Quantum theory of transition scattering

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A microscopic quantum theory of transition scattering is constructed under the assumption that the scattered permittivity wave has a classical character, i.e., that $\hbar q_0 \ll E_{\mathbf{p}}$ and $\hbar_{\mathbf{q}} \ll \mathbf{p}$, where q_0 and $\mathbf{q}$ are the frequency and wave vector of the scattered wave, while $E_{\mathbf{p}}$ and $\mathbf{p}$ are the energy and momentum of the particle. It is shown that, for particle energies $E_{\mathbf{p}} \gg E_{cr} = (mc^2)^2/2qc\hbar$, the scattering spectrum for electromagnetic waves extends to a maximum frequency $\omega_{\max} \sim E_{\mathbf{p}}/\hbar$, whereas for $E_{\mathbf{p}} \ll E_{cr}$ (i.e., in the classical limit) this spectrum extends, as is well known, to $2qc(E_{\mathbf{p}}/mc^2)^2$. The spectral intensity of the transition scattering is computed in the ultraquantum limit. It is shown that the total radiation intensity virtually does not change on going from the energy region $E_{\mathbf{p}} \ll E_{cr}$ into the region $E_{\mathbf{p}} \gg E_{cr}$. The problem of the detection of high-energy particles (cosmic rays) with counters based on transition scattering is considered.

§1. INTRODUCTION

The classical (nonquantum) theory of transition radiation and transition scattering is now a thoroughly developed theory (see Ref. 1). But the problem of the quantum corrections to transition scattering has thus far not been discussed, and a quantum theory of transition radiation has been constructed only in the special case of a sharp boundary between the two media, using a phenomenological approach,[2] the limits of applicability of which need to be further examined. It should be noted that transition radiation, like transition scattering, and unlike the Vavilov-Čerenkov radiation has no upper emission-frequency bound. For the Vavilov-Čerenkov radiation, the existence of a threshold condition—the Vavilov-Čerenkov condition—makes the quantum corrections to the radiation intensity small, in view of the absence from the radiation spectrum of high frequencies on the order of $E_{\mathbf{p}}/\hbar$, where $E_{\mathbf{p}}$ is the particle energy. But in the case of transition radiation and transition scattering the classical theory does not yield a frequency spectrum that is bounded from above, and this compels us to regard as pressing the problem of the construction of a quantum theory of the corresponding processes, i.e., the problem of quantum recoil during radiation emission, to be a pressing one. In Ref. 1 it is noted that transition radiation can, in a certain approximation, be treated as a superposition of transition scatterings at those harmonics (in the wave-number spectrum) that are spanned by the jump (or fairly smooth variation) of the permittivity at the boundary between the two media. Therefore, the problem of constructing microscopic quantum theories of transition scattering and transition radiation is actually equivalent to the problem of finding the analog of the Klein-Nishina formula for charged-particle scattering by permittivity waves with emission of electromagnetic waves. The well-known difficulty encountered in the solution of this problem is that the process itself should, in contrast to ordinary Compton scattering, which is described by the diagram shown in Fig. 1, be described by the diagram in Fig. 2, which includes a nonlinear material vertex due to the nonlinear interaction of the waves. If this is so, and the quasiclassical limit indicates that it is, it is necessary to carry out the rather tedious nonlinear-response calculation in the general quantum case. Strictly speaking, the problem consists in the demonstration that, in the quantum approach, transition scattering is described precisely by the diagram in Fig. 2, since thus far such diagrams have been drawn only to explain the nature of the process in the quasiclassical limit (see Ref. 1).

Thus, the formulation of the problem should include the question of the modification of the density matrix of the medium in which the charged particle moves, a modification which is caused by the field of the indicated charge, and the dielectric polarization of the medium in the inhomogeneity wave.

If we do not consider the case of strong fields, we do not encounter any fundamental difficulties in the derivation of the general expressions for the amplitudes of particular processes, but do encounter appreciable difficulties in the computation of specific quantities. The main complication lies in the extremely unwieldy expression for the nonlinear response in the quantum case. The problem is eased by the fact that, in practice, the quantum corrections are important only in the region of very high radiation frequencies $\hbar\omega \sim E_{\mathbf{p}}$. In the quasiclassical limit the highest frequencies emitted in transition scattering on a permittivity wave with wave vector $\mathbf{q}$ (with frequency $q_0 \ll qc$, or, in particular, with zero frequency, i.e., on a static permittivity wave) are of the order of $2qc(E_{\mathbf{p}}/mc^2)$.[2] Setting, for the purpose of making estimates, $\hbar\omega_{\max} \sim E_{cr}$, we find that $E_{cr} = (mc^2)^2/2qc\hbar$,

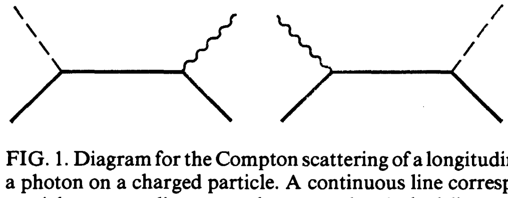

FIG. 1. Diagram for the Compton scattering of a longitudinal quantum into a photon on a charged particle. A continuous line corresponds to a spinor particle; a wavy line, to a photon; and a dashed line, to the field of the medium.

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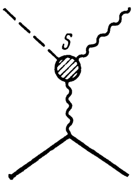

FIG. 2. Diagram for transition scattering. The lines have the same meanings as in Fig. 1.

i.e., when the particle itself has energy $E_{\mathbf{p}} \gtrsim E_{cr}$, the emitted frequencies are of the order of the energy $E_p/\hbar$, the particle motion is strongly perturbed in the course of the emission, and a quantum-mechanical computation is necessary. Furthermore, it can be assumed that the particle is ultrarelativistic, i.e., that $E_p \gg mc^2$, since $2qc\hbar/mc^2 \ll 1$, and the characteristic permittivity modulation length $a = 2\pi/q$ is usually much greater than the Compton wavelength of the particle (for the electron $\hbar/m_e c \sim 10^{-11}$ cm). Thus, the quantum effects show up when a particle with ultrarelativistic energy emits radiation in the frequency region where the permittivity can be described by the plasma formula $\varepsilon(\omega) = 1 - \omega_{pe}{}^2/\omega^2$. Because the frequencies are high, the particles of the medium can be considered to be free, and therefore in the calculations we shall use the plasma approximation not only for the linear, but also for the nonlinear response of the medium.

