Topologically stable configurations with singular cores in ordered media

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Singular cores of solitons—a new type of topological excitations in ordered media—are investigated by the methods of homotopic topology. A classification of such excitations is found. The singular cores of solitons and defects in superfluid ^{3}He-A and nematic liquid crystals are investigated within the framework of the proposed approach.

The techniques of homotopic topology are extremely useful in the study of nontrivial configurations of the order parameter in ordered media.[1–3] By means of the method of homotopic groups the existence of an extensive class of stable nonuniform states in liquid crystals, ferromagnets, and superfluid ^{3}He has been predicted. Among the most interesting types of such states are solitons terminating on defects. The soliton is a cylindrical (respectively, planar) region that has a strongly nonuniform distribution of the order parameter in it and terminates on a point (respectively, line) singularity.[4] Here the degeneracy space in the given cylindrical (planar) region, called the soliton core, is much richer than the degeneracy space characterizing the symmetry of the order parameter in the rest of the medium. In the present paper we give a topological analysis of a type of stable configurations of the order parameter that has not been considered previously—solitons with singular cores.

We shall consider a soliton terminating on a point singularity. The degeneracy space outside the soliton core $\tilde{V}\subset V$, where V is the degeneracy space in the soliton core. In the core there can be point singularities differing from that on which the soliton terminates. For the characteristic of one such singularity we enclose it by a sphere S^2 in such a way that the circle S^1 belonging to S^2 lies in the region outside the soliton core (Fig. 1). To classify this configuration it is then necessary to find homotopic classes of mappings $f: S^2\to V$ under which the restriction of f to S^1 has the form $f|_{S^1}: S^1\to\tilde{V}$. The set of homotopic classes of such mappings is the direct product of sets A, B, and C. Here A is the set of homotopic classes of mappings $g: S^1=\partial S^2_\uparrow=\partial S^2_\downarrow\to V$, that are homotopic to the trivial mapping in the extension of $\tilde{V}$ to V. Here $S^2_\uparrow$ is the upper hemisphere of the sphere S^2 and lies "above" the singularity (Fig. 1), $S^2_\downarrow$ is the lower hemisphere, and $\partial S^2_\uparrow$ ($\partial S^2_\downarrow$) is the boundary of the upper (lower) hemisphere; B is the set of homotopic classes of mappings $(S^2_\uparrow,\partial S^2_\uparrow)\to(V,\tilde{V})$ for a fixed homotopic class of mapping $g{:}\partial S^2_\uparrow\to\tilde{V}$; C is the set of homotopic classes of mappings $(\partial S^2_\downarrow,\partial S^2_\downarrow)\to(V,\tilde{V})$ for a fixed class of mapping $g{:}\partial S^2_\downarrow\to\tilde{V}$.

The set $A\times B$ (or $A\times C$) is the relative homotopic group[1)] $\pi_2(v,\tilde{V})$. The set C (or B) is the group

$$\mathrm{Ker}\left[\pi_2(V,\tilde{V})\xrightarrow{\varphi}\pi_1(\tilde{V})\right],$$

where the homomorphism φ belongs to the exact sequence $\to\pi_r(\tilde{V})\to\pi_r(V)\to\pi_r(V,\tilde{V})\to\pi_{r-1}(\tilde{V})\to$. Thus, the stable states of a linear soliton with an internal point defect are classified by the elements of the group

$$R_2(V,\tilde{V})=\pi_2(V,\tilde{V})\times\mathrm{Ker}[\pi_2(V,\tilde{V})\to\pi_1(\tilde{V})].\qquad(1)$$

In the case when the classes of mappings $(S^2_\uparrow,\partial S^2_\uparrow)\to(V,\tilde{V})$ and $(\partial S^2_\downarrow,\partial S^2_\downarrow\to V,\tilde{V})$ coincide, the point singularity is removable and the configuration is equivalent to a soliton with a continuous core.

An analysis similar to that given above shows that planar solitons with line singularities in the cores are classified by the elements of the group

$$R_1(V,\tilde{V})=\pi_1(V,\tilde{V})\times\mathrm{Ker}[\pi_1(V,\tilde{V})\to\pi_0(\tilde{V})].\qquad(2)$$

Examples. 1) Superfluid ^{3}He-A placed in a magnetic field. In this case, $V=[S^2\times SO(3)]/Z_2$, $\tilde{V}=S^1\times S^1$.[2–4] Then $\pi_2(V,\tilde{V})=Z\times Z\times Z$, and $\mathrm{Ker}[\pi_2(V,\tilde{V})\to\pi_1(V)]=Z$. Consequently, $R_2=Z\times Z\times Z\times Z$.

2) Superfluid ^{3}He-A placed in a small volume whose linear dimensions l are smaller than the dipole length ξ_D. In the presence of a magnetic field such that l is much greater than the magnetic length ξ_m, the degeneracy space is[2–4]

$$V=[S^2\times SO(3)]/Z_2,\qquad \tilde{V}=[S^1\times SO(3)]/Z_2.$$

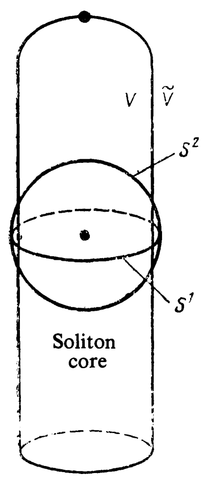

FIG. 1. Linear soliton terminating on a point singularity. In the soliton core there is a second point singularity, enclosed by the sphere $S^2=S^2_\uparrow\cup S^2_\downarrow$. The circle $S^1=\partial S^2_\uparrow=\partial S^2_\downarrow$, which belongs to S^2, is outside the core. V and $\tilde{V}$ are the degeneracy spaces in and outside the soliton core, respectively.

