Intermittency of passive fields in random media

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A characteristic feature of the distribution of a passive scalar or vector quantity (e.g., concentration or magnetic field) in a random medium is intermittency, i.e., the appearance of sharp peaks in which the main part of the intensity is concentrated. This is demonstrated for the example of a scalar field in random stationary and nonstationary potentials with statistically uniform initial conditions.

1. INTRODUCTION

A number of problems concerning the evolution of scalar and vector fields (distribution of concentration or temperature, magnetic field or vorticity) are treated against the background of the random motion of a fluid. A typical example is the turbulent flow of a carrier fluid.[1] Problems concerning the motion of matter relative to a medium under the action of a force equal to the gradient of a potential, a random distribution of the potential is typical.[1] The local rate of reproduction or death of a given species in biological problems is often a random function of space and time.[3] In the problems enumerated, as a rule, it is necessary also to take into account molecular diffusion or the equivalent migration under the action of a concentration gradient. In the case of a vector quantity, e.g., a magnetic field, random transport of the vector can be accompanied by change of its length, and, in particular, by exponential amplification of the field.[4]

The classical way of approaching the solution of these problems is to obtain equations for the average quantities. The Reynolds expressions for the average flux of momentum or heat in turbulent flow, and the Taylor formula for the turbulent viscosity, are well known. In magnetohydrodynamics an important role is played by the average helicity of the velocity field.[5,4] In this case the problem reduces to the study of deterministic equations similar to or modified in comparison with the evolution equations in the nonrandom medium. Usually one confines oneself to considering the first two statistical moments: the mean, and the mean square or correlation function. It is assumed that these averages characterize the field sufficiently well. It is also customary to assume that the system is ergodic, so that averaging over space or time is equivalent to averaging over realizations of the random medium.

In reality, with such a simplified approach, very important properties of the field that arise from the random distribution of the medium are left outside the picture. As Mark Twain remarked in his story *My Watch*, "a correct average is only a mild virtue in a watch . . . ". Mathematical experiments of recent years,[6] and also detailed experimental studies of turbulent flows[1,7] and astronomical observations of the structure of the universe,[2] have shown that a typical distribution of sclar and vector fields is one in which there appear characteristic structures accompanied by high peaks or spikes with large intensity and small duration or spatial extent. The intervals between the spikes are characterized by small intensity and large extent.

The general name for such a situation is "intermittency." This phenomenon has been studied for turbulence,[20,8,9] in particular, in connection with the refinement, stimulated by Landau, of the Kolmogorov-Obukhov theory (the subsequent development is reflected in Ref. 1), and also in the theory of wave propagation in random media.[10] Another well known example is the phenomenon of localization in the theory of disordered media, which has been comprehensively studied by Liftshitz and his students.[11] From the physical point of view, intermittency arises as a result of the random, fluctuating nature of the medium. For example, in a stream of a conducting liquid with a random velocity field and with an applied initial magnetic field one can find places at which the flow will most effectively intensify the magnetic field, say, by turning magnetic loops into a figure eight and doubling them.[4] Of course, the appearance of such regions is a rare, improbable event. But since almost all the energy of the generated field will be concentrated in these rate maxima we cannot neglect them; they will make the main contribution to the mean and mean square. However, the first two moments are not adequate for a full characterization of the peaks. The main indicator of intermittency is an anomalous (e.g., in comparison with the Gaussian) relation between successive statistical moments. In Fourier-analysis terms intermittency is characterized not only by a slow decrease of the amplitudes of the Fourier harmonics with increase of the wave number, but also by a definite phase relationship between the harmonics. A sum of high harmonics with random phases would give something similar to a Weierstrass function, or, in modern terminology, a fractal curve, instead of individual high peaks.[12]

As we shall show, in a medium that is spatially uniform in the statistical sense, intermittency is a very strongly pronounced phenomenon: In the presence of instability the ratio of the mean squares of the field that is concentrated in the peaks and the field that is not concentrated in the peaks grows exponentially.

In the case of spatially bounded media it turns out that the characteristic spacing between high peaks begins, after a certain time, to exceed the size of the system. After this it is found that spatial averages cease to coincide with ensemble

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averages, with the former growing more slowly than the latter: Intermittency is expressed less sharply in a bounded medium.

It is also of great interest to take the non-one-dimensionality into account and to analyze the structure of the field distribution. Asymptotically in time, one typically observes the formation of high isolated spots—field peaks separated by extensive regions of reduced intensity. As an intermediate asymptotic form, however, it is possible (and, in many cases, typical) to find the formation of a cellular or network structure—thin channels of raised intensity (the rich phase), separating isolated islands of the poor phase.