The quantum theory will be constructed for spin-$\frac{1}{2}$ particles with the aid of the Dirac equation, an approach which is dictated by the necessity to take the relativistic effects into consideration, since, as has been noted above, the quantum corrections are important only for ultrarelativistic particles. At the same time, as shown in Ref. 3, the Compton scattering of plasma waves (an example of permittivity waves) is usually much more intense than transition scattering. The latter predominates when $E_{\mathbf{p}}/mc^2 \ll m/m_e$, where m is the particle mass and m_e is the mass of the electrons of the medium. In the present paper we shall consider transition scattering without allowance for the Compton scattering. This makes sense for heavy particles or quasineutral permittivity waves (of the type of acoustic, etc., waves), in which there is no appreciable electric-field component resulting from charge separation in the wave. An example of such waves is the low-frequency ion—sound wave. It is clear that many inhomogeneities of a medium are quasineutral. The estimate $E_{\mathbf{p}}/mc^2 \ll m/m_e$ holds true only for those waves in which the energy is divided equally between the field and the particles of the medium (such are, for example, the Langmuir waves in a plasma).

In the calculations we use the following procedure. In the high-frequency limit the permittivity depends only on the electron density of the medium, and its modulation is determined by $\delta n_{\mathbf{q}}$, the Fourier transform of the concentration variation. It is sufficient to compute the amplitude of the process for a quasineutral perturbation, in which $\delta n_{\mathbf{q}}$ is uniquely connected with the field $E_{\mathbf{q}}$, and express the final result in terms of $\delta n_{\mathbf{q}}$. The result thus obtained will not depend on the specific relation between the quantities $E_{\mathbf{q}}$ and $\delta n_{\mathbf{q}}$, since this result is valid, in particular, for a true neutral density perturbation: the effect, in final analysis, is determined by $\delta n_{\mathbf{q}}$. This simplifies the calculations in the sense that the vertex depicted in Fig. 2 should be computed for current induced by two fields: $\mathbf{E}_{\mathbf{q}}$ and $\mathbf{E}'$ (the latter has the meaning of the self-field of the charge); we go through a similar procedure in both the classical and quantum cases, using the method of kinetic equation for the distribution function (density matrix) of the electrons of the medium.

§2. ANALYSIS OF THE CONSERVATION LAWS

Let us analyze the conservation laws that obtain in the scattering of a particle of mass m, energy $E_{\mathbf{p}} = (\mathbf{p}^2c^2 + m^2c^4)^{1/2}$, and momentum $\mathbf{p}$ by a permittivity wave of frequency q_0 and wave vector $\mathbf{q}$. With no smallness limitation imposed on the recoil, the energy conservation law has the form (the kinematic scheme is depicted in Fig. 3)

$$\hbar q_0 + E_{\mathbf{p}} = \hbar\omega_{\mathbf{k}} + E_{\mathbf{p}+\hbar\mathbf{q}-\hbar\mathbf{k}}.$$

If, first, the frequency of the emitted photon is high, so that $\omega_{\mathbf{k}} \gg \omega_{pe}$ (where $\omega_{pe} = (4\pi e^2 n_0/m_e)^{1/2}$ is the plasma frequency, n_0 is the unperturbed electron density in the medium, and $\omega_{\mathbf{k}} \simeq kc$), second, the charged particle is ultrarelativistic and, as is normal in this case, $\theta \lesssim mc^2/E_{\mathbf{p}} \ll 1$, i.e., the radiation is concentrated in a narrow angle cone along the direction of motion of the particle, and third, the scattered wave is quasistatic (i.e., $q_0 \ll qc$), the relation between the emitted frequency $\omega_{\mathbf{k}}$ and the angle θ of emission of the photon will be the following:

$$\frac{\omega_{\mathbf{k}}}{2}\left[\left(\frac{mc^2}{E_{\mathbf{p}}}\right)^2 + \frac{\omega_{pe}{}^2}{\omega^2} + \theta^2\right] = qc\left(1 - \frac{\hbar\omega_{\mathbf{k}}}{E_{\mathbf{p}}}\right), \qquad (1)$$

where the vectors $\mathbf{p}$ and $\mathbf{q}$ were, for simplicity, assumed to be oppositely directed; the conservation law cannot be obeyed under the indicated assumptions in the case when the vectors are oriented in the same direction. In the formula (1) q is the magnitude of the vector $\mathbf{q}$. Thus, for an ultrarelativistic particle, emission is possible only in the case of scattering by a wave whose wave vector is opposite to that of the particle momentum.

The expression (1) differs from the corresponding classical analog by the factor $(1 - \hbar\omega_{\mathbf{k}}/E_{\mathbf{p}})$. At a given emission angle θ, two frequencies are radiated:

$$\omega_\pm(\theta) = \omega_{pe}\left\{\frac{qc}{\omega_{pe}} \mp \left[\left(\frac{qc}{\omega_{pe}}\right)^2 - \left(\frac{mc^2}{E_{\mathbf{p}}}\right)^2 - \frac{2qc\hbar}{E_{\mathbf{p}}} - \theta^2\right]^{1/2}\right\}^{-1}. \qquad (2)$$

The radiation itself is possible only when the particle energy

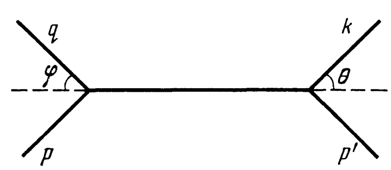

FIG. 3. Kinematic scheme of the scattering process.

is higher than some threshold value

$$E_p \geqslant E_{th} \approx mc^2 \frac{\omega_{pe}}{qc}\left(1+\frac{\hbar\omega_{pe}}{mc^2}\right) = E_{th}^{cl}\left(1+\frac{\hbar\omega_{pe}}{mc^2}\right). \quad (3)$$

It can be seen that the quantum effects almost do not alter this expression, since for media with a solid-state electron density the electron rest energy is 10^5 times greater than the energy of the plasma photon. In order for the threshold particle energy to be also ultrarelativistic, we must have $\omega_{pe}/qc \gg 1$; for a medium with the same parameters this condition is satisfied by permittivity modulation lengths of the order of, or greater than, the optical wavelength.

