Growth rate of a percolation cluster

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(Submitted 31 January 1985)
Zh. Eksp. Teor. Fiz. **89**, 974–984 (September 1985)

A new variational technique is used to analyze a branching subcritical diffusion process which simulates percolation and has no self-intersections. The mean square distance between a wetted site and the coordinate origin is found as a function of the number of the generation in which the site is wetted. This dependence yields critical index values for percolation on a d-dimensional lattice which are in close agreement with results found in numerical experiments for $2 \leqslant d \leqslant 6$.

Much work has been devoted to percolation theory (see the original papers and reviews[1–5]). Three approaches to describing percolation in d-dimensional lattices may be distinguished—exact solutions, the mean-field technique (solution on a Bethe lattice), and the renormalization group method. Exact solutions are currently available only for two-dimensional systems.[6–8] The mean-field theory is valid for spatial dimensions greater than or equal to the critical dimension $d_c = 6$. This leaves the interval $2 < d < 6$, for which no microscopic theory is available for calculating critical indices that agree reasonably well with values found from numerical experiments.

In this paper we study the time evolution of the following percolation process. Let a wetted site be present at the coordinate origin. In unit time (one generation of the branching process), there is a certain probability that this site will wet anywhere from 0 to z lattice sites (where z is the number of nearest neighbors in the first coordination sphere). Each of the wetted sites may then wet their nearest neighbors during the next generation, etc. We will study how the mean-square size of the wetted cluster depends on time. (We will amplify this model below and discuss how this dependence is related to the definition of the correlation radius.)

For a quasi-one-dimensional Bethe lattice (tree), the percolation process is a Markov branching process whose properties are well-established.[9] The situation for an actual d-dimensional lattice differs, because in general a single wetted site may correspond to several wetted Bethe lattice sites. Thus, if we use the Bethe model to describe actual percolation processes (this is tantamount to the mean-field approximation), errors occur because the "daughters" of a single wetted site ("parent") may be erroneously counted several times as the descendents of different sites. The branching percolation can go back to a previously wetted site, and this recurrence is responsible for the nonmarkovian nature of the process.

Percolation in nonideal lattices can be modeled as a branching diffusion process[9] subject to a "volume exclusion" principle which forbids returns to previously wetted sites (i.e., no self-intersections are allowed). Such processes may be referred to as "tree growth with excluded volume"; their properties were studied by numerical simulation in Refs. 10–12.

For any instance of the process one can find a unique sequence of parent-daughter sites which joins any site wetted at the Nth generation to the coordinate origin. In this paper we regard the spatial distribution of parent-daughter sequences as a distribution of random-walk trajectories on a lattice. The "Hamiltonian"[13] of this distribution allows for the average number of times the unbranched random walk trajectory crosses the branches of the percolation trajectory (this number is calculated for a fixed trajectory position). The Hamiltonian thus takes into account the average number of times the random-walk trajectory crosses the sites in the cluster. It describes a highly nonmarkovian process, and the percolation model in this paper is therefore not equivalent to the mean-field approximation.

We will analyze the spatial distribution of the random-walk trajectories by using a variational principle with the noval test Hamiltonian proposed in Ref. 14 trajectories. The mean-square distance R_{ij} between sites wetted in the ith and jth generations of the model process is found to be

$$\begin{aligned} &\langle R_{ij}^2\rangle = |i-j| \quad \text{for small} \quad |i-j|, \\ &\langle R_{ij}^2\rangle \infty |i-j|^{2\nu} \quad \text{for large} \quad |i-j|, \\ &\nu = 4/(d+2), \end{aligned} \tag{1}$$

and for large $|i-j|$ the site distribution is scale-invariant. According to Refs. 10–12, the mean lifetime T of an excluded-volume branching diffusion process is given by $T \sim \Delta^{-\nu_t}$, where ν_t is a new critical index and $\Delta = p_c - p$, where p is the fraction of broken bonds and p_c is the threshold for percolation. For $\Delta \to +0$ and $|i-j| = T$, the formulas derived below determine how the cluster radius R depends on Δ: $R^2 \propto \Delta^{-\nu} p$. The estimate for the corresponding critical index ν_p yields an exact value for the upper critical dimension d_c and gives results in close agreement with ν_p found in numerical experiments for $2 \leqslant d \leqslant 6$.

