Stochastic behavior in the equations of motion for the quasimomentum

V. L. Bonch-Bruevich

M. V. Lomonosov State University, Moscow

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The stability of the stationary solution of the equations of motion for the quasimomentum is investigated. Conditions are found for instability to occur within an energy interval and for the quasimomentum components to undergo regular or stochastic self-oscillations with time. Some physical consequences are discussed.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

It now seems clear that it is the rule rather than the exception for stochastic self-oscillations to occur in nonequilibrium nonlinear systems with phase spaces of dimension $\geqslant 3$. In particular, in Refs. 1, 2 and Ref. 3 conditions were found for the onset of stochastic self-oscillations in the electric field and free charge carrier concentration in crystals and unordered semiconductors. The field dependence of the carrier trapping and/or production cross sections, and possibly also the field dependence of the carrier mobility, played a key role. In the latter case the electric field no longer necessarily heats the carriers, as may be seen, e.g., from the fact that the Poole-Frenkel effect occurs when the mobility is determined by carrier trapping in shallow traps followed by ejection into the conduction (or valence) band.

In this paper we study the even simpler situation when the concentration of free carriers (assumed for definiteness to be electrons) is specified and the motion of an individual electron is governed by the system of equations for the components of the momentum (by which we will always mean the quasimomentum). In this kinetic model the interaction among the electrons can be significant only when one allows for screening, as is required to calculate the probability of scattering by a charged impurity.

We confine ourselves to a system with an isotropic parabolic spectrum and take the energy minimum to lie at the center of the Brillouin zone.[1)] The carrier scattering is assumed to be nearly elastic. The system of equations for the components of the momentum $\mathbf{p}$ then has the standard form

$$\dot{\mathbf{p}}=e\mathbf{E}-\tau^{-1}(W)\,\mathbf{p}. \tag{1}$$

Here $\mathbf{E}$ is the electric field vector, and the momentum relaxation time $\tau(W)$ depends on the energy $W=p^2/2m$. It is this dependence that produces the nonlinearity of interest; we will see below that stochastic self-oscillations in p do not occur if $\tau(W)$ is arbitrary.

Equation (1) (with the relaxation term in the right-hand side) has been studied repeatedly in the literature (see, e.g., Refs. 4, 5). It can be derived in the standard way by evaluating the time derivative of the velocity operator $\hat{\mathbf{v}}$ and averaging the resulting expressions quantum mechanically (for a specified electron energy W). We need only include a term $\hat{H}_r$ in the Hamiltonian to describe the quasielastic scattering processes. As in laser physics,[6] we define

$$\frac{i}{\hbar}\langle[\hat{H}_r,\hat{\mathbf{v}}]_-\rangle=-\frac{1}{\tau(W)}\mathbf{v}.$$

Here the brackets denote a quantum mechanical average, and $\mathbf{v}=\mathbf{v}(W)$ is the electron velocity (also averaged quantum mechanically). It is clear that $\tau(W)$ should be identical to the transport relaxation time (not averaged over the distribution function) that appears in the solution of the kinetic equation (provided that the conditions for its validity are satisfied).

Obviously,

$$W=W_{th}+\Delta W, \tag{2}$$

where $W_{th}=p_{th}^2/2m$ and $\mathbf{p}_{th}$ are the thermal energy and momentum of an electron and ΔW is the field correction. Clearly

$$\Delta W=\frac{(\mathbf{p}+\mathbf{p}_{th})^2}{2m}-W_{th}=\frac{p^2}{2m}+\frac{\mathbf{p}\mathbf{p}_{th}}{m}. \tag{3}$$

Here the components of $\mathbf{p}_{th}$ appear as parameters. We note two things regarding Eqs. (1) and (2).

First, although system (1) is isotropic it still cannot be reduced to two equations (for the component of $\mathbf{p}$ parallel to $\mathbf{E}$ and for the energy correction ΔW). Indeed, we will be interested not only in the stationary state but also in the fluctuations about it. The latter are random and the direction of $\mathbf{p}$ may be arbitrary.

Second, ΔW determines only the "influx" of energy gained by the electron from the field. The energy "loss" to the lattice becomes appreciable over times of order $\tau_e(W)$ (the energy relaxation time) and does not figure in Eq. (2).[2)] When using (2) to calculate ΔW we must therefore remember that the result is meaningful only for times $\lesssim\tau_e(W)$.