In the present paper we shall study the phenomenon of intermittency for the example of a scalar impurity in a steady state and in a nonstationary (transient) regime. In the absence of diffusion this example admits an elementary interpretation. To take diffusion into account we shall make use of the technique of integration over random Wiener trajectories (see, e.g., Refs. 13 and 14).

2. THERMODYNAMIC EQUILIBRIUM

We shall consider a random medium, characterized by a random potential $\varphi(\mathbf{x},\omega)$. The parameter ω labels the realizations of the potential, so that for a fixed ω the potential is an ordinary deterministic function. We suppose, for definiteness, that φ has a Gaussian distribution with zero mean and variance σ^2. Such a potential can be viewed as a sum of phase-incoherent Fourier harmonics. The equilibrium concentration of the substance in such a medium,

$$n=n_0\exp(-\varphi/kT) \tag{1}$$

is not Gaussian, by virtue of its nonlinear dependence on φ. This is obvious for $\sigma/kT\gtrsim 1$. But it seems natural that for $\sigma/kT\ll 1$ the dependence is linear:

$$n\approx n_0(1-\varphi/kT),$$
$$\langle n\rangle\approx n_0.$$

In fact, we shall find the value of the potential that corresponds to the most probable concentration of the substance. Since

$$P_\varphi(n)=\exp(-\varphi/kT-\varphi^2/2\sigma^2),$$

the maximum of the exponential corresponds to $\varphi_m/\sigma=-\sigma/kT$, where $P_{\max}\sim\exp(\sigma^2/2k^2T^2)$. However, in an exact treatment of the successive moments

$$\langle n\rangle=n_0\exp(\sigma^2/2k^2T^2),$$
$$\langle n^2\rangle^{1/2}=n_0\exp(\sigma^2/k^2T^2),\ldots,\quad \langle n^p\rangle^{1/p}=n_0\exp(p\sigma^2/2k^2T^2) \tag{2}$$

it is found that they are larger the larger their label p: $\langle n^2\rangle\gg\langle n\rangle^2, \langle n^4\rangle\gg\langle n^2\rangle^2$, i.e., the successive averages are determined not by the most probable value σ/kT, but by $p^{1/2}\sigma/kT$. Therefore, even for small σ/kT, generally speaking, we cannot use the linear approximation to (1) (if we are interested in sufficiently high moments). The behavior of the moments (2) is explicable only if there are rare, high peaks in the concentration distribution. In principle, there are also high peaks in the Gaussian potential itself, for which $\langle\varphi^p\rangle^{1/p}$ grows like $p^{1/2}$ at large p. But this growth is weak in comparison with the exponential growth in (2).[1)] In this way, weak intermittency in φ, which in itself could be neglected, turns out to be sharply expressed in a concentration distribution that depends nonlinearly on the potential.

3. A STATIONARY MEDIUM

The equilibrium concentration (1) arises as the stationary solution of the equation describing the transport of an impurity in a medium with constant density and diffusion coefficient D:

$$\partial n/\partial t=-\operatorname{div}\mathbf{j},\quad \mathbf{j}=-D\nabla n+n\mathbf{V},$$

where in the mobility approximation we have $V=(D/kT)\nabla\varphi$. The probability of the appearance of high peaks is determined by the quantities $\sigma/kT\equiv\mathrm{Re}$, which has a direct similarity to the Reynolds number. In fact, the root-mean-square velocity $v=(D/kT)[(\overline{\nabla\varphi})^2]^{1/2}=D\sigma/kTl$, and therefore, $\mathrm{Re}=vl/D$.

We shall investigate how intermittency arises from an initial smooth distribution $n_0(x)$, say, $n_0=$ const. In the initial stage we can neglect the term $\nabla n\cdot\nabla\varphi$ in comparison with $n\nabla\varphi$. In the equation obtained after this, in contrast to the exact equation, the total number of particles is no longer conserved, and essentially we are dealing with an increase in the number of particles at the peaks on account of the (disregarded) decrease of the concentration in the space between the peaks. The growth of the peaks ceases upon increase of the concentration gradients, when the concentration approaches the stationary solution (1). The approach to equilibrium is slow, and by the time t equilibrium is established in cells with spacing $R\sim(Dt)^{1/2}$ and depth of the order of $kT(3\ln R)^{1/2}$. Thus, with time, equilibrium is established in ever deeper cells (on account of particles escaping from the less deep cells).