The dependence of the frequency on the angle of emission of the photon is a function with two branches: ω_+ and ω_-. At the zero value of θ these functions correspond to the frequencies

$$\omega_{\substack{max\\min}} = \omega_{pe}\left\{\frac{qc}{\omega_{pe}} \mp \left[\left(\frac{qc}{\omega_{pe}}\right)^2 - \left(\frac{mc^2}{E_p}\right)^2 - \frac{2qc\hbar}{E_p}\right]^{1/2}\right\}^{-1}, \quad (4)$$

which give the limits of the range in which the emitted frequencies lie.

When the particle energy is much higher than the threshold energy, the lowest emitted frequency almost does not depend on the particle energy, specifically,

$$\omega_{min} \approx \omega_{pe}^2/2qc, \quad (5)$$

while the highest frequency depends on the energy in the following manner:

$$\omega_{max} = \frac{2qc(E_p/mc^2)^2}{1+E_p/E_{cr}}, \quad (6)$$

where E_{cr} is a critical energy value separates arbitrarily those regions in the particle-energy range where a classical and a quantum description of the process are essential. For energies $E_p \ll E_{cr}$ the value of ω_{max} coincides with the value obtained in the classical limit, where the frequency is a quadratic function of the energy. In the opposite (ultraquantum) limiting case, as expected, the highest frequency is approximately equal to $E_p/\hbar$. This difference between the classical and quantum situations alters the transition-scattering spectrum in the region of high frequencies of the order of $E_p/\hbar$.

§3. QUANTUM THEORY OF TRANSITION SCATTERING

We shall assume that the density variation in the permittivity wave (the variation $\delta n_{\mathbf{q}} \ll n_0$, so that the computation is carried out with the aid of perturbation theory) is uniquely connected with the wave field $\mathbf{E}_{\mathbf{q}}^m$. The total field then consists of two terms:

$$A_\mu^t(x) = A_\mu^m(x) + A_\mu(x),$$

where $A_\mu(x)$ is the operator field describing the radiation field and the self-field of the particle and the potential $A_\mu^m(x)$ can be considered to be a c-number function. The charged particle itself is described by a spinor field $\psi(x)$ obeying the Dirac equation. Finally, the medium is described by a density matrix that takes account of both the linear and quadratic polarization of the medium by the field $A_\mu^t(x)$. The linear part of the polarization current gives rise to the effects connected with the photon propagation in the medium (the modification of the dispersion law and the form of the field propagators), so that there remains on the right-hand side of the Maxwell equations only the quadratic polarization current

$$J_\mu^{Nl}(x) = 2\int S_{\mu\nu\rho}(x,x',x'') A_\nu^{m*}(x') A_\rho(x'')\,dx'\,dx'', \quad (7)$$

$$S_{\mu\nu\rho}(x,x',x'') = \dot{S}_{\mu\rho\nu}(x,x'',x').$$

Here S is the nonlinear vertex (response) of the medium in the coordinate representation. For a medium that is homogeneous in the absence of perturbation, the response possesses the natural symmetry property

$$S_{\mu\nu\rho}(x,x',x'') = S_{\mu\nu\rho}(x-x', x-x''). \quad (8)$$

The response S is found by solving the kinetic equation for the density matrix of the medium with the aid of perturbation theory (see Ref. 4).

The occurrence of a polarization current in the Maxwell equations can be described in the field-theoretic language with the aid of an additional (to the usual quantum-electrodynamic) interaction Lagrangian

$$L' = \int dx\, A_\mu^t(x) J_\mu^{Nl}(x).$$

In terms of the action functional, the additional term in the action is

$$I' = \int dx_1\,dx_2\,dx_3\, S_{\mu\nu\rho}(x_1,x_2,x_3) A_\mu(x_1) A_\nu^{m*}(x_2) A_\rho(x_3). \quad (9)$$

In this expression we have omitted the part proportional to $J_\mu^{Nl} A_\mu^m$ as being unessential to a scattering process accompanied by radiation: it describes particle scattering in the field of the medium without radiation.

The form in which the formula (9) is written implies that S possesses another symmetry property, besides those noted above:

$$S_{\mu\nu\rho}(x,x',x'') = S_{\rho\nu\mu}(x'',x',x). \quad (10)$$

Setting

$$I = \int dx L(x) + I', \quad L(x) = e\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x),$$

into which enters the ordinary Lagrangian density for the interaction in QED (in $L(x)$ we do not take account of the term $j_\mu A_\mu^m$, which describes the emission by the particle during scattering on the fields produced by the inhomogeneities of the medium, i.e., as it undergoes "Compton" scattering), we write the scattering matrix in the form

$$S = T\exp(iI). \quad (11)$$

The initial and final states in the problem are $|in\rangle = \hat{a}_{\mathbf{p}\sigma}^+|0\rangle$ (one particle) and $|f\rangle = \hat{a}_{\mathbf{p}'\sigma'}^+, \hat{c}_{\mathbf{k}s}^+|0\rangle$ (a particle and a photon), i.e., there exists in the initial state a particle with momentum $\mathbf{p}$ and polarization σ; while in the final state we have a particle with $\mathbf{p}'$ and σ' and an emitted photon with wave vector $\mathbf{k}$ and polarizaiton s (in this section we use the unit with $\hbar = c = 1$).