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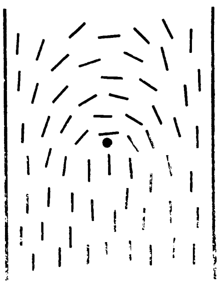

FIG. 2. A (0, 0, 1) point singularity in a nematic liquid crystal placed in a cylindrical vessel. The figure shows a section of the vessel by a plane passing through the central axis of the cylinder.

Then

$$\pi_2(V, \tilde{V})=Z\times Z, \qquad \mathrm{Ker}[\pi_2(V, \tilde{V})\to\pi_1(\tilde{V})]=Z.$$

Consequently, $R_2 = Z\times Z\times Z$.

3) Superfluid ^{3}He-A in a cylinder. Near the boundaries of the cylinder we have $\tilde{V}=S^1$ (Ref. 4). The classification of the stable point singularities inside such a system is identical to the classification of linear solitons (formula (1)). For $V=[S^2\times SO(3)]/Z_2$ we have $R_2(V, \tilde{V}) = Z\times Z\times Z$.

4) A uniaxial nematic (degeneracy space $V=RP_2$) in a cylinder. Under certain conditions, near the boundaries of the cylinder we have $\tilde{V}=S^1$ (Ref. 5). The interior point singularities are classified by the elements of the group $R_2(RP_2,S^1) = Z\times Z\times Z$. For illustrative purposes we shall consider this system in more detail. Each configuration of the director field of a nematic liquid crystal inside a cylindrical vessel is characterized by elements of the group $Z\times Z\times Z$, i.e., by a set of three integers (z_1,z_2,z_3), where $z_1\in A=Z, z_2\in B=Z, z_3\in C=Z$. If $z_2=z_3$ the point singularity is removable by a continuous deformation of the director field. In the case $z_2\neq z_3$ the singularity is topologically stable. Figure 2 depicts the configuration characterized by the set (0, 0, 1).

It is known that the dependence of the degeneracy space on λ—the scale of the nonuniformities of the order parameter—is also governed by the existence of nontrivial cores of defects.[6] The nonuniformities of the order parameter are large (λ is small) near the center of any topologically stable defect, and therefore there exists inside the defect a region whose degeneracy space differs from that in the whole medium. This region is called the defect core. It is easy to see that the approach described above is also applicable for the analysis of singularities in the cores of topologically stable defects. An example of such a singularity—the so-called "cockroach"—has been discussed in Ref. 2.

In the framework of our scheme we shall investigate superfluid ^{3}He-A placed in a magnetic field. Since the degeneracy space $\tilde{V}=S^1\times S^1$, topologically stable line defects, characterized by the elements of the group $\pi_1(\tilde{V}) = Z\times Z$, can be present in the system. The form of V—the degeneracy space in the defect core—depends in an essential way on the relative magnitudes of the character lengths ξ_m and ξ_D. If the magnetic field is so strong that $\xi_m \ll \xi_D$, then $V=[S^1\times SO(3)]/Z_2$, whereas if the magnetic field is weak, i.e., $\xi_m \gg \xi_D$, then $V=SO(3)$ (we shall not consider the intermediate case $\xi_m\sim\xi_D$). In both cases there exists only one (the trivial) homotopic class of mappings of the form $(S^2_\uparrow,\partial S^2_\uparrow)\to(V,\tilde{V})$ and $(S^2_\downarrow,\partial S^2_\downarrow)\to(V,\tilde{V})$ for a fixed class of mapping $\partial S^2_\uparrow = S^2_\downarrow\to\tilde{V}$, i.e., the mappings of the upper and lower hemisphere are always homotopically equivalent. This means that any point singularity is topologically unstable and can be removed by means of a continuous deformation of the order parameter.

We now consider strongly nonuniform superfluid ^{3}He-A placed in a strong magnetic field. Here $\xi_m \ll \lambda \lesssim \xi_D$, and the degeneracy space $\tilde{V}=[S^1\times SO(3)]/Z_2$.

In such a system topologically stable line defects, characterized by the elements of the group $\pi_1([S^1\times SO(3)]/Z_2) = Z\times Z_4$, can exist. In the core of each defect the degeneracy space is larger than $\tilde{V}$, and is $V=[S^2\times SO(3)]/Z_2$. In contrast to the previous case, topologically stable point singularities can be present in the cores. They are classified by the elements of the group

$$R_2(V, \tilde{V})=Z\times Z\times Z. \tag{3}$$

Thus, in this paper we have demonstrated the possibility of the existence of topologically stable singular cores of solitons and defects in ordered media. To classify them we have proposed an algorithm that takes into account the dependence of the degeneracy space of the medium on the nonuniformities of the order-parameter field.

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