In a number of problems in biology or in the kinetics of chemical and nuclear reactions we are concerned directly with random proliferation and diffusion (see, e.g, Refs. 15 and 16). In these case the initial growth is suppressed by nonlinear effects. Here, also, one finds the onset of intermittency, which grows with time and tends, evidently, to a stationary state of the type described above. We shall consider the onset of such intermittency in the linear approximation, using the example of the simple equation

$$\partial\psi/\partial t=\varkappa\Delta\psi+U(\mathbf{x},\omega)\psi,$$
$$\psi(\mathbf{x},0)=\psi_0(\mathbf{x}). \tag{3}$$

In the general case, the solution of Eq. (3) can be written in the form of an integral over random trajectories[13,4] ξ_t describing diffusion with diffusion coefficient $\varkappa$:

$$\psi(\mathbf{x},t)=M_\mathbf{x}\exp\left(\int_0^t U(\xi_s)\,ds\right)\psi_0(\xi_t), \tag{4}$$

where $M_\mathbf{x}$ denotes averaging over all trajectories arriving at the point $\mathbf{x}$ at time t. The solution (4) is similar to (1), except for the fact that the exponent is nonstationary.

Let the potential U have a Gaussian distribution with a

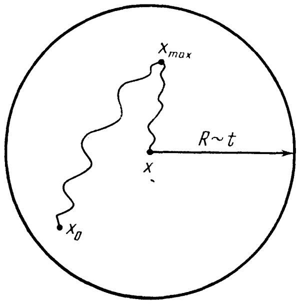

FIG. 1. Schematic illustration of an "optimal" trajectory, starting from a point $\mathbf{x}_0$ at which $\psi_0 > 0$ and arriving, at time t, at the point $\mathbf{x}$. At the point $\mathbf{x}_{\max}$ of the maximum of the potential the trajectory also spends a time of the order of t.

certain characteristic decay length of the spatial correlation, zero mean, and variance σ^2. For simplification, it is convenient to divide space into cells with dimensions equal to the correlation length of the potential and to replace the diffusion by a discrete random walk of the particle, which, in time Δt, crosses with probability $\varkappa\Delta t$ into one of the six (in the three-dimensional case) neighboring cells, and remains with probability $1-6\varkappa\Delta t$ in the initial cell. In other words, we replace the continuous integral (4) by a close, finite-dimensional integral of very high multiplicity.

We shall show that for any bounded non-negative function $\psi_0(x)$ and any $\varkappa > 0$, with probability unity the solution grows asymptotically like $\exp[t(6\sigma^2 \ln t)^{1/2}]$; more precisely, there exists the limit

$$\gamma = \lim_{t\to\infty} \frac{\ln \psi(x,t)}{t(\ln t)^{1/2}} = (6\sigma^2)^{1/2}. \tag{5}$$

This time dependence is explained by the fact that the decisive contribution to the solution (4) is made by trajectories that fall rapidly into the high maximum of the potential. The magnitude of the maximum of the potential in a region of size $R\to\infty$ is of the order of [2]

$$\max U \sim (2\sigma^2 \ln V)^{1/2} \sim (6\sigma^2 \ln R)^{1/2},$$

as is easily established directly from the form of the Gaussian distribution. For a typical trajectory $R \equiv t^{1/2}$, and therefore it would seem that $\psi \propto \exp[t(3\sigma^2 \ln t)^{1/2}]$. In reality, a large contribution is given not by a typical trajectory but by a less probable, so-called (by I. M. Liftshitz) optimal trajectory, which in a time t moves away over a large distance $\sim R \propto t$ (see Figure 1), compensating its small statistical weight by the large factor max U. This weight can be determined as follows. The number of jumps of a particle in time t is a Poisson process with parameter $6\varkappa t$, and therefore

$$P\{\max|\xi_t - \mathbf{x}| > at\}$$
$$= \sum_{k>at} \frac{(6\varkappa t)^k}{k!} \exp(-6\varkappa t) \approx (6e\varkappa/a)^{at} \exp(-6\varkappa t).$$

Thus, the statistical weight of a trajectory deviating by a distance at is of the order of $e^{-\delta t}$, where $\delta = 6\varkappa + a\ln(a/6e\varkappa)$. We shall estimate the solution (4) for $t\to\infty$:

$$\psi(\mathbf{x},t) \approx \psi_0 \exp(-\delta t)\exp(\max Ut)$$
$$\propto \exp[t(6\sigma^2 \ln t)^{1/2}],$$

which proves (5) (compare with the contribution of a typical trajectory!).