In the first nonvanishing approximation the scattering matrix has the form

$$S^{(2)}=-2eT_A\int dx\,dx_1\,dx_2\,dx_3\,\overline{\psi}(x)\gamma_\mu \times\psi(x)A_\mu(x)S_{\lambda\nu\rho}(x_1,x_2,x_3)A_\nu^{m*}(x_2)A_\lambda(x_1)A_\rho(x_3') \quad (12)$$

(it is sufficient to retain the ordering with respect to only the photon operators). Using the standard notation (see Ref. 5), we obtain the following expression for the required matrix element:

$$M_{if}^{(2)}=-2e\bar{u}_{p\sigma}\gamma_\mu u_{p'\sigma'}D_{\mu\nu}(k-q)S_{\lambda\nu\rho}(q-k,q,-k)\times A_\nu^{m*}(q)(4\pi)^{1/2}e_\rho^s; \quad (13)$$

here $u_{p\sigma}$ is a bispinor and $D_{\mu\nu}(k)$ is the photon propagator in the medium. The field due to the medium inhomogeneity has been written in the form of a plane wave, e_ρ^s is the polarization vector of the emitted photon, and $S_{\lambda\nu\rho}(q-k,q,-k)$ is the nonlinear response of the medium for the relations between the fields and the current in the k representation:

$$J_\mu^{Nl}(k)=\int dk_1\,dk_2\delta(k-k_1-k_2)S_{\mu\nu\rho}(k,k_1,k_2)A_\nu^*(k_1)A_\rho(k_2). \quad (14)$$

The result of the computation of the vacuum averages contains, besides the indicated terms, another term that results from the contraction of the photon operators contained in the additional Lagrangian L'; to this term corresponds the diagram in Fig. 4.

It is easy to describe a new modified diagrammatic technique that takes account of the interaction of the electromagnetic field not only with the charged particles, but also with itself as a result of the nonlinearity of the medium (and not only the interaction terms quadratic in the field, but the terms of all orders—cubic, etc.—in the field). There is no need to dwell at length on the rules of this technique: they will not be needed in the present paper. The diagram shown in Fig. 4 is the first in the perturbation theory series of new diagrams of the unconnected type (as usual, the unconnected diagrams diverge, but they do not make any contribution to the scattering matrix). Let us further note that Melrose[6,7] has developed similar ideas in quantum electrodynamics.

We shall, for the sake of simplicity, describe the field A_μ^m by a scalar potential φ_q. In the case of high-frequency radiation emission the photon propagator reduces in the Coulomb gauge to the following transverse propagator:

$$D_{ij}(k)=D_{ij}^t=\frac{4\pi}{\mathbf{k}^2-\omega^2\varepsilon^t(\omega,\mathbf{k})}\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right), \quad (15)$$

$$D_{0i}=D_{i0}=0,\quad D_{00}=D^l\ll D^t, \quad (16)$$

where $\varepsilon^t(\omega,\mathbf{k})$ is the transverse permittivity of the medium. The expression for the matrix element (13) assumes the form

$$M_{if}^{(2)}=-2e\bar{u}_{p\sigma}\gamma_j u_{p'\sigma'}D_{ji}(k-q)S_{i0l}(q-k,q,-k)(4\pi)^{1/2}\varphi_q^* e_l^s. \quad (17)$$

The expression for S_{i0l} is quite unwieldly (see the Appendix). But if the emitted-wave frequency $\omega\gg\omega_{pe}$, then we can use the following simple expression,[8] which is valid in a broad range of frequencies:

$$S_{i0l}(k-q,q,-k)=\frac{e\omega_{pe}^2}{8\pi m_e}\frac{q^2}{q_0^2}\delta_{il} \quad (18)$$

(for its derivation, see the Appendix).

Let us further take account of the fact that the potential φ_q is connected with the Fourier transform of the density variation by the Poisson equation:

$$\varphi_q=-4\pi e\delta n_q/\mathbf{q}^2\varepsilon^l(q_0,q);$$

here $q=(q_0,\mathbf{q})$, $q_0\neq0$, and $\varepsilon^l(q)$ is the longitudinal permittivity. Let us assume that $q_0\ll\omega_{pe}$, and that

$$\varepsilon^l(q)=1-\omega_{pe}^2/q_0^2\approx-\omega_{pe}^2/q_0^2$$

(we neglect the spatial dispersion in ε^l).

Collecting all the results, we obtain the following expression for the probability, averaged over the initial and summed over the final polarization states, for scattering of a particle by an inhomogeneity wave during which an electromagnetic wave of frequency ω and wave vector $\mathbf{k}$ is emitted:

$$dw=\frac{e^2\omega_{pe}^4}{8\pi}\left(\frac{\delta n_q}{n_0}\right)^2\left\{(E_pE_{p'}-\mathbf{pp}'-m^2)\frac{\mathbf{q}^2}{\mathbf{p}^2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}+\frac{2(\mathbf{k}(\mathbf{k}-\mathbf{q}))^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^4}\right\}\frac{[\mathbf{pk}]^2}{(2qk)^2}\delta(E_p+q_0-E_{p'}-\omega)\frac{d\mathbf{k}}{E_pE_{p'}\omega}, \quad (19)$$

where

$$\mathbf{p}'=\mathbf{p}+\mathbf{q}-\mathbf{k},\quad qk=\mathbf{qk}-q_0\omega,\quad \omega_\mathbf{k}=(\omega_{pe}^2+\mathbf{k}^2)^{1/2},\quad \omega_\mathbf{k}\equiv\omega,$$

the vectors $\mathbf{p}$ and $\mathbf{q}$ being now considered to be antiparallel to each other (the case of arbitrary orientation is, of course, in no way more complicated; it is just more tedious). Since $\omega\gg\omega_{pe}\gg q_0$, we can assume $q_0=0$ in (19).