We will now proceed to find the stationary solutions of system (1), i.e., its singular points, and study the time behavior of the random fluctuations about them.

2. STATIONARY SOLUTION

Setting $\dot{\mathbf{p}}=0$ in (1) and using a 0 subscript to denote the corresponding values of $\mathbf{p}$ and ΔW, we obtain

$$\mathbf{p}_0=e\tau(W_{th}+\Delta W_0)\,\mathbf{E}, \tag{4}$$

where

$$\Delta W_0=\frac{\mathbf{p}_0\mathbf{p}_{th}}{m}+\frac{p_0^2}{2m}. \tag{5}$$

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We find that

$$\tau^{-1}(W)=aW^{\gamma/2}, \tag{6}$$

where γ and a (apart from a possible logarithmic factor) are wellknown constants (see, e.g., Ref. 7, 8). The equation for ΔW_0 then reads

$$\Delta W_0=\frac{e^2E^2}{2ma^2}W^{-\gamma}+2\left[\frac{e^2E^2}{2ma^2}W^{-\gamma}W_{th}\right]^{1/2}\cos\theta, \tag{7}$$

where θ is the angle between $\mathbf{p}_{th}$ and $\mathbf{E}$.

Equation (7) may have one or several real positive solutions, depending on the parameters of the problem. If the field is sufficiently weak that $W\sim T$ (where T is the lattice temperature in energy units), the solution is of course unique and is obtained by replacing W by W_{th} in the right-hand side of (7). However, we will be interested in the behavior in a relatively narrow energy interval when $W_{th}\approx\Delta W_0$ and $\gamma<0$. Negative γ correspond to scattering of electrons by charged impurities ($\gamma=-3$) or by longitudinal acoustic phonons interacting piezoelectrically with electrons ($\gamma=-1$).[3)]

Setting $W_{th}=c\Delta W_0$ where c is of order unity and writing

$$W_e=\left(\frac{2ma^2}{e^2E^2}\right)^{1/\gamma}, \tag{8}$$

we find from Eq. (7) that

$$\Delta W_0=W_e\frac{-(c(1+c)^3)^{1/\gamma}\cos\theta+((c\cos^2\theta+1)(1+c)^3)^{1/2}}{(1+c)^3}. \tag{9}$$

In this case there is a unique singular point.

The dependence $\Delta W_0\sim E^{-1}$ is rather unusual. However, we note that E must not approach zero. Indeed, for large ΔW_0 the scattering by charged impurities becomes ineffective and is superseded, e.g., by scattering by acoustic phonons via the deformation potential; Eq. (9) then breaks down and the energy gained during time τ decreases as E decreases. Moreover, we have noted that the energy gained is limited by dissipation, and this may be the most important factor of all. At steady state we have

$$\Delta W_0/\tau=(W-W_{th})/\tau_e(W_0). \tag{10}$$

According to Ref. 7, for energy dissipation by acoustic phonons via a deformation potential we have (for $W\ll T$)

$$\tau_e(W)=\frac{m^{5/2}E_1^2}{2^{7/2}\pi^3\rho\hbar^4}W^{1/2}\equiv\xi W^{1/2}. \tag{11}$$

Here E_1 is the deformation potential constant, ρ is the crystal density, and the right-hand side defines ξ.

Together with relations (6), (8), (9), and (11), Eq. (10) yields

$$\Delta W_0\equiv W-W_{th}=\left(\frac{a}{\xi^2}\frac{eE}{(2m)^{1/\gamma}}\right)^{1/\gamma}. \tag{12}$$

By assumption $\tau_e\gg\tau$, i.e., $\Delta W\gg\Delta W_0$. We will consider the constraints imposed by these inequalities in Sec. 4.

3. CONDITIONS FOR INSTABILITY OF THE STATIONARY STATE

The conditions for the stationary state [i.e., the singular point of system (1)] to be stable are found in the standard way by writing

$$\mathbf{p}=\mathbf{p}_0+\delta\mathbf{p},\quad \delta\mathbf{p}\sim e^{\lambda t}, \tag{13}$$

and expanding the right-hand side of (1) in $\delta\mathbf{p}$ to get the secular equation for λ:

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0. \tag{14}$$

A simple calculation gives

$$b_1=\tau^{-1}(W_0)(3+\nu),\quad b_2=\tau^{-2}(W_0)(3+2\nu),\quad b_3=\tau^{-3}(W_0)(1+\nu), \tag{15}$$

where

$$\nu=\gamma\Delta W_0/(W_{th}+\Delta W_0). \tag{16}$$

Equation (14) has the roots

$$\lambda_3=-\tau^{-1}(W_0)(1+\nu),\quad \lambda_1=\lambda_2=-\tau^{-1}(W_0). \tag{17}$$

We see that for $\nu<-1$, i.e., for

$$\gamma<0,\quad (|\gamma|-1)\Delta W_0>W_{th}, \tag{18}$$

the root λ_1 is positive, so that the stationary state is unstable.