We call attention to the fact that, although the rate of growth does not depend on $\varkappa$, in the case $\varkappa = 0$ the solution will grow only exponentially.[15] This is connected with the fact that the time at which the emergence to the superexponential solution occurs depends in an essential way on $\varkappa$. Thus, depending on the order of the limits ($\varkappa\to 0$, $t\to\infty$, or $t\to\infty$, $\varkappa\to 0$), different results are obtained.

We now find the rate of growth of successive statistical moments of ψ. For this we make use of the fact that, in discrete space,

$$\int_0^t U(\xi_s)\,ds = \sum U(\mathbf{x})\,\tau(\mathbf{x},t),$$

where $\tau(\mathbf{x},t)$ is the time spent at the point $\mathbf{x}$ before the time t, and also use the well known formula for averaging an exponential: $\langle \exp\varphi\rangle = \exp(\langle\varphi^2\rangle/2)$, which we have already used in the preceding Section. Then from (4) we obtain

$$\langle\psi(\mathbf{x},t)\rangle = M_{\mathbf{x}} \exp\left(\frac{\sigma^2}{2}\sum_{\mathbf{y}} \tau^2(t,\mathbf{y})\right)\langle\psi_0\rangle.$$

Obviously, $\tau \leqslant t$, and therefore the maximum value $\sigma^2 t^2/2$ of the exponent is attained on that trajectory which sits in the same cell (at point $\mathbf{x}$) for the whole time t. The statistical weight of this trajectory is equal to $\exp(-6\varkappa t)$. Therefore,

$$\exp(\sigma^2 t^2/2) \geqslant \langle\psi\rangle \geqslant \exp(-6\varkappa t + \sigma^2 t^2/2),$$

whence we obtain

$$\lim_{t\to\infty} \frac{\ln\langle\psi(\mathbf{x},t)\rangle}{t^2} = \frac{\sigma^2}{2}. \tag{6}$$

The pth moment is estimated analogously:

$$\langle\psi^p\rangle = M_{\mathbf{x},\ldots,\mathbf{x}} \exp\left\{\frac{\sigma^2}{2}\sum_{\mathbf{y}} [\tau^{(1)}(t,\mathbf{y}) + \ldots + \tau^{(p)}(t,\mathbf{y})]^2\right\},$$

where $\tau^{(i)}$ is the time spent at the point $\mathbf{y}$ by the ith trajectory before the time t. Consequently,

$$\gamma_p \equiv \lim_{t\to\infty} \frac{\ln\langle\psi^p\rangle}{t^2} = \frac{p^2\sigma^2}{2}. \tag{7}$$

Thus, the statistical moments $\langle\psi^p\rangle^{1/p}$ grow much more rapidly, like $\exp(p\sigma^2 t/2)$, than the function ψ itself, and in such a way that the rate of growth increases with the order of the moment. This progressive increase of the moments is explained by the presence of sharp peaks in the solution $\psi(\mathbf{x},t)$, i.e., by the presence of intermittency in the distribution of ψ.

The appearance of superexponential growth of the moments can be explained by the fact that we have considered, in an infinite space, a potential that can take arbitrarily large values. For a bounded potential the solution and its moments will grow exponentially. It can be shown that

$$\gamma \equiv \lim_{t\to\infty} \frac{\ln\psi}{t} = \sup|U(x)| \equiv \bar{U},$$

$$\gamma_p \equiv \lim_{t\to\infty} \frac{\ln\langle\psi^p\rangle}{t} = p\bar{U}. \tag{8}$$

However, when the next term of the asymptotic form is taken into account we have

$$\langle\psi^p\rangle=\exp\left\{p\overline{U}t-\beta(p)\frac{t}{\ln t}\right\},$$

where $\beta(p)/p$ falls with increase of p. Thus, for a bounded potential the intermittency is expressed much more weakly.

We note that for a Gaussian potential in a bounded volume we have $\psi\propto\exp(\max Ut)$, where $\max U$ is a random quantity. As before, the moments grow like $\exp(p^2\sigma^2t^2/2)$.

4. A SCALAR FIELD IN A RANDOM NONSTATIONARY POTENTIAL

We shall consider now the case of a nonstationary potential $U(t,\mathbf{x},\omega)$. We can foresee that in this case the solution will grow more slowly, since the deep cells that ensure the maximum growth will exist only for a finite time.

We shall examine the simple case of an extremely nonstationary potential, when the potential is white noise in time, with independent values in different spatial cells of a certain characteristic size:

$$U(t,\ \mathbf{x},\ \omega)=dw_t(\mathbf{x})/dt. \tag{9}$$

where w_t is a Wiener (Brownian) process, $\langle w_t\rangle=0$, $\langle w_t^2\rangle=\sigma t$, and $\langle\dots\rangle$ denotes the average at the given point $\mathbf{x}$. This problem was considered in Ref. 17 in relation to the Burgers equation, which in its pure form reduces to (3). When generalized potentials are used it is necessary to specify more precisely the meaning of $\partial/\partial t$ in (3), i.e., to specify the order of the limits: First, $dt\to0$ and then the correlation time $\tau_0\to0$, or vice versa. We shall consider the second case, i.e., we shall understand (3) as the limit of a difference equation (for the first case, see Ref. 17).

