_{ll'}, \tag{28}$$

In terms of which the expressions for the cross section (22)

$$\left\langle\frac{d\sigma^{imp}}{d\Omega}\right\rangle=\frac{2\pi^2}{k_F^2}\sum_{Jl,Mm}\mathrm{Tr}(\Lambda_{Jl}^+\Lambda_{Mm})\,[\alpha_{ll'}^{*}\alpha_{mm'}-\mathrm{Re}(\beta_{ll'}^{*}\gamma_{mm'})] \tag{29}$$

coincide with (B.13) of Ref. 12 if the latter is converted to an explicitly real form. For a normal liquid, where $\Delta=0$, we get from (29) the usual expression

$$\frac{d\sigma^N}{d\Omega}=k_F^{-2}\left|\sum_l(2l+1)e^{i\delta_l}\sin\delta_lP_l(\hat{\mathbf{k}}\hat{\mathbf{k}}')\right|^2. \tag{30}$$

To find the values of the function on the surface of the sphere at a finite potential U_0, we note that at $r<a$ there are two independent vector functions that are regular at the origin, viz., the solutions (3), which take at $U_0\gg\varepsilon_F\gg\Delta$ the form

$$\mathbf{f}(k_-')=\begin{pmatrix}\bar{u}I_{l+1/2}(k_-'r)\\ \bar{v}I_{l'+1/2}(k_-'r)\end{pmatrix},\quad \mathbf{f}(k_+')=\begin{pmatrix}\bar{v}I_{l+1/2}(k_+'r)\\ -\bar{u}I_{l'+1/2}(k_+'r)\end{pmatrix}$$
$$\bar{u}=\left\{\frac{1}{2}\left[\frac{1}{E}\left(\frac{k_0^2\Delta^2}{k_F^2}+E^2-\Delta^2\right)^{1/2}+1\right]\right\}^{1/2}$$
$$\bar{v}=\left\{\frac{1}{2}\left[\frac{1}{E}\left(\frac{k_0^2\Delta^2}{k_F^2}+E^2-\Delta^2\right)^{1/2}-1\right]\right\}^{1/2},$$
$$k_0^2=2mU_0,\quad (k_\pm')^2=k_0^2-k_F^2\pm2m\left(\frac{k_0^2\Delta^2}{k_F^2}+E^2-\Delta^2\right)^{1/2}. \tag{31}$$

The particular solution (7) and (31) can be used to construct the radial part of the general solution of (3). For the case of incident particles ($k>k_F$) the solution that satisfies (9) is

$$\mathbf{f}_{\mathbf{k}}^{out}(r)=\theta(a-r)\,[B_-\mathbf{f}(k_-')+B_+\mathbf{f}(k_+')]+$$
$$+\theta(r-a)\gamma[i\mathbf{f}_r(k)+a^{out}(\mathbf{f}_r(k)+i\mathbf{f}_s(k))+b^{out}(\mathbf{f}_r(\bar{k})-i\mathbf{f}_s(\bar{k}))]. \tag{32}$$

The conditions that this function and its derivative be continuous on the sphere yield equations for $B_\pm$

$$\alpha_{11}B_-+\alpha_{12}B_+=\beta_1^{out},\quad \alpha_{21}B_-+\alpha_{22}B_+=\beta_2^{out},$$
$$\alpha_{ij}=\begin{pmatrix}(k_-'a)\bar{u}I_{l-1/2}(k_-'a) & (k_+'a)\bar{v}I_{l-1/2}(k_+'a)\\ (k_-'a)\bar{v}I_{l'-1/2}(k_-'a) & -(k_+'a)\bar{u}I_{l'-1/2}(k_+'a)\end{pmatrix},$$
$$\beta^{out}=\gamma\left\{ka(2\pi k)^{1/2}\begin{pmatrix}u(k)\,[iJ_{l-1/2}(ka)+a^{out}H^{(1)}_{l-1/2}(ka)]\\ \operatorname{sgn}\varkappa v(k)\,[iJ_{l'-1/2}(ka)+a^{out}H^{(1)}_{l'-1/2}(ka)]\end{pmatrix}\right.$$
$$\left.+b^{out}\bar{k}a(2\pi\bar{k})^{1/2}\begin{pmatrix}u(\bar{k}) & H^{(2)}_{l-1/2}(\bar{k}a)\\ \operatorname{sgn}\varkappa v(\bar{k}) & H^{(2)}_{l'-1/2}(\bar{k}a)\end{pmatrix}\right\}. \tag{33}$$