In the case of ultrarelativistic particles, the radiation is concentrated in a narrow angle cone along p, with aperture angle $\theta\lesssim m/E_\mathbf{p}$, and, using the results obtained in §2, we can write the δ function responsible for the energy conservation law in the form

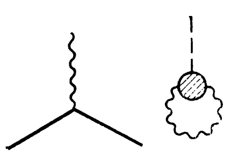

FIG. 4. Unconnected diagram, which makes no contribution to the scattering matrix.

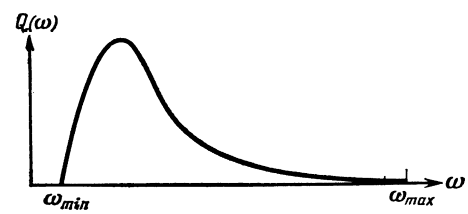

FIG. 5. Shape of the transition-scattering spectrum ($E_\mathbf{p}\gg E_{th}$).

$$\delta(E_p-E_{p'}-\omega)=\left(1-\frac{\omega}{E_p}\right)\delta\left\{\frac{\omega}{2}\left[\left(\frac{m}{E_p}\right)^2+\frac{\omega_{pe}^2}{\omega^2}+\theta^2\right]-q\left(1-\frac{\omega}{E_p}\right)\right\} \quad (20)$$

The total radiation intensity can be written in the following form:

$$Q_q=\frac{e^2\omega_{pe}^4}{8\pi(2q)^2}\left(\frac{\delta n_q}{n_0}\right)^2\int_0^\infty d\omega\int_0^\infty \pi\theta^2\,d\theta^2\,\delta\left\{\frac{\omega}{2}\left[\left(\frac{m}{E_p}\right)^2+\frac{\omega_{pe}^2}{\omega^2}+\theta^2\right]\right. \quad (21)$$

$$\left.-q\left(1-\frac{\omega}{E_p}\right)\right\}=\int_{\omega_{min}}^{\omega_{max}}\omega w(\omega)\,d\omega=\frac{e^2\omega_{pe}^4}{4(2q)^2}\left(\frac{\delta n_q}{n_0}\right)^2$$

$$\times\int_{\omega_{min}}^{\omega_{max}}\left[\frac{2q}{\omega}-\left(\frac{m}{E_p}\right)^2-\frac{\omega_{pe}^2}{\omega^2}-\frac{2q}{E_p}\right]\frac{d\omega}{\omega} \quad (22)$$

$$=\frac{e^2\omega_{pe}^4}{2c(2qc)^2}\left(\frac{\delta n_q}{n_0}\right)^2\left\{\frac{2qc}{\omega_{pe}}\left[\left(\frac{qc}{\omega_{pe}}\right)^2-\left(\frac{mc^2}{E_p}\right)^2-\frac{2qc\hbar}{E_p}\right]^{1/2}-\left(\frac{mc^2}{E_p}\right)^2\left(1+\frac{E_p}{E_{cr}}\right)\ln\frac{\omega_{max}}{\omega_{min}}\right\} \quad (23)$$

(the last formula has been written in the usual units).

The spectral distribution is depicted in Fig. 5. It can be seen from formula (22) that the spectrum undergoes appreciable modifications, caused by the recoil, only in the region of energies of the order of E_p, where the energy attains the critical value E_{cr}. In the region of initial and center frequencies (i.e., in the frequency region where the emission is most intense) the radiation intensity is only slightly lower than the intensity that follows from the formulas of the classical theory (see Ref. 1). At the highest frequencies in the region of energies $E_p \gg E_{cr}$ the spectrum extends up to $\omega\approx E_p/\hbar$, instead of $2qc(E_p/mc^2)^2$, the limit predicted by the classical theory. According to the expression (22) for the emission probability, the most intensely emitted photons are the ones with frequency $3\omega_{pe}^2/2qc$; the average frequency emitted is equal to $3\omega_{pe}^2/qc$. All this is an indication of the weakness of the influence of the quantum-mechanical recoil effects on transitions scattering in the case of media the frequencies of whose proper motions are low compared to $E_p/\hbar$ no matter how high the energy E_p of the scattered charged particle is. In the plasma approximation used in the present paper the role of the proper motions is played by the collective plasma waves.

At particle energies much higher than the threshold energy (the discussions carried out above pertain precisely to this case) the total radiated power comes, on the basis of the formula (23), to

$$Q_q=\frac{e^2\omega_{pe}^2}{8c}\left(\frac{\delta n_q}{n_0}\right)^2\left\{1-\frac{1}{2}\left(\frac{E_{th}^{cl}}{E_p}\right)^2\left(1+\frac{E_p}{E_{cr}}\right)\times\left[\ln\frac{(2E_p/E_{th}^{cl})^2}{1+E_p/E_{cr}}+1\right]\right\}, \quad (24)$$

where $E_{th}^{cl}=mc^2(\omega_{pe}/qc)$. In the classical limit, where $E_p \ll E_{cr}$, we have

$$Q_q=\frac{e^2\omega_{pe}^2}{8c}\left(\frac{\delta n_q}{n_0}\right)^2\left[1-\left(\frac{E_{th}^{cl}}{E_p}\right)^2\left(\ln\frac{2E_p}{E_{th}^{cl}}+\frac{1}{2}\right)\right]; \quad (25a)$$

in the ultraquantum limit, where $E_p \gg E_{cr}$,

$$Q_q=\frac{e^2\omega_{pe}^2}{8c}\left(\frac{\delta n_q}{n_0}\right)^2\times\left\{1-\frac{\hbar\omega_{pe}}{2mc^2}\frac{E_{th}^{cl}}{E_p}\left[\ln\left(\frac{4E_p}{E_{th}^{cl}}\frac{mc^2}{\hbar\omega_{pe}}\right)+1\right]\right\}, \quad (25b)$$

i.e., the total radiation intensity at $E_p \gg E_{th}$ depends weakly on the particle energy, and we can assume that

$$Q_q=\frac{e^2\omega_{pe}^2}{8c}\left(\frac{\delta n_q}{n_0}\right)^2, \quad (26)$$

the quantum corrections to this formula being at most a quantity of the order of $(\hbar\omega_{pe}/mc^2)^2\ln(mc^2/\hbar\omega_{pe})$, which, for solid-state densities $n_0 = 10^{23}\,\mathrm{cm}^{-3}$ and $m = m_e$, yields a relative value of 10^{-9} for the quantum corrections; the corrections will have their maximum value at a particle energy of $E_p = 2E_{cr}$.