According to Shil'nikov,[9,10] the growth of instability in this case may lead either to a unique periodic stable mode or to a periodic motion of the saddle type (in the latter case we anticipate a homoclinic structure). The first case occurs when $\lambda_1+\lambda_3<0$, i.e., for

$$W_{th}(|\gamma|-1)^{-1}<\Delta W_0<2W_{th}(|\gamma|-2)^{-1}, \tag{19a}$$

and the second when $\lambda_1+\lambda_3>0$, i.e., when

$$\Delta W_0>2W_{th}/(|\gamma|-2). \tag{19b}$$

For $\gamma=-3$ these inequalities take the form

$$^1/_2\Delta W_0<W_{th}<2\Delta W_0 \tag{19'a}$$

and

$$W_{th}<^1/_2\Delta W_0. \tag{19'b}$$

Of course, ΔW_0 must be less than the energy at which the momentum dissipation mechanism changes.

4. LIMITING ASSUMPTIONS

We have made the following assumptions in the calculation:

1) Charged impurities rather than phonons are primarily responsible for the momentum scattering.

2) The energy relaxation time is long compared to the momentum relaxation time.

3) Most of the electrons are unheated.

The condition for the first assumption to be valid follows by comparing the corresponding relaxation times. As before, we confine ourselves to a system with a parabolic isotropic spectrum. Using the well-known formulas for the relaxation times (see Refs. 7 and 8 or any other textbook on semiconductor physics), we get the following condition:

$$W\ll 2\pi zE_1^{-1}W_B^2\left[\frac{2ms^2}{T}\frac{\rho a_B^3}{m}N_i a_B^3\ln\eta\right]^{1/\gamma}. \tag{20}$$

Here z is the charge of the scattering center in electron charge units, W_B and a_B are the Bohr energy and radius in the crystal, s is the speed of sound, N_t is the charged impurity concentration, and $\eta = 8mW\hbar^{-2}r_0^2$, where r_0 is the screening radius; we can substitute W_e (8) for W in the left-hand side of (20) and in the formula for η. We then get

$$E > \frac{E_1}{ea_B} z^2 \ln\eta \left(2\frac{mN_t}{\rho}\frac{T}{ms^2}\right)^{1/2}. \qquad (20')$$

The condition for the validity of the second assumption follows by comparing the times $\tau(W_0)$ and $\tau_e(W_0)$ (11); this leads to the inequality

$$E \gg \frac{E_1}{ea_B}\frac{z^2}{2\pi}\left(\frac{mN_t}{\rho}\ln\eta\right)^{1/2}, \qquad (21)$$

which is certainly satisfied if (20′) holds.

Finally, the condition for the third assumption to be valid is that

$$\Delta W \ll T, \qquad (22)$$

where ΔW is the energy gained by the electron during time $\tau_e(W_e)$. Using Eqs. (11), (12) and the familiar expression for the coefficient a in relation (6), we get

$$E \ll \frac{27}{512}\frac{T^3E_1^4}{ea_BW_B^6}\left[z^2\frac{\rho^2a_B^6}{m^2}N_ta_B^3\ln\eta\right]^{-1}. \qquad (22')$$

If we asume the rough values $N_t = 10^{15}$ cm^{-3}, $M/m = 10^4$, $V_0 = \rho/M = 10^{-22}$ cm^{-3}, $a_B = 10^{-7}$ cm, and $E_1W_b^{-1} = 10^{+3}$, we obtain $E \ll 10^3$V/cm from (22′).

5. PHYSICAL CONSEQUENCES

Inequalities (19′a), (19′b) show that in weak fields, instability occurs only in a narrow interval of low energies whose upper bound is determined by (19′a). We observe that for $\Delta W_0 \ll T$ the instability develops against a "global" resistive background (most of the electrons are not involved in the self-oscillations, whether regular or stochastic).