DESCRIPTION OF THE MODEL

We regard percolation as a non-self-intersecting branching diffusion process (return to previously wetted sites is not allowed). Assume that a wetted site is located at the coordinate origin (by definition, it belongs to generation zero). The probability that his site will wet s of its nearest neighbors (first-generation sites) is given by

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$$P(s)=C_z^s p^s (1-p)^{z-s},$$

where the C_z^s are the binomial coefficient and z is the coordination number of the lattice. Each of the k^{th}-generation sites has a probability $P(s)$ of wetting s different sites in its first coordination sphere, which by definition belong to the $(k+1)$th generation. If any of the $(k+1)$-generation sites coincides with a j-generation site with $j \leqslant k$, we remove it from the $(k+1)$-generation set. If more than two sites belonging to the same generation coincide, all but one of them (chosen arbitrarily) are removed. Equivalently, we can define percolation as a branching diffusion process, each instance of which is non-self-intersecting (i.e., as the growth of a tree with excluded volume). Figure 1 shows an example of such a process in two dimensions; it provides an adequate percolation model if the number of generation N is less than the mean lifetime T of the process.[12]

Although the above-defined process is nonmarkovian it does not contain any cycles. In any given instance, each wetted site is wetted only once and is joined to the zero-generation site by a unique path.

Consider an arbitrary site x_N which is wetted in the Nth generation for a given instance of the process. Then for this instance each generation contains one and only one site which belongs to the chain of sites joining x_N to the 0-generation site. We write x_i for the radius vector of the ith-generation site belonging to the chain.

In the continuous limit the joint probability distribution of the d-dimensional vectors x_i can be written in the form

$$\mathscr{P}\{x_i\}=(2\pi\varepsilon)^{-dN/2} I_0^{-1}(N) \exp(-H_0\{x_i\}), \tag{2}$$

where

$$I_0(N)=(2\pi\varepsilon)^{-dN/2}\int \dots \int \exp(-H_0\{x_j\}) \prod_{i=1}^{N} dx_i, \qquad x_0=0.$$

If self-intersections are allowed, the sequence $\{x_i\}$ is an unrestricted random walk, and the Hamiltonian H_0 is therefore given by

$$E=(1/2\varepsilon)\sum_{i=1}^{N}(x_i-x_{i-1})^2,$$

where ε is the persistence length.[13]

The distribution (2) is known to correctly describe the excluded-volume effect for an unbranched chain if the Hamiltonian is given by[13]

$$H_0=E+\sum_{i=1}^{N}\sum_{j=1}^{i-1} V_\delta(|x_i-x_j|), \tag{3}$$

where $V_\delta(r)$ is a short-range potential.

The second sum on the right is over all self-intersections of a given instance of a self-avoiding random walk.

In analogy with self-avoiding walks, we assume that the spatial distribution density for the distingushed chain $\{x_i\}$ is given by (2) with the Hamiltonian

$$H_0=E+W. \tag{4}$$

By definition, the functional $W\{x_i\}$ is equal to the mean number of sites on the distinguished trajectory which are nearest neighbors of sites wetted in previous generations (this average must be calculated for a specified position of each of the sites x_i in the distinguished chain).