Equation (3) is solved explicitly for $\varkappa=0$ in the manner of Ito[13]:

$$\psi(t,\ \mathbf{x},\ \omega)=\psi_0(\mathbf{x})\ \exp\ (w_t-\sigma t/2). \tag{10}$$

The appearance of the additional factor[3)] $-\sigma t/2$ in the exponential is connected with the fact that in differentiation of a Wiener process it is necessary to consider the square of its differential and to use the equality $(dw_t)^2=\sigma dt$ (Ref. 13). Therefore,

$$\begin{aligned}d\exp\ (w_t-\sigma t/2)&=\exp\ (w_t-\sigma t/2)\ [(dw_t-\sigma dt/2)\\&+(dw_t-\sigma dt/2)^2]=\exp\ (w_t-\sigma t/2)\ [dw_t-\sigma dt/2+dw_t^2/2]\\&=\exp(w_t-\sigma t/2)\,dw_t.\end{aligned}$$

Since a typical value of the process w_t for large t is of the order of $t^{1/2}$, the solution (10) decays with probability unity like $\exp(-\sigma t/2)$ at any spatial point. However, with a small probability a Wiener process takes values exceeding $t^{1/2}$ by an arbitrary amount. Therefore, on the background of the general solution it cannot be doubted that there are rare high peaks in the solution ψ.

To verify this, we shall calculate successive statistical moments, assuming $\psi_0(\mathbf{x})$ to be a random quantity that is deterministic or distributed independently of U. Applying the formula

$$\langle\exp pw_t\rangle=\exp\ (p^2\sigma t/2),$$

we find

$$\langle\psi^p\rangle=\langle\psi_0^{\ p}\rangle\langle\exp(pw_t-p\sigma t/2)\rangle=\langle\psi_0^{\ p}\rangle\exp[p(p-1)t\sigma/2].$$

Thus, the exponential-growth rates γ_p/p increase like $\sigma(p-1)/2$ with increase of the order of the moment. The mean value, corresponding to $p=1$, coincides with $\langle\psi_0\rangle$, the mean square grows like $\exp(\sigma t/2)$, the fourth moment grows like $\exp(6\sigma t)$, etc. We recall that a typical realization $\psi(t)$ decays like $\exp(-\sigma t/2)$ with increase of t. Therefore, the increase of the moments is explained by the nontrivial contribution of rare events. This means that amongst the complete set of realizations $\psi(t,\mathbf{x},\omega)$, at any time t, functions growing with time at certain spatial points $\mathbf{x}$ can be found. As is clear from the solution (10) and the properties of a Wiener process, these functions are functions of intermittent growth. This can also be elucidated by calculating the averages not over the statistical ensemble but over space (sample averages). It is usually assumed that these averagings give identical results. In the present case the result of the spatial averaging depends in an essential way on the form of the initial function.

It is easily verified that the sampling moments coincide with the statistical moments when the initial function $\psi_0(x)$ is distributed statistically uniformly in an unbounded volume. Now let the initial distribution be localized in a finite volume V. Then, with probability unity,

$$\mu_p=\frac{1}{V}\int\psi_0^{\ p}(x)\exp\left(pw_t(x)-\frac{p\sigma t}{2}\right)d^3x\underset{t\to\infty}{\sim}\exp\left(-\frac{p\sigma t}{2}\right).$$

We draw attention to the fact that the sampling moments μ_p, unlike the statistical moments, are random quantities. However, their rate of growth is a nonrandom quantity and is equal to $\tilde{\gamma}_p/p=-1/2$. In the Stratonovich approach or for a potential with a small but finite restoration time one obtains $\tilde{\gamma}_p/p=0$. Thus, this difference affects the result. We note that the correction to $\tilde{\gamma}t$ is of the order of $\eta t^{1/2}$, where η is a random quantity.