Under conditons (19′a) the crystal momentum of the electrons within this narrow energy range varies periodically with time. Formally, one would expect electromagnetic waves of frequencies $\sim\tau^{-1}(W_0)$ to be emitted. However, we note that the electron energy "drops" at random times separated on the average by intervals of the order of $\tau_e(W)$ (for $W \approx \Delta W_0$). For this reason, it is likely that regular self-oscillations can be observed in this way only in experiments with very good time resolution. On the other hand, under conditions (19′b), stochastic oscillations should be excited and show up experimentally as "noise" in the measurement channel. This noise is superposed on the regular self-oscillations and makes their detection more difficult.

In estimating the amplitude of the self-oscillations or the time-averaged energy of the stochastic oscillations ("noise intensity"), it is helpful to reduce system (1) to dimensionless form. We limit ourselves to the case $\gamma = -3$ and divide the dimensional momentum and time by

$$p_e = (2m)^{1/4}\left(\frac{a}{eE}\right)^{1/2}, \quad t_0 = a^{1/2}(2m)^{1/4}(eE)^{-1/2} \qquad (23)$$

respectively. Setting

$$\mathbf{p} = p_e\mathbf{q}, \quad \mathbf{p}_{th} = p_e\mathbf{q}_{th}, \quad \boldsymbol{\xi} = \mathbf{E}/E, \qquad (24)$$

we find

$$\dot{\mathbf{q}} = \boldsymbol{\xi} - \mathbf{q}[(\mathbf{q}+\mathbf{q}_{th})^2]^{-3/2} \qquad (25)$$

in place of (1).

The (nonlinear) equations for the deviations $\delta\mathbf{q}$ form the stationary solution $\mathbf{q}_0$ are similar (the vector ξ appears only through $\mathbf{q}_0$). System (25) contains only one vector parameter $\mathbf{q}_{th}$, which is also dimensionless. This means that in terms of dimensional quantities the oscillation amplitude of the quasimomentum can be expressed solely in terms of p_e and $\mathbf{p}_{th}$. Under conditions (19′a) and (19′b), p_e and $\mathbf{p}_{th}$ are comparable in order of magnitude, so that the oscillation amplitude will be proportional to p_e and the noise intensity to $p_e^2/2m = W_e$.

Calculation of the electron density in the energy intervals (19′a) and (19′b) (as well as the other quantities expressed as integrals) requires integrating over the momenta $\mathbf{p}_{th}$ between appropriate limits. The parameter $\mathbf{p}_{th}$ then drops out of the equations and the result is expressed solely in terms of W_e. Indeed, since we are assuming that W_e/T is small, the lattice temperature can affect the result only by influencing the total electron density n. Assuming the electron gas to be nondegenerate, we find that the electron density in the "active energy intervals" is given by

$$n_{\text{акт}} = \text{const}\,(W_e/T)^{3/2}n, \qquad (26)$$

where const denotes a numerical factor of order unity [similar formulas also hold for the separate intervals (19′a) and (19′b)]. Similar arguments yield the result

$$(\delta\mathbf{j})^2 \sim \frac{e^2}{m^2}n^2p_e^2\left(\frac{W_e}{T}\right)^3 \sim n^2E^{-4} \sim n^6j_0^{-4}, \qquad (27)$$

for the square of the current density fluctuation $\delta\mathbf{j}$; here $\mathbf{j}_0$ is the average current density in the sample. It is thus plain that the effect of the stochastic self-oscillations of the quasimomentum on the current noise in semiconductors should be to increase the noise (as compared with ordinary carrier production and recombination noise) and to alter the current dependence of the noise at low currents (of course, for the reason indicated above, the limiting case $j_0 \to 0$ cannot occur). The magnitude of the small parameter $(W_e/T)^3$ determines how pronounced this effect will be.

We note in closing that the third assumption made in Sec. 4 is by no means essential. Indeed, without it the energy interval of the instability expands and the effects discussed above may become much more pronounced. However, in this case one must also analyze the (possibly time-dependent) distribution function of the hot electrons.

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[1)] It is easy to see that in this case nothing fundamental is changed if we give up the isotropic and/or parabolic approximation (at least in materials which have only one energy minimum).

[2)] The time $\tau_e(W)$ should not be confused with the usual average energy relaxation time. An expression for $\tau_e(W)$ can be derived, e.g., by considering the right-hand side of Eq. (5.10) in Ref. 7.

[3)] We also have $\gamma < 0$ for scattering of electrons by optical (polarization) phonons; however, in this case the electron energy must be large compared to T.

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