All instances of the process are free of cycles, i.e., they represent trees. We call the distinguished trajectory the trunk of the trees, and the functional W is the number of times the trunk crosses itself and the branches. In order to obtain a closed definition of the density $\mathscr{P}$ in (2)–(4), we define W self-consistently by assuming that the average spatial distribution of the sites on a branch relative to its branching point (this is the unique point at which the branch and tree intersect) is the same as the distribution of the sites on the trunk relative to the root (the zero-generation site). With this assumption, W is given by

$$W=\alpha\sum_{i=1}^{N}\sum_{j=1}^{i-1}\sum_{\tau=0}^{i-j} g(\tau)\, V(\tau, x_i-x_j). \tag{5}$$

Figure 2 illustrates the summation in the right-hand side of Eq. (5); i indexes the site on the trunk, j is the number of the site at which the branch occurs, and τ is the generation number within the branch (regarded as a new process with the branch point x_j as its root). The function $g(\tau)$ is equal to the

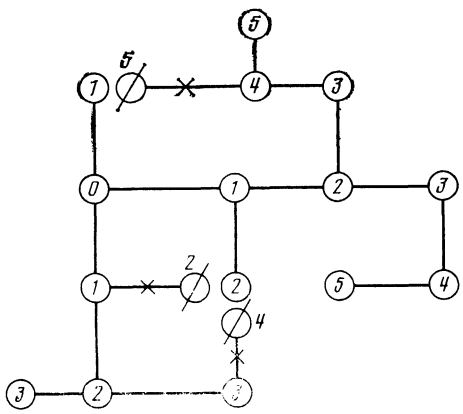

FIG. 1. Branching diffusion process for $d=2$, $N=5$. Each circle is labeled by the number of the generation at which the site is wetted; the crossed-out circles and bonds are eliminated.

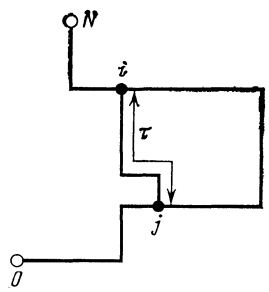

FIG. 2. Diagram illustrating how the functional W is calculated: i is a site on the trunk which is crossed by a branch with branching site j; τ is the number of generations within the branch.

mathematical expectation of the number of sites wetted in generation τ. In this paper we assume that $g(\tau)$ is equal to the corresponding distribution for a Bethe lattice,

$$g(\tau)=e^{-\tau/T}.$$

Since we assume self-consistency, we have

$$V(\tau, r)=G(0, 0; \tau, r), \tag{6}$$

where $G(0,0;\tau,r)$ is the Green's function for the trunk site distribution; it is equal to the probability density for site τ to lie near point r when the root lies at the coordinate origin.

The functional W is defined up to a constant factor, which depends on the branching density of the various instances of the branching diffusion process. We set $\varepsilon=\alpha=1$, because the critical indices are independent of the constants ε and α and no method is yet available for comparing the numerical factors in scaling laws of the type (1) with results from numerical simulations.

VARIATIONAL PROCEDURE

In this section we suggest a variational procedure whose main idea is to construct a test Hamiltonian $H\{x_i\}$ which allows us to independently vary the scale of the fluctuations $\langle(x_i-x_j)^2\rangle$ for different scales of the "time" intervals $|i-j|$ (this requires introducing infinitely many parameters). The parameters of the test Hamiltonian, the Green's function G, and the asymptotic scaling laws are found by maximizing the right-hand side of the variational inequality[15]

$$I_0(N)\geqslant I(N)\exp(-\langle H_0-H\rangle), \tag{7}$$

where

$$I(N)=(2\pi\varepsilon)^{-dN/2}\int\ldots\int e^{-H}\prod_{i=1}^{N}dx_i,$$

and the mean $\langle F\rangle$ of the functional $F\{x_i\}$ is defined as

$$\langle F\rangle=I^{-1}(2\pi\varepsilon)^{-dN/2}\int\ldots\int Fe^{-H}\prod_{i=1}^{N}dx_i. \tag{8}$$

Following Ref. 14, we make the change of variables $x_i\to y_l^m$, which enables us to explicitly separate the additive contributions in E and W from the fluctuations. The latter are observed on segments of the trajectory of length between 2^m and 2^{m+1}, where $m=1,2,\ldots,M(M=[\log_2 N]$ and $[a]$ denotes the integral part of a). To simplify the calculations we assume that M is an integer.