If, on the other hand, the particle energy is not much higher than the threshold energy, then the total intensity depends on the energy in a root manner:

$$Q_q\propto(E_p-E_{th})^{3/2}=(\delta E)^{3/2}, \quad (27)$$

typical of threshold processes; this same singularity (the power 3/2) occurs in the classical limit.[1] The quantum effects do not alter the form of the singularity, as might have been expected in the case of a quantum excess of $\delta E = \hbar\omega_{pe}$ over the threshold energy.

Further, let us consider what fraction of the energy is radiated in the cutoff-frequency region $\omega_{max}-\delta\omega\leqslant\omega\leqslant\omega_{max}$ (where $\delta\omega\ll\omega_{max}$) in the spectrum. This question has a bearing on the problem of high-energy-photon detection in transition scattering in a counter with $\propto \sin \mathbf{q}\cdot\mathbf{x}$ permittivity modulation (such a modulation can be produced by a laser in a nonlinear medium). A simple computation carried out on the basis of the formula (22) yields

$$\delta Q_q=\frac{e^2}{8c}\left(\frac{\omega_{pe}}{2qc}\right)^4\left(\frac{\delta n_q}{n_0}\right)^2\left(\frac{mc^2}{E_p}\right)^6\left(1+\frac{E_p}{E_{cr}}\right)^3(\delta\omega)^2. \quad (28)$$

In the two limiting cases we obtain:

$$\delta Q_q=\frac{e^2}{8c}\left(\frac{\omega_{pe}}{2qc}\right)^4\left(\frac{\delta n_q}{n_0}\right)^2\left(\frac{mc^2}{E_p}\right)^6(\delta\omega)^2, \quad (29)$$

for $E_p \ll E_{cr}$, and

$$\delta Q_q=\frac{e^2}{8c}\left(\frac{\omega_{pe}}{2qc}\right)\left(\frac{\delta n_q}{n_0}\right)^2\left(\frac{\hbar\omega_{pe}}{E_p}\right)^3(\delta\omega)^2, \quad (30)$$

for $E_p \gg E_{cr}$.

For the emission distance for a single photon with energy lying in the interval $(\omega_{max}-\delta\omega,\omega_{max})$, the formulas (29) and (30) yield respectively

$$L_q=\frac{\hbar\omega_{max}}{\delta Q_q}c=8\frac{\hbar c}{e^2}\frac{1}{2q}\left(\frac{2qc}{\omega_{pe}}\right)^6\left(\frac{n_0}{\delta n_q}\right)^2\left(\frac{E_p}{mc^2}\right)^4\left(\frac{\omega_{pe}}{\delta\omega}\right)^2 \quad (31)$$

and

$$L_q=8\frac{\hbar c}{e^2}\frac{1}{2q}\left(\frac{2qc}{\omega_{pe}}\right)^2\left(\frac{n_0}{\delta n_q}\right)^2\left(\frac{E_{\mathbf{p}}}{mc^2}\right)^4\left(\frac{mc^2}{\hbar\omega_{pe}}\right)^4\left(\frac{\omega_{pe}}{\delta\omega}\right)^2. \quad (32)$$

For the purpose of making the crudest estimate, we can set $\delta\omega\sim\omega_{\max}$; then the characteristic distance over which one photon is emitted by a bunch of particles with mean energy $E_{\mathbf{p}}\gg E_{cr}$ (the emission distance) will, in order of magnitude, be given by the following expression

$$L_q\approx 10^{-1}\frac{c}{\omega_{pe}}\left(\frac{E_{\mathbf{p}}}{E_{cr}}\right)^2\left(\frac{mc^2}{\hbar\omega_{pe}}\right)^4\frac{1}{N}.$$

Of course, in the case of individual particles the emission distance for one photon is excessively large, so that the detection of particles with energy $E_{\mathbf{p}}\gg E_{cr}$ is not possible; in fact for an electron density of $n_0 = 10^{23}$ cm^{-3} in the medium and a modulation length $a = 5000$ Å in a "sinusoidal" counter, the distance L_q is equal to 10^{12} $(E_{\mathbf{p}}/E_{cr})^2N^{-1}$ cm (for the estimate we set $\delta n_q\sim 1/3\, n_0$ and $m = m_e$). However, the effect may turn out to be appreciable in the case of emission by a large bunch of particles, i.e., in the case of bunches with $N\sim 10^{14}$ and higher.

But on the whole the quantum effects are small, and the particle counters can be designed on the basis of the classical formulas. It is shown above that, even at energies $E_{\mathbf{p}}\gg E_{cr}$, the emitted radiation consists largely of low-energy photons with frequency ω_{pe}^2/qc; for such frequencies $\hbar\omega\ll E_{\mathbf{p}}$, this being due to the smallness of the quantity $(\hbar\omega_{pe}/mc^2)$ (ω_{pe}/qc) for ordinary media in comparison with $E_{\mathbf{p}}/mc^2$. Indeed, $\hbar\omega_{pe}/mc^2\sim 10^{-5}$, while $E_{\mathbf{p}}/mc^2\gg 1$, i.e., ω_{pe}/qc should be of the order of 10^6 and the frequencies ω_{pe}^2/qc lie in the x-ray region. The distance over which one photon is emitted is large irrespective of the frequency:

$$L_q\approx 8\frac{\hbar c}{e^2}\frac{1}{q}\left(\frac{n_0}{\delta n_q}\right)^2. \quad (33)$$

For $\delta n_q\sim\frac{1}{3}n_0$ and $q\sim\omega_{pe}/10c\sim 10^4$ cm^{-1}, we find $L_q\sim 10$ cm, i.e., the counter can be fairly simple and suitable for the detection of fast charged particles (cosmic rays).