This difference between the sampling moments and the statistical moments should be understood as follows. For a localized initial distribution or in a bounded body, in the overwhelming majority of realizations the rare intermittency peaks exist only for a finite time. After a certain time the characteristic spacing between these peaks exceeds the size of the body (or the characteristic size of the region occupied by the field), and only a certain, exponentially decaying probability of reappearance of a peak remains. Nevertheless, this probability is sufficient to change qualitatively the statistical average. It is not difficult to understand that the time for the peaks to die out is of the order of (we do not write out the dependence on p)

$$t_d=\tau\ln\ (V/r_0^{\ 3}),$$

where r_0 is the characteristic correlation length of the potential and τ is the characteristic growth time of the given moment. Thus, for $t>t_d$ the given volume turns out to be too small for ergodicity to be realized in respect of the given order of the moment.

5. LIMIT OF SMALL DIFFUSION

We shall show that the intermittency effect is preserved even in the presence of diffusion, at least in the limit of small $\varkappa$. To this end, it is sufficient to prove the continuity of the rates of growth of the moments as $\varkappa \to 0$.

This result is by no means obvious. We shall think of the diffusion of ψ as an average of an aggregate of transports of this scalar along distinct random Brownian trajectories. Then for $\varkappa \neq 0$ there arrives at each point in space a beam consisting of an infinite number of trajectories, each of which "brings" a solution of the type (10). Although the statistical contribution of a typical trajectory is small, viz., of the order of $\exp[(\sigma t)^{1/2} - \sigma t/2]$, one can find an exponentially small number of optimal trajectories that carry a contribution of the order of $\exp At$ with $A > -\frac{1}{2}\sigma$. The problem is to show that in the limit $\varkappa \to 0$ the contribution of these optimal trajectories will not change the results of the preceding Section.

We note that in the above problem with a stationary potential diffusion slows the growth of the solution. In a nonstationary problem, as will be shown below, the inclusion of diffusion has the opposite effect. The point is that, because of the nonstationary character, a well at a given point always disappears. But thanks to the diffusion there are always trajectories that are situated for the maximum time in regions of growth of the solution. Of course, there is also a multiplicity of paths passing mainly through regions where ψ is decreasing. However, because of the exponential dependence of the solution on the potential the contribution of the constructive paths turns out to be the more important.

The solution of the problem (3) for $\varkappa \neq 0$ can be represented in the form of a Wiener integral over trajectories that arrive, at time t, at a given point $\mathbf{x}$. In the discrete space-time described, this solution naturally generalizes (10):

$$\psi(\mathbf{x}, n) = M_{\mathbf{x}}^{(\varkappa)} \psi_0(\xi_n) \exp\Big(\sum_{s=1}^{n} \Delta w^{(\xi_s)} - n/2\Big),$$

where n is the discrete time, $\xi_s(\varkappa)$ is the trajectory of a random walk through the lattice, $M_{\mathbf{x}}$ is an average over the trajectories ξ_s but not over the realizations of the potential U, and $\Delta_s w^{(\mathbf{x})} = w_s^{(\mathbf{x})} - w_{s-1}^{(\mathbf{x})}$ are the increments of the Wiener processes at the times s at the given point. These increments are indepenent Gaussian random quantities with zero mean and unit variance.

Let the initial distribution $\psi_0(\mathbf{x})$ be localized at zero:

$$\psi_0(\mathbf{x}) = 1, \quad \mathbf{x} = 0; \quad \psi_0(\mathbf{x}) \neq 0; \quad \mathbf{x} \neq 0.$$

We shall find bounds on the pth sampling moment

$$\mu_p(n) = \sum_{\mathbf{x}} \psi^p(n, \mathbf{x}).$$

For the lower bound we can consider the point $\mathbf{x} = 0$ and the trajectory which, having begun its motion at this point, does not leave it before the time n. The statistical weight of this trajectory is equal to $(1 - 6\varkappa)^n$. Therefore,

$$\mu_p(n) \geqslant (1 - 6\varkappa)^n \exp(p w_n(0) - pn/2).$$

Hence, with probability unity,

$$\gamma_p = \lim_{n\to\infty} \frac{\ln \mu_p(n)}{n} \geqslant \ln(1 - 6\varkappa) + \frac{p w_n}{n} - \frac{p}{2} \approx -6\varkappa - \frac{p}{2}, \quad n \to \infty.$$