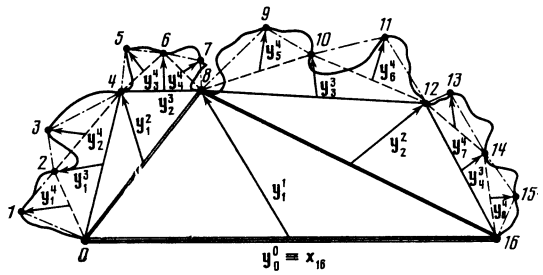

FIG. 3. Illustration of the change of variables $x_i\leftrightarrow y_l^m$ for $M=4$, $N=16$. The indices $i=1,2,\ldots,16$ of the points x_i are indicated on the trajectory. The dashed-and-dotted, dashed, single, double, and triple lines approximate the trajectory to within the time scales t_m, $m=0,1,\ldots,4$, respectively.

We define the vector y_l^m to point from the midpoint of the segment joining the points $x_{2lNk_m^{-1}}$ and $x_{2(l-1)Nk_m^{-1}}$ to the "intermediate" point $x_{(2l-1)Nk_m^{-1}}$ on the trajectory (Fig. 3). It is easy to see that the functional $E\{x_i\{y_l^m\}\}$ is equal to

$$E=\frac{1}{2}\left\{\left(y_0{}^0N^{-1}+2\sum_{m=1}^{M}t_m{}^{-1}\sum_{l=1}^{2^{m-1}}(y_l{}^m)^2\right\}\right., \tag{9}$$

where $t_m=Nk_m^{-1}$, $y_0^0=x_N$, $k_m=2^m$.

If we include y_0^0, the number of independent variables $y_l^m, m=1,2,\ldots,M, l=1,2,\ldots,k_{m-1}$, is equal to N and thus to the number of independent variables x_i. Since the linear transformation $x_i\to y_l^m$ is norm-preserving, its Jacobian is equal to 1. Figure 3 shows the vectors x_i and y_l^m for $M=4$, while Table I gives the inverse transformation.

In general, the inverse transformation $\{y_l^m\}\to\{x_i\}$ is given by

$$x_l=\sum_{m=0}^{M}\mu^m(l)\,y_{[k_{m+1}^{-1}l]}^{M-m},$$

where

$$\mu^m(l)=2r_m,\quad 0\leqslant r_m\leqslant{}^1/_2,$$
$$\mu^m(l)=2(1-r_m),\quad {}^1/_2\leqslant r_m\leqslant 1,$$
$$r_m=k_{m+1}^{-1}l-[k_{m+1}^{-1}l].$$

Physically, the sum

TABLE I. The inverse transformation $\{y_l^m\}\to\{x_i\}$ for $N=16$.