Let us note that, in the case of counters that are sufficiently thick (so that we can neglect the radiation connected with the boundary), from 10^{-4} to 10^{-6} bremsstrahlung photons are emitted, depending on the nuclear charge of the material used in the counter, for each emitted transition photon. This estimate was obtained without allowance for multiple scattering, which should be taken into account at high particle energies; but allowance for the multiple scattering (see Ref. 9) only reduces the bremsstrahlung power. Thus, the bremsstrahlung can be entirely ignored at frequencies of the order of ω_{pe}^2/pc.

Finally, let us state again that, according to the classical formulas, the emission distance for a photon with frequency $\omega_{\max}$ (at $E_{\mathbf{p}}\ll E_{cl}$) is, for a particle energy $E_{\mathbf{p}} = 10E_{th}$, equal to about 400 m, and therefore the counter will be ineffective in the detection of the high-frequency classical photons as well.

§4. CONCLUSION

Thus, the quantum corrections to the intensity of transition scattering are negligible in ordinary media even at very high particle energies. By ordinary media we mean those media whose internal-motion energies (the energies of the quanta of the collective oscillations, for example) are small compared with the energy of the scattered extraneous particles. This is physically quite explicable: the medium exerts a strong influence on the particle motion when the transmitted agent carries energy comparable to the particle energy (the condition for a strong recoil in the course of the emission) and to the energy associated with the proper motions of the medium (condition for resonance excitation of the medium).

For the considered case of an electron plasma the relative corrections do not, in order of magnitude, exceed the quantity $(\hbar\omega_{pe}/mc^2)^2\ln(mc^2/\hbar\omega_{pe})$, or, for densities $n_0 = 10^{23}$ cm^{-3} and $m = m_e$, the value 10^{-9}.

But for dense media the quantum effects can be appreciable. The densities at which the plasma photon has an energy comparable in magnitude to the electron rest energy are roughly equal to 10^{34} cm^{-3}. A material of this density is usually not described by electrodynamics alone.

In the case of scattering of fast charged particles by heavy many-electron atoms, ions, or molecules, transition bremsstrahlung can be emitted as a result of the polarization of the electron shell of the scatterer by the field of the particle passing through the material; the role of the internal motions is played here by the reconstruction of the electron configurations. For example, the ionization potential for the K-shell electrons can be as high as tenths and higher fractions of the electron rest energy.

APPENDIX

A method that allows us to find the responses of a quantum-mechanical system of charged particles (an electron plasma in a background of stationary ions) to an electromagnetic field is presented in Ref. 4. Formally, the problem reduces to solution; by the perturbation theory method, of a chain of recurrence equations for the medium's density matrix, knowing which, we can compute the current induced by the field.

The complete expression for the quadratic response is extremely unwieldy; therefore, we present here only the result of the computations without derivation; the computational scheme is expounded in Ref. 4. The essential components of the response have the following form (the γ and β are the Pauli matrices):

$$S_{i0j}(k,q,k-q)=\frac{e^3}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}A_{ij}(\mathbf{p},k,q), \quad (A.1)$$

$$A_{ij}(\mathbf{p},k,q)=\frac{\Phi_{\mathbf{p}+\mathbf{k}/2}}{q_0+E_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}-E_{\mathbf{p}+\mathbf{k}/2}}\mathrm{Sp}\Big[\gamma_i\Lambda_+^{+}\beta\Lambda_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}^{+}\gamma_j$$

$$\times\left(\frac{\Lambda_-^{+}}{\omega-E_++E_-}-\frac{\Lambda_-^{-}}{\omega-E_+-E_-}\right)\Big]-\frac{\Phi_{\mathbf{p}+\mathbf{k}/2}}{q_0-E_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}-E_{\mathbf{p}+\mathbf{k}/2}}$$

$$\times\mathrm{Sp}\left[\gamma_i\Lambda_+^{+}\beta\Lambda_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}^{-}\gamma_j\left(\frac{\Lambda_-^{+}}{\omega-E_++E_-}-\frac{\Lambda_-^{-}}{\omega-E_+-E_-}\right)\right]$$

$$-\frac{\Phi_{\mathbf{p}-\mathbf{k}/2}}{q_0+E_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}+E_{\mathbf{p}-\mathbf{k}/2}}\mathrm{Sp}\Big[\gamma_i\left(\frac{\Lambda_+^{+}}{\omega-E_++E_-}-\frac{\Lambda_+^{-}}{\omega+E_++E_-}\right)$$

$$\times\gamma_j\Lambda_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}^{-}\beta\Lambda_-^{+}\Big]+\frac{\Phi_{\mathbf{p}-\mathbf{k}/2}}{q_0-E_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}+E_{\mathbf{p}-\mathbf{k}/2}}\mathrm{Sp}\Big[\gamma_i\Big(\frac{\Lambda_+^{+}}{\omega-E_++E_-}$$

$$-\frac{\Lambda_+^-}{\omega+E_++E_-}\Big)\gamma_j\Lambda^+_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}\beta\Lambda_-^+\Big]$$

$$-\frac{\Phi_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}}{q_0-E_{\mathbf{p}+\mathbf{k}/2}+E_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}}\mathrm{Sp}\Big[\gamma_i\Lambda_+^+\beta\Lambda^+_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}\gamma_j$$

$$\times\Big(\frac{\Lambda_-^+}{\omega-E_++E_-}-\frac{\Lambda_-^-}{\omega-E_+-E_-}\Big)\Big]+\frac{\Phi_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}}{q_0+E_{\mathbf{p}+\mathbf{k}/2}+E_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}}$$