This bound is sufficient to establish the continuity of the rate of growth. However, it would be possible to conclude from it that diffusion only decreases the rate of growth. In fact, allowance for a nonzero $\varkappa$ increases the rate of growth, since, owing to the diffusion, the particle is able to visit sites with a very large value of the potential. For the proof we divide the time segments n into intervals with a length k that depends on $\varkappa$. We shall choose this dependence later. We consider a trajectory that starts from the coordinate origin and passes into that neighboring cell which has the maximum value of the quantity $\sum_1^k U$. This random quantity has a Gaussian distribution with variance $k\sigma^2$ and zero mean. Therefore, the average value of $\sum_1^k U$ is equal to $C(k\sigma^2)^{1/2}$, where C is a constant that depends on the number of neighboring cells. After the time k the trajectory passes into a new cell—the cell with the maximum $\sum_k^{2k} U$, and so on. The statistical weight of this trajectory is equal to

$$\varkappa^{n/k}(1 - 6\varkappa)^{n - n/k} \approx \exp\left[\frac{n}{k} \ln \varkappa - 6\varkappa\left(n - \frac{n}{k}\right)\right],$$

and its contribution gives the following bound on the solution:

$$\mu_p(n) > \exp\left[\sum_1^n U + \frac{n}{\varkappa} \ln \varkappa - 6\varkappa\left(n - \frac{n}{k}\right) - \frac{pn}{2}\right].$$

Since, by the law of large numbers,

$$\sum_1^n U = \sum_1^k + \sum_k^{2k} + \ldots \approx \frac{n}{k} (k\sigma^2)^{1/2} C,$$

the maximum of the exponent as a function of k is reached at $k_{\max} = (9/4\, \sigma C) |\ln \varkappa|$. Consequently,

$$\tilde{\gamma}_n = \lim_{n\to\infty} \frac{\ln \psi_n}{n} > \frac{\text{const}\,(>0)}{|\ln(1/\varkappa)|} - \frac{p}{2}.$$

To obtain an upper bound we draw attention to the fact that a typical value of the quantity

$$\sum_{s=0}^{n} \Delta_s w(\xi_s)$$

is of order $n^{1/2}$. The danger arises from those rare trajectories that are moving, so far as this is possible, through local maxima of the potential and making a higher (in comparison with $n^{1/2}$) contribution to this quantity. The probability that for one fixed trajectory the sum of the increments exceeds n by a factor of δ is exponentially small:

$$P\left\{\left|\sum^n \Delta w(\xi_s)\right| > \delta n\right\} = P\left\{\frac{1}{n^{1/2}}\left|\sum^n \Delta_s w\right| > \delta n^{1/2}\right\}$$

$$= \frac{2}{(2\pi)^{1/2}} \int_{\delta n^{1/2}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \leqslant \exp\left(-\frac{\delta^2 n}{2}\right).$$

However, the number of trajectories is exponentially large—of the order of 6^n. Obviously, $6^n \exp(-\delta^2 n/2)$ tends to zero as $n \to \infty$ only for $\delta > (2 \ln 6)^{1/2}$. This crude bound is not satisfactory, since it gives only an upper limit $\tilde{\gamma}_p \leqslant [(2 \ln 6)^{1/2} - \frac{1}{2}]$ that does not depend on $\varkappa$ and so does not solve the problem of the limit $\varkappa \to 0$. This estimate is crude—we have not taken into account the fact that the statistical weights of the different trajectories are different. We shall allow for this circumstance.

In the time n, a typical trajectory ξ_n executes, not n but only $6\varkappa n \equiv \langle \nu_n \rangle$ transitions to other cells. The probability that m transitions occur is equal to

$$P\{\nu_n=m\}=C_n^m(6\varkappa)^n(1-6\varkappa)^{n-m}$$

$$\approx \frac{n^n}{m^m(n-m)^{n-m}}(6\varkappa)^m(1-6\varkappa)^{n-m}$$

$$\approx \exp\left\{n\left[r\ln\frac{6\varkappa}{r}+q\ln\frac{1-6\varkappa}{q}\right]\right\},$$

where

$$r=m/n, \quad q=(n-m)/n, \quad r+q=1.$$

The entropy $S(\varkappa,r)$ in the square brackets in the exponential is always negative, except at the point $r=6\varkappa$, where it vanishes. For a given m the number of trajectories arriving at the point $x=0$ at the time n is equal to

$$C_n^m 6^m \approx \frac{n^n 6^m}{m^n(n-m)^{n-m}} \approx \exp\{n[-r\ln r-q\ln q]+p\ln 6\}.$$