i	$x_i=\sum_{m,l}\beta_{m,l}y_l^m$	i	$x_i=\sum_{m,l}\beta_{m,l}y_l^m$
1	${}^1/_{16}y_0{}^0+{}^1/_8y_1{}^1+{}^1/_4y_1{}^2+{}^1/_2y_1{}^3+y_1{}^4$	8	${}^1/_2y_0{}^0+y_1{}^1$
2	${}^1/_8y_0{}^0+{}^1/_4y_1{}^1+{}^1/_2y_1{}^2+y_1{}^3$	9	${}^9/_{16}y_0{}^0+{}^7/_8y_1{}^1+{}^1/_4y_2{}^2+{}^1/_2y_3{}^3+y_5{}^4$
3	${}^3/_{16}y_0{}^0+{}^3/_8y_1{}^1+{}^3/_4y_1{}^2+{}^1/_2y_1{}^3+y_2{}^4$	10	${}^5/_8y_0{}^0+{}^3/_4y_1{}^1+{}^1/_2y_2{}^2+y_3{}^3$
4	${}^1/_4y_0{}^0+{}^1/_2y_1{}^1+y_1{}^2$	11	${}^{11}/_{16}y_0{}^0+{}^5/_8y_1{}^1+{}^3/_4y_2{}^2+{}^1/_2y_3{}^3+y_6{}^4$
5	${}^5/_{16}y_0{}^0+{}^5/_8y_1{}^1+{}^3/_4y_1{}^2+{}^1/_2y_2{}^3+y_3{}^4$	12	${}^3/_4y_0{}^0+{}^1/_2y_1{}^1+y_2{}^2$
6	${}^3/_8y_0{}^0+{}^3/_4y_1{}^1+{}^1/_2y_1{}^2+y_2{}^3$	13	${}^{13}/_{16}y_0{}^0+{}^3/_8y_1{}^1+{}^3/_4y_2{}^2+{}^1/_2y_4{}^3+y_7{}^4$
7	${}^7/_{16}y_0{}^0+{}^7/_8y_1{}^1+{}^1/_4y_1{}^2+{}^1/_2y_2{}^3+y_4{}^4$	14	${}^7/_8y_0{}^0+{}^1/_4y_1{}^1+{}^1/_2y_2{}^2+y_3{}^4$
		15	${}^{15}/_{16}y_0{}^0+{}^1/_8y_1{}^1+{}^1/_4y_2{}^2+{}^1/_2y_4{}^3+y_8{}^4$
		16	$y_0{}^0$

$$\sum_{l=1}^{2^{m-1}} (y_l^m)^2$$

measures the deviation of the trajectory constructed to time scale t_m from the more refined $(m+1)$th approximation, which is twice as accurate.

We will take the test Hamiltonian to be the following functional, which is constructed by analogy with $E\{y_l^m\}$ in (9):

$$H\{y_l^m\}=\frac{1}{2}\left\{(y_0^0)^2\sigma_0^{-2}+2\sum_{m=1}^{M}\sigma_m^{-2}\sum_{l=1}^{2^{m-1}}(y_l^m)^2\right\}, \quad (10)$$

where the variable parameters σ_m $(m=0,1,2,\ldots,M)$ determine the variances of the y_l^m, i.e., the scale of the fluctuations over time intervals t_m.

For this test Hamiltonian

$$\langle H_0-H\rangle\equiv\langle E-H\rangle+\langle W\rangle,$$

$$I(N)=\exp\left(\frac{3}{2}\ln\sigma_0^2+\frac{3}{4}N\sum_{m=1}^{M}t_m^{-1}\ln\sigma_m^2\right), \quad (11)$$

$$\langle E-H\rangle=\frac{3}{2}\left\{\sigma_0^2N^{-1}-1+N\sum_{m=1}^{M}(2t_m)^{-1}(\sigma_m^2t_m^{-1}-1)\right\}. \quad (12)$$

The corresponding "Green's function" is

$$G(i,r;j,r')=\exp\left[-\frac{(r-r')^2}{2\tau(i,j)}\right]\{2\pi\tau(i,j)\}^{-d/2}, \quad (13)$$

where

$$\tau(i,j)=\sum_{m,l}\beta_{m,l}^2(i,j)\sigma_m^2,$$

and the $\beta_{m,l}$ are the coefficients in the expansion of the vector x_i-x_j in terms of the y_l^m:

$$x_i-x_j=\sum_{m,l}\beta_{m,l}(i,j)y_l^m. \quad (14)$$

An expression for $\beta_{m,l}(i,j)$ was presented and analyzed in Ref. 9, where it was shown that if the variance σ_m^2 varies as a power of t_m, i.e.,

$$\sigma_m^2\propto t_m^{2\nu} \quad (15)$$

then

$$\tau(i,j)=\beta|i-j|^{2\nu}, \quad (16)$$

and the numerical factor β oscillates between 1 and $2^{2\nu-1}$ as i and j vary and reaches a maximum for $i=N$.