$$\times\mathrm{Sp}\Big[\gamma_i\Lambda_+^-\beta\Lambda^+_{\mathbf{p}+\mathbf{k}/2-\mathbf{q}}\gamma_j\Big(\frac{\Lambda_-^+}{\omega+E_++E_-}-\frac{\Lambda_-^-}{\omega+E_+-E_-}\Big)\Big]$$

$$-\frac{\Phi_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}}{q_0-E_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}+E_{\mathbf{p}-\mathbf{k}/2}}\mathrm{Sp}\Big[\gamma_i\Big(\frac{\Lambda_+^+}{\omega-E_++E_-}-\frac{\Lambda_+^-}{\omega+E_++E_-}\Big)$$

$$\times\gamma_j\Lambda_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}\beta\Lambda_-^+\Big]+\frac{\Phi_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}}{q_0-E_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}-E_{\mathbf{p}-\mathbf{k}/2}}$$

$$\times\mathrm{Sp}\Big[\gamma_i\Big(\frac{\Lambda_+^+}{\omega-E_+-E_-}-\frac{\Lambda_+^-}{\omega+E_+-E_-}\Big)\gamma_j\Lambda^+_{\mathbf{p}-\mathbf{k}/2+\mathbf{q}}\beta\Lambda_-^-\Big]$$

$$+\text{symmetrization}\ (\beta\rightleftarrows\gamma_i,\ q\rightleftarrows k-q),$$

where we have, for convenience, temporarily introduced the notation:

$$E_\pm\equiv E_{\mathbf{p}\pm\mathbf{k}/2},\quad \Lambda_\pm^\pm\equiv\Lambda^\pm_{\mathbf{p}\pm\mathbf{k}/2},$$

$$\Lambda_\mathbf{p}^\pm=\frac{m_e-i\boldsymbol{\gamma}\mathbf{p}\pm\beta E_\mathbf{p}}{2E_\mathbf{p}}\qquad E_\mathbf{p}^2=m_e^2+\mathbf{p}^2,$$

and Sp denotes the trace over the matrix indices. In the classical limit $|\mathbf{k}|$, $|\mathbf{q}|\ll p_{\text{therm}}$, where p_{therm} is the mean thermal momentum of the electrons of the medium; this expression coincides with the one that is obtained from the classical kinetic equation (see, for example, Ref. 8). The latter has a rather unwieldy form, but it is significantly simpler in the limit $\omega\gg\omega_{pe}$:

$$S_{i0j}(k,q,k-q)=\frac{e\omega_{pe}^2}{8\pi m_e}\frac{\mathbf{q}^2}{q_0^2}\delta_{ij},$$

it being assumed that $q_0\gg qp_{\text{therm}}/m_e$ (the case of a cold plasma).

In the ultraquantum limit $|\mathbf{k}|,\omega\gg m_e$, the projection operators in the expression for S_{i0j} assume the form

$$\Lambda_{\mathbf{p}+\mathbf{k}}\approx\Lambda_\mathbf{k}^\pm=\frac{1}{2\omega}(m_e-i\boldsymbol{\gamma}\mathbf{k}\pm\beta\omega),\quad \Lambda_\mathbf{p}^\pm\approx\frac{1}{2m_e}(m_e-i\boldsymbol{\gamma}\mathbf{p}\pm\beta m_e)$$

and the sum of the first, second, fifth, and seventh terms in (A.1) yields in the limit $q\ll p_{\text{therm}}$ (the plasma is considered to be nonrelativistic and nondegenerate) the expression

$$\Phi_p\Big(\mathbf{q}\frac{\partial}{\partial\mathbf{p}}\Big)\Big\{\frac{1}{q_0-\mathbf{q}\mathbf{v}}\mathrm{Sp}\Big[\gamma_i\Lambda_\mathbf{p}^+\gamma_j\Big(\frac{\Lambda^+_{\mathbf{p}-\mathbf{k}}}{\omega-E_\mathbf{p}+E_{\mathbf{p}-\mathbf{k}}}-\frac{\Lambda^-_{\mathbf{p}-\mathbf{k}}}{\omega-E_\mathbf{p}-E_{\mathbf{p}-\mathbf{k}}}\Big)$$

$$-\gamma_i\Big(\frac{\Lambda^+_{\mathbf{p}+\mathbf{k}}}{\omega-E_{\mathbf{p}+\mathbf{k}}+E_\mathbf{p}}-\frac{\Lambda^-_{\mathbf{p}+\mathbf{k}}}{\omega+E_{\mathbf{p}+\mathbf{k}}+E_\mathbf{p}}\Big)\gamma_j\Lambda_\mathbf{p}^+\Big]\Big\},\quad \mathbf{v}=\frac{\mathbf{p}}{m_e}$$

(we have made the appropriate change of integration variables). In the ultraquantum case the indicated terms yield $2\delta_{ij}(\mathbf{q}^2/q_0^2)m_e^{-2}$; as to the contribution of the remaining twelve terms in (A.1), it is small on account of the fact that

$$\frac{e^2}{\hbar c}\Big(\frac{m_ec}{\hbar}\Big)^{-3}n_0\frac{T_e}{m_ec^2}\ll1,\quad T_e=\frac{p_T^2}{2m_e}.$$

Here the electron density

$$n_0=2\int\Phi_p\frac{d\mathbf{p}}{(2\pi)^3}.$$

Thus, the form of the nonlinear response is the same as the form found in the classical high-frequency limit. This is entirely similar to the situation in which the permittivity always has the plasma form in the region of high frequencies, including the ultraquantum limit ω, $k\gg m_e$ (Ref. 10). Since S_{i0j} has the same asymptotic form in the two limiting cases, we can use a single expression for S_{i0j} in the entire frequency interval provided $\omega\gg\omega_{pe}$.

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