Since $B_\pm\sim(k_0a)^{-1}$, it suffices to substitute in (33) the amplitudes a_0 and b_0 obtained from the solution of the problem with $U_0=\infty$. Recognizing that for $U_0\to\infty$ and $l\sim k_Fa\gg1$ we have

$$k_\pm'\to k_0,\quad \bar{u}\bar{v}/(\bar{u}^2+\bar{v}^2)\to{}^1/_2,\quad I_{l\pm1/2}(z)\to e^z/(2\pi z)^{1/2}, \tag{34}$$

it follows from (32) and (33) that

$$\mathbf{f}_{\mathbf{k}}^{out}(a)=\beta^{out}/(k_0a). \tag{35}$$

The Fermi momenta can now be substituted, accurate to T_c/ε_F, everywhere except in the coherence factors $u(k)$ and $v(k)$, and the quasiclassical asymptotic form of the Bessel functions can be used:

$$H^{(1)}_{l+1/2}(x)\approx-i(2/\pi)^{1/2}(x^2-p^2)^{-1/4}\exp(-i\delta_l),$$
$$p=l+{}^1/_2,\quad x=k_Fa,\quad \delta_l=\pi l/2-(x^2-p^2)^{1/2}-p\operatorname{arctg}(p/x),$$
$$\sin(\delta_l-\delta_{l-1})\approx[1-(l/k_Fa)^2]^{1/2}. \tag{36}$$

As a result, the wave functions of the scattering states are reduced to the form (which is valid for all $k\gtrless k_F$)

$$\langle r|\Psi_{\mathbf{k},\lambda}\rangle|_{r=a}=\frac{(2/\pi)^{1/2}}{k_0a}\sum_{jln}\langle\Omega_{jln}(\hat{\mathbf{k}})|\eta^{(\lambda)}\rangle\times\begin{pmatrix}{}^{(1)}B(k;jl)\,i^l\Omega_{jln}(\hat{\mathbf{r}})\\ -{}^{(2)}B(k;jl)\,i^{l'}\sigma_2\Omega_{jl'n}(\hat{\mathbf{r}})\end{pmatrix},$$
$$B(k)^{out}=\left|\frac{\xi}{E}\right|\left[\cos(\delta_{l'}-\delta_l)\left|\frac{\xi}{E}\right|+i\sin(\delta_{l'}-\delta_l)\right]^{-1}$$
$$\times\begin{pmatrix}u(k)\left[1-\left(\dfrac{l}{k_Fa}\right)^2\right]^{1/4}\exp[i\delta_{l'}\operatorname{sgn}(k-k_F)]\\ v(k)\left[1-\left(\dfrac{l'}{k_Fa}\right)^2\right]^{1/4}\exp[i\delta_l\operatorname{sgn}(k-k_F)]\end{pmatrix},$$
$$B^{in}=B^{*\,out}. \tag{37}$$

It is important that, apart from phase factors, the phase shifts δ_l for scattering by a sphere in a normal liquid enter the obtained functions only in the combination $\delta_l-\delta_{l\pm1}$. It can be seen from (36) that this circumstance, the complicated expression for δ_l notwithstanding, permits effective summation over the partial waves, replacing the sums over l by easily calculated integrals.