The expansion (14) involves at most two vectors y_l^m with the same index m. The dependence (16) can be traced to the fact that the k th term, for which $|i-j|=t_k$, gives the dominant contribution to $\tau(i,j)$. The contribution from larger m (smaller t_m) decreases as $2^{2\nu|m-k|}$ because the factor σ_m^2 decays, while the contribution from smaller m (larger t_m) drops as $2^{-2(1-\nu)|m-k|}$ because the factor $\beta_{m,l}^2$ decreases. Figure 4 plots τ as a function of $|i-j|$ for $\nu=4/5$. The oscillations of β are insignificant (they average out in the formula for W and do not alter the index ν).

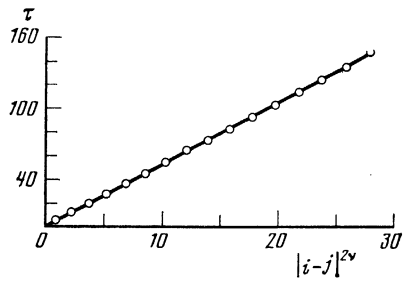

FIG. 4. The dependence of $\tau(i,j)$ on $|i-j|^{2\nu}$ found by numerically summing the series with $\sigma_m^2=t_m^{\nu}$, $\nu=4/5$, $i=$ const.

The postulated dependence (15) is self-consistent in the sense that we use it to calculate the dependence of $\tau(i,j)$ on $|i-j|$, while at the same time Eq. (15) is a consequence of the requirement that the test Hamiltonian be a minimum with respect to the σ_m for large t_m (see below).

We now explicitly identify the contribution to $\langle W\rangle$ from the effective interaction between trajectory sites i and j for which $k_m\leqslant|i-j|\leqslant k_{m+1}$. By Eqs. (4), (8), and (13),

$$\langle W\rangle=\sum_{m=1}^{M}\sum_{i=t_m}^{N}\sum_{s=t_{m+1}}^{t_m}\sum_{\tau=0}^{\cdot}\int drG(i,0;i-s,r)V(\tau,r)g(\tau). \quad (17)$$

If we replace the sums over s and τ by integrals and substitute $g(\tau)$ $V(\tau,r)$, $G(i,0;i-s,r)$, and $\tau(i,j)$ from Eqs. (5), (6), (14), and (16), we get

$$\langle W\rangle=\sum_{m=0}^{M}(N-t_m)g_mt_m^2\sigma_m^{-d}, \quad (18)$$

$$g_m=(2\pi\beta)^{-d/2}\int_0^1\frac{\exp(-\tau t_m/T)}{(1+\tau^{2\nu})^{d/2}}, \quad (19)$$

The coefficient g_m is independent of t_m and T both for $t_m\ll N$ and for $t_m=N=T$. In general, for $N\leqslant T$ g_m does not change the scaling dimension of the mth contribution to $\langle W\rangle$.

Substituting the expressions for $I(N)$, $\langle E-H\rangle$, and $\langle W\rangle$ from (11), (12), and (18) into the variational inequality (7), we obtain

$$I_0(N)\geqslant\exp\left\{-\left[\frac{3}{2}\left(\frac{\sigma_0^2}{N}-1\right)-\frac{3}{2}\ln\sigma_0^2+\frac{1}{2}g_0\frac{N^3}{\sigma_0^d}+\sum_{m=1}^{M}\left(\frac{N}{t_m}\left(\frac{3}{4}\left(\frac{\sigma_m^2}{t_m}-1\right)-\frac{3}{4}\ln\sigma_m^2+g_m\frac{t_m^3}{\sigma_m^d}\right)-\frac{t_m^3}{\sigma_m^d}g_m\right)\right]\right\}. \quad (20)$$