We recall that the sampling moment is a random quantity. The probability that μ_p takes anomalously large values is estimated as follows:

$$P\{\mu_p(n)\}>\sum_m \exp\{nS(\varkappa,r)+k\delta(r)n-pn/2\}$$

$$\leqslant \sum_m \exp\{n[-r\ln r-q\ln q+r\ln 6-\delta^2(r)/2]\}.$$

We choose $\delta(r)$ so that the maximum of the function in the square brackets in the right-hand side of the inequality is negative; i.e., we let

$$\delta(r)>[2r\ln 6-2r\ln r-2q\ln q]^{1/2}.$$

Therefore, after a certain time n the inequality

$$\mu_p(n)<\exp\{n\max[S(\varkappa,r)+\delta n-pn/2]\}$$

will be fulfilled. Consequently, with probability unity,

$$\tilde{\gamma}_p(\varkappa)\leqslant\max[S(\varkappa,r)+r\delta(r)-p/2]$$

$$\approx -p/2+p^{1/2}[\zeta(\varkappa)/2\ln(1/\varkappa)]^{1/2},$$

where $\zeta(\varkappa)$ is a quantity of the order of $\ln\ln(1/\varkappa)$. Thus,

$$\left|\ln\frac{1}{\varkappa}\right|^{-1}<\tilde{\gamma}_p+\frac{p}{2}\leqslant\left|\ln\frac{1}{\varkappa}\right|^{-1/2},$$

i.e., the continuity of the growth rates is proved. The bounds show that for $\varkappa\to 0$ the graph of the function $\tilde{\gamma}_p(\varkappa)$ has a sharp tangency to the vertical axis. The lower bound shows that small diffusion increases the rate of growth.

The continuity of the rates of growth of statistical moments that are not random quantities has been demonstrated in Ref. 4. In this case,

$$\gamma_p(\varkappa)\to p(p-1)/2, \quad \varkappa\to 0.$$

Thus, $\gamma_p(\varkappa)>0$ and $\tilde{\gamma}_p(\varkappa)<0$ for $\varkappa\to 0$.

6. COMMENTS ON THE VECTOR AND MULTIDIMENSIONAL CASES

The derived properties of the growth rates for statistical and sampling moments are also typical, in our opinion, of more complicated problems, e.g., for the problem of hydromagnetic dynamos in random flows.[4] The main difference in this case is that for sufficiently small diffusion coefficients even a typical realization of the field can grow exponentially. The intermittency in this case takes the following form. Against the background of the exponential growth of a typical realization there are peaks which ensure advanced growth of the moments. By a fixed time, the statistical and sampling averages of order p smaller than a certain critical value p_0 coincide, while those of large orders p are different. This means that the peaks that give rise to the advanced growth of the statistical moments in comparison with that of the sampling moments have already become so rare that, as a rule, not one of them appears in the volume occupied by the field. The quantity p_0 decreases exponentially with time (here we are considering also the noninteger statistical and sampling moments), so that the contrast of the field outside the peaks in a typical realization grows nonexponentially or even stabilizes. A mathematical justification of the picture we have drawn requires calculations more complicated and cumbersome than those carried out for the simple example studied in this paper. We note that in the vector case the nonuniformities of the solution have a structure in the form of plaits of magnetic lines or layers.

The simple model of the type (3) considered above does not convey an important feature typical of the evolution of intermittency in the multidimensional case. In the model (3), the peaks are concentrated from the outset near individual points—local maxima of the potential. More typical is an initial formation of features with cellular or network structures on the caustics of the Lagrange trajectories.[19] In the course of time, however, separationj of the individual sites in these structures will certainly occur. This was first exhibited in the numerical experiments of Shandarin for the example of gravitational problems; see Ref. 2. This separation is due to the fact that the characteristic size of the structures grows exponentially in time, and even the low-probability trajectories, which we should take into account, move away in a time t to a distance that has only a power-law dependence on the time and therefore are unable to give rise asymptotically to the long-range correlations necessary for the formation of a cellular or network structure.

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[1)] One can cite examples of distributions with peaks that are even less pronounced than a Gaussian peak, e.g., the potential distribution $\varphi^2/(1+\varphi^2)$, where φ is a Gaussian quantity. In this sense, for the characteristic of weak intermittency it is not sufficient to confine oneself to calculating or measuring the excess $\langle n^4\rangle/\langle n^2\rangle^2$. The weak intermittency may appear mainly in the higher moments.

[2)] In the region there are $N\propto R^3$ correlation cells. The probability that a certain U_0 is reached in one cell is of the order of $P\sim\exp(-U_0^2/2\sigma^2)$. The condition $PN\sim 1$ then gives the estimate of max U.

[3)] This factor vanishes if, instead of a potential that is δ-correlated in time, we consider a potential that is restored after a correlation time τ_0. In this case all the growth rates calculated below increase correspondingly. It turns out that, in this way, the limits $t \to \infty$, $\tau_0 \to 0$, and $\tau_0 \to 0$, $t \to \infty$ lead to different results. We note that the solution without the factor $\exp(-\sigma t/2)$ is also obtained from (9) if one uses the approach of Stratonovich.[18]

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