Equations (37) generalize the equations introduced in Refs. 9 and 21 for a normal Fermi gas to include a superfluid liquid. In the limit as $\Delta\to0$ they go over into the corresponding equations of a normal system, where at $k > k_F$

$$\langle \mathbf{r} | \Psi_{\mathbf{k},\lambda} \rangle^{out} \to \psi_{\mathbf{k}}^{(+)},$$

and at $k < k_F$

$$\langle \mathbf{r} | \Psi_{\mathbf{k},\lambda} \rangle^{out} \to \psi_{\mathbf{k}}^{(-)},$$

and the definition of $\psi_{\mathbf{k}}^{(\pm)}$ is the same as in Ref. 22. This corresponds to the treatment of holes as time-reversed particles.

3. EXCHANGE SCATTERING

The exchange-interaction operator in the Bogolyubov equation that corresponds to (1) takes the form

$$H_{ex} = \frac{1}{2} \Gamma(\mathbf{r}-\mathbf{R}) \sigma_{ij} \Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \begin{pmatrix} \sigma_{\alpha\beta} & 0 \\ 0 & -\sigma_{\beta\alpha} \end{pmatrix}. \tag{38}$$

The contribution of the exchange scattering to the mobility is calculated in the same way as the contribution of the potential scattering[12]:

$$\frac{1}{\mu_{ex}^B} = \frac{k_F^2}{3} \sum_{\mathbf{k}_{1,2}\lambda_{1,2}\nu_2} (1-\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2) \left(-\frac{\partial n}{\partial E_{k_1}}\right) \times 2\pi\delta(E_{k_1}-E_{k_2}) \sum_{\mathbf{R}_n} \overline{|T_{\mathbf{k}_2\lambda_2\nu_2;\mathbf{k}_1\lambda_1}|^2}, \tag{39}$$

where ν_2 is the index of the branch of the final state of $\mathbf{k}_2$ and allowance is made for the fact that the transition probability does not depend on the branch of the incident quasiparticle. The bar over the T matrix denotes averaging over the spin states of the surface atom located at the point $\mathbf{R}_n$. It is convenient first to calculate the quantity

$$S(E) = \sum_{\lambda_1\lambda_2\nu_2} \int d\hat{\mathbf{k}}_1\, d\hat{\mathbf{k}}_2 (1-\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2) \overline{|T_{\mathbf{k}_2\lambda_2\nu_2,\mathbf{k}_1\lambda_1}(E)|^2}, \tag{40}$$

and next find the mobility from the relation

$$\frac{1}{\mu_{ex}^B} = \frac{4\pi k_F^2}{3} (mk_F)^2 \int_\Delta^\infty dE \left(-\frac{\partial n}{\partial E}\right) \left|\frac{E}{\xi}\right|^2 S(E) N_s. \tag{41}$$

In the first Born approximation in V_0 we have

$$T_{\mathbf{k}_2\lambda_2\nu_2,\mathbf{k}_1\lambda_1} = \langle \Psi^{out}_{\mathbf{k}_2\lambda_2\nu_2} | H_{ex} | \Psi^{in}_{\mathbf{k}_1\lambda_1} \rangle. \tag{42}$$

The value of $S(E)$ is obtained using (37) via straightforward but laborious calculations. Incidentally, on going to higher corrections in V_0 these results can be used without change. Let us describe a more efficient procedure. It is first necessary to sum explicitly in (37) over $j = l \pm 1/2$ and over the projection of the angular momentum n. We must next, in succession, sum over $\lambda_{1,2}$, find the related traces, and average them over the states of the impurity spin. It is necessary to consider separately and add the contributions made to $S(E)$ from the usual (k_1, $k_2 > k_F$) and Andreev ($k_2 < k_F < k_1$) scattering. The integration in (40) over the momentum directions reduces, by use of the identities

$$\int d\hat{\mathbf{k}} \hat{k}^i \mathcal{F}(\hat{\mathbf{k}}\hat{\mathbf{R}}) = \hat{R}^i \int d\hat{\mathbf{k}} (\hat{\mathbf{k}}\hat{\mathbf{R}}) \mathcal{F}(\hat{\mathbf{k}}\hat{\mathbf{R}}), \tag{43}$$