For $N\to\infty$ the variable parameters σ_m of the test Hamiltonian are determined by maximizing the argument of the exponential in the right-hand side of (20):

$$\sigma_m^{d+2}-t_m\sigma_m^d-g_0dt_m^4/6=0 \quad \text{for} \quad t_m\to N, \quad (21)$$

$$\sigma_m^{d+2}-t_m\sigma_m^d-2g_mdt_m^4/3=0 \quad \text{for} \quad t_m\ll N. \quad (22)$$

For small t_m Eq. (22) has the solution $\sigma_m=t_m^{1/2}$,

which is the same as the corresponding parameter in the functional E in (9). The function $\tau(i,j)$ determining the Green function in (13) is therefore equal to $\tau(i,j) = |i-j|$ for small $|i-j|$. For large t_m

$$\sigma_m^2 \infty t^{2\nu}, \quad \nu = 4/(d+2) \tag{23}$$

and hence $\tau(i,j) \sim |i-j|^{2\nu}$. By minimizing the test Hamiltonian H (10) we thus define a process which coincides with an unrestricted random walk for short times and is scale-invariant for long times. The critical index ν in (23) determines how the squared radius of the growing percolation cluster depends on the number of generations N (on the growth time):

$$R^2 \infty N^{2\nu}. \tag{24}$$

Recent numerical studies of percolation cluster growth[14-16] indicate that the mean lifetime of an excluded-volume branching diffusion process is equal to

$$T \infty \Delta^{-\nu_t}, \tag{25}$$

where ν_t is a new critical index, $1 \leqslant \nu_t < 2$, $\nu_t|_{d=6} = 1$.

For $N = T$, expressions (24) and (25) determine how the mean square radius of the percolation cluster depends on Δ in the subcritical region:

$$R^2 \infty \Delta^{-\nu\nu_t}. \tag{26}$$

Our percolation model thus gives the result

$$\nu_p = 4\nu_t/(d+2) \tag{27}$$

for the critical index ν_p of the correlation radius.

Table II lists values of ν_p calculated by Eq. (27), together with results from numerical experiments for $2 < d < 6$ and exact values for $d = 2$ and $d = 6$. Our calculated values for ν_p agree with the simulation results to within the computational error.

Equations (21), (22) imply that the structure of the trunk for each instance of the process is self-similar.[16] For small times t_m the distribution of the sites is the same as in the mean-field approximation, which corresponds to $\nu = 1/2$. For large t_m ($t_m \gg 1, t_m \ll T$) the distribution is determined by the statistics at the critical point $p = p_c$, while the $t_m \to T$ it is determined by the statistical behavior in the subcritical region $\Delta \to +0$.

For $N \gg T$, the distribution changes because $g(\tau)$ in the functional $\langle W \rangle$ decreases abruptly. This qualitative behavior agrees with the finding in Ref. 12, where it was shown that for $N \gg T$ the process does not describe percolation clusters but instead obeys the statistics of lattice figures.

DISCUSSION OF THE ASSUMPTIONS AND RESULTS

The reduction of the percolation process to a branching diffusion process is exact. The principal assumption made in this paper is contained in Eq. (6). Although the assumption is a conjecture of the self-consistent potential type, the proposed method does not reduce to the mean-field technique. The Hamiltonian of the distribution (2) modeling the behavior of the trunk is defined self-consistently, whereas the test Hamiltonian (10) used to study (2) contains an infinite set of variable parameters which determine the spectrum of the fluctuations in the model spatial distribution.

The full strength of conjecture (6) is not needed, because the final expression for ν_p is unchanged if we take $V(\tau,r)$ equal to any scaling potential normalized to unity:

$$V_s(\tau, r) = \tau^{-d\nu'} f(r/\tau^{\nu'})$$

with $\nu' \leqslant \nu$.

In terms of a systematic variational analysis, it might be more logical to carry out the variational procedure for a Hamiltonian with the potential V_s. However, since this does not alter the result we have used the simpler conjecture (6) in the text. According to Ref. 17, the scaling dependence on Δ for the characteristic length of the dangling (dead-end) branches leaving the backbone of a percolation cluster is the same as for the backbone itself, which supports (6).

Although we have analyzed the properties of the spatial distribution of a distinguished sequence of sites in this paper, we have not studied how the number of wetted sites changes with time (generation number). For this reason, the function $g(\tau)$ is essentially prescribed in the formulation of the problem. The choice of $g(\tau)$ is determined by the fact that the quasi-one-dimensional Bethe model is the only percolation model for which the number of wetted sites is known exactly as a function of generation number. The close agreement between the results in this paper and the results of numerical experiments could result from the fact that the presence of a cluster generated by the distinguished sequence of wetted sites increases the probability that branching sites will "disappear," which in turn effectively increases the distance from the percolation threshold when $g(\tau)$ is calculated.

TABLE II.

d	2	3	4	5	6
ν_t	1,35 [a]	1,11 [b]	1,04 [b]	1 [c]	1
ν	1	0,8	0,67	0,57	0,5
ν_p	1,35	0,89	0,69	0,57	0,5
$\nu_p^{\exp}$	(4/3) [d]	0,88	0,7	0,6	1/2

[1)]Values denoted by [a] and [b] are taken from Refs. 11 and 10, respectively; for lack of experimental data, the values indicated by [c] are taken equal to the exact values on a Bethe lattice; [d] indicates values calculated in Refs. 6–8. The other results for $\nu^{\exp}$ are taken from Ref. 18.

We note that the triple summation in the functional W enumerates the pair interactions between sites in the distinguished sequence; the average number of sites wetted in previous generations can be calculated under the condition that the number of N^{th}-generation sites is greater than zero (because there exists an N th site of the distinguished sequence). The conditional mean found in this way is quite different from the average number of sites in an instance of the process. For a Markov branching process (a process on a Bethe lattice), the latter number is proportional to

$$\sum_{\tau=0}^{\infty} g(\tau).$$

Our description of the excluded volume effect in terms of the functional W in the hamiltonian is based on analogy with the statistics of self-avoiding random walks, for which Eq. (6) is now generally employed.

The assumptions discussed above are necessary in order to reduce the problem to an analysis of the scaling properties of the distribution $\mathscr{P}\{x_i\}$. The variational analysis of $\mathscr{P}\{x_i\}$ for a specified fixed functional $W\{x_i\}$ is quite rigorous.

It is noteworthy that according to perturbation theory, in which $W\{x_i\}$ is assumed to be small compared with $E\{x_i\}$, the average number of times, $N^{-1}W\{x_i\}$, an instance of the process encounters a wetted site is equal to

$$N^{-1}\langle W\rangle_E = \int\ldots\int We^{-E}\prod_{i=1}^{N} dx_i \Big(\int\ldots\int e^{-E}\prod_{i=1}^{N} dx_i\Big)^{-1},$$

which for $d<6$ tends to infinity as $N^{3-d/2}$ as $N\to\infty$, while the corresponding average with the density $\mathscr{P}\{x_i\}$ is finite. Similarly, for $d\leqslant 2$ an unrestricted random walk has self-interactions, unlike self-avoiding walks of arbitrary dimension.[14]

CONCLUSIONS

In this paper we have treated percolation as a self-similar branching diffusion process that obeys the excluded volume condition. Our model is intermediate between the Bethe lattice model, which completely neglects self-intersections, and the exact treatment. Our expression for the critical index ν predicts the exact value $d_c = 6$ for the critical dimension and yields results in close agreement with numerical experiments and exact values of ν_p for $2\leqslant d\leqslant 6$.

I am deeply grateful to A. A. Ovchinnikov for his unflagging interest in this work and for valuable advice, and also to V. Ya. Krivnov, A. Yu. Grosberg, S. P. Obukhov, A. R. Khokhlov, and I. Ya. Erukhimovich for a fruitful discussion.

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Translated by A. Mason