$$\int d\hat{\mathbf{k}} \hat{k}^i \hat{k}^j \mathcal{F}(\hat{\mathbf{k}}\hat{\mathbf{R}}) = \frac{1}{2} \hat{R}^i \hat{R}^j \int d\hat{\mathbf{k}} [3(\hat{\mathbf{k}}\hat{\mathbf{R}})^2 - 1] \mathcal{F}(\hat{\mathbf{k}}\hat{\mathbf{R}}) + \frac{1}{2}\delta^{ij} \int d\hat{\mathbf{k}} [1-(\hat{\mathbf{k}}\hat{\mathbf{R}})^2] \mathcal{F}(\hat{\mathbf{k}}\hat{\mathbf{R}}) \tag{44}$$

to finding a certain set of integrals of the Legendre polynomials P_l and P_l^1. As a result we get

$$\begin{aligned} S(E) \propto 2\left|\frac{\xi}{E}\right|^4 &\left\{\left[\sum w_l\right]^2 + \left[\sum w_l \sin\psi_l\right]^2\right\} \\ + \frac{\Delta^2}{3E^2}\left|\frac{\xi}{E}\right|^4 &\left\{\left[\sum w_l (\cos^2\psi_l - \sin^2\psi_l)\cos^2\psi_l\right]^2\right. \\ &+ 2\left[\sum w_l(\cos^2\psi_l - \sin^2\psi_l)\sin^2\psi_l\right]^2 \\ &\left.+ 3\left[\sum w_l 2\sin^2\psi_l\cos^2\psi_l\right]^2\right\}, \end{aligned} \tag{45}$$

where

$$\begin{aligned} &w_l = l|Z_l|^{-2}[1-(l/k_Fa)^2]^{1/2}, \quad \psi_l = \delta_{l+1} - \delta_l, \\ &Z_l = \cos\psi_l |\xi/E| + i\sin\psi_l. \end{aligned} \tag{46}$$

The omitted constant factor can be obtained by comparison with the analogous expression for a normal liquid.[9] For a qualitative comparison with experiment it suffices to study the mobility at $\Delta/T_c \ll 1$, namely calculate in the expansion of $(\mu^N/\mu^B)_{ex}$ the numerical coefficient of the term linear in Δ. In this limit, expression (41), which takes the form

$$\frac{1}{\mu_{ex}^B} = \int_\Delta^\infty dE \left(-\frac{\partial n}{\partial E}\right) \mathcal{K}\left(\frac{\Delta}{E}\right), \tag{47}$$

can, by subtracting and adding under the integral sign the value $\mathcal{K}$ of the function $\mathcal{K}^N$ in the normal phase, be rewritten as

$$\frac{1}{\mu_{ex}^B} = \frac{1}{2}\mathcal{K}^N\left(1-\frac{\Delta}{2T}\right) + \frac{\Delta}{4T}\int_1^\infty dx \left[\mathcal{K}\left(\frac{1}{x}\right) - \mathcal{K}^N\right],$$

$$x = \frac{E}{\Delta}. \tag{48}$$

The integral converges and is determined by the region $x \sim 1$. Were it possible to neglect the influence of the pairing on the character of the quasiparticle scattering and assume the cross sections in the normal and superfluid phases to be identical,[23] the presence of a gap in the excitation spectrum would give rise to the increase in mobility described by the first term of (48). The first term thus yields the correction due to the statistical factor (the change of the distribution function), and the entire influence of the altered quantum mechanics is contained in the second term. From (48) we get

$$\left(\frac{\mu^N}{\mu^B}\right)_{ex} = 1 - \frac{\Delta}{2T}(1-c_{ex}),$$

$$c_{ex} = \int_1^\infty dx \left[\mathcal{K}\left(\frac{1}{x}\right) - \mathcal{K}^N\right] \Big/ \mathcal{K}^N. \tag{49}$$

Numerical calculation yields $c_{ex} = -0.6$.

It is convenient to represent the structure of the succeeding Born approximation in graphic form, using the

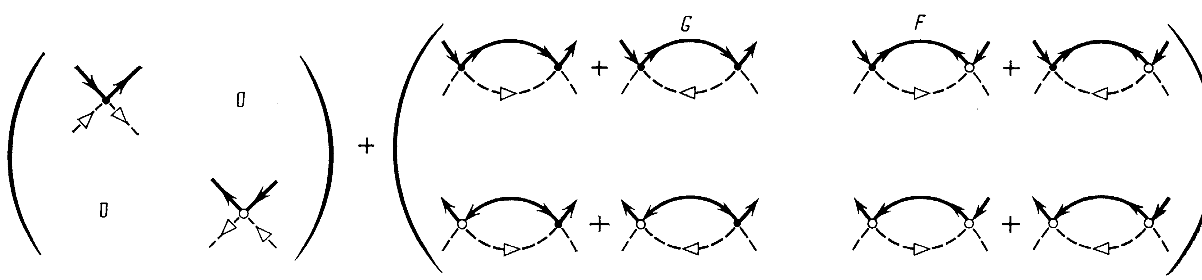

FIG. 1.

methods of Ref. 10. The sum (38) in the first iteration is shown in Fig. 1, where a dashed line denotes the Green's function of the preudofermion corresponding to the spin of the surface atom, and the solid lines denote normal and anomalous Green's functions of the quasiparticles. They are connected with the components of the Bogolyubov equation solutions by the relations

$$G(E)=\sum_{\alpha}\frac{^{(1)}|\Psi_\alpha\rangle\langle\Psi_\alpha|^{(1)}}{E-E_\alpha},\quad F(E)=\sum_{\alpha}\frac{^{(1)}|\Psi_\alpha\rangle\langle\Psi_\alpha|^{(2)}}{E-E_\alpha}\tag{50}$$

where the subscript α stands for $\mathbf{k}$, λ, or the sign of the energy. Solutions of (3) with negative energy are obtained from these by making the substitutions $(u(k)\to -v(k)$, $v(k)\to u(k),\eta^{(\lambda)}\to\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\sigma_2\eta^{(\lambda)}$. Although in the second-order approximation the matrix amplitude of the exchange scattering is no longer diagonal, it can be shown by direct calculations that the off-diagonal components are smaller than the diagonal ones relative to the parameter $(k_F a)^{-1}$.

CONCLUSION

In the model calculated, the exchange interaction was assumed weak compared with the potential one. There are no particular reasons for this in real ^{3}He. It is therefore meaningful to compare only the qualitative predictions with experiment. It follows from the calculations for potential scattering 12 that near T_c

$$(\mu^N/\mu^B)_{rot}=1-1.9(\Delta/2T).\tag{51}$$

In the first Born approximation (49) the contribution from the exchange scattering depends somewhat less on Δ. It can be shown that if the exchange is weak the use of the second Born approximation decreases this dependence even more. This means that when simultaneous account is taken of both interactions, the $\mu^B_+(T)$ dependence is weaker than $\mu_-{}^B(T)$, in qualitative agreement with the experimental results. Calculation of the higher Born corrections is made difficult by the fact that starting with the third approximation it is necessary to take into account the rescattering of the quasiparticles by various surface atoms. In addition it was assumed everywhere that the spins of the iceberg atoms are free. There are experimental data (see Ref. 24 and the bibliography therein) indicating that the surface layer of solid ^{3}He bordering on the liquid has a tendency to acquire ferromagnetic ordering at $T\approx 2$ mK. If the exchange scattering by an iceberg with a ferromagnetic shell is larger than by one with a paramagnetic shell, allowance for the resultant ferromagnetic correlations will also improve the agreement with experiment.

The contribution of the exchange to the mobility can be determined quantitatively from experiments in strong magnetic fields $H\approx 6$ T. The magnetic field aligns the spins of the quasiparticles and thereby, excludes exchange scattering. Under these conditions the behavior of ions of different signs will be determined only by their dimensions. An additional possibility of studying ion structure, mentioned in Refs. 25 and 26, is to compare their interactions with vortices.

The author is indebted to V. I. Mel'nikov and V. P. Mineev for helpful discussions.

[1)] *Out*-states in the case $U_0=\infty$ were first obtained in equivalent form in Ref. 16.

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Translated by J. G. Adashko