Collapse of nonlinear Langmuir waves

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The dispersion of intense Langmuir waves is determined by an intrinsic (electron) nonlinearity. Generally, the wave energy density needed for the electron nonlinearity to appear is reached, prior to the development of dissipative processes. Until now, the influence of the electron nonlinearity on the dynamics of the collapse and on the spectrum of strong Langmuir turbulence, which may be quite significant, has not been investigated extensively, because of the difficulty in describing nonlinear Langmuir waves. The positive definiteness of the Hamiltonian of an electron nonlinearity is proven in the present work. The growth rate of the modulational instability of a localized condensate of nonlinear Langmuir waves in a caviton is calculated and a universal collapse law is found.

1. INTRODUCTION

The first work on the collapse of Langmuir waves[1] and most of the subsequent studies of this subject describe an electrostatic potential φ by a Schrödinger-type linear equation, with the perturbed ion density n playing the role of the potential. The nonlinearity of the basic equations of Ref. 1 is revealed only via the dependence of n on the Langmuir wave pressure. Such a description is justified if the energy density W of the waves is much less than the thermal energy density n_0T of the electrons. When $W > n_0T$, the oscillation velocity of the electrons exceeds their thermal velocity and the Langmuir waves become intrinsically nonlinear. A nonlinear equation for the electrostatic potential was derived in Ref. 2. This equation turned out to be quite complicated, and, as a result, the influence of the nonlinearity of Langmuir waves on their collapse could not be adequately analyzed. This question can be bypassed by making certain assumptions. For example, in the "sonic" model of collapse,[3,4] the flattening of the caviton increases in such a way that the threshold for the trapped waves instability $W_* \sim n_0T(r_D^2/a^2)$ (a and r_D being the smallest size of the caviton and Debye radius, respectively) remains always of order W. Then the Landau damping becomes significant before W reaches the value n_0T and thus the electron nonlinearity does not appear. Numerical computations do not confirm the sonic model of the collapse, but rather provide evidence, supporting the assumption that all the dimensions of the caviton have identical time dependence (the "supersonic" regime).[5–7] In the supersonic regime of the collapse, the trapped-wave energy density grows as a^{-3} and, formally, reaches the value n_0T at $a \gg r_D$. This indicates that electron nonlinearity may arise. There exist, however, several factors limiting its development. Firstly, Landau damping becomes significant even for a Maxwellian distribution, when the size of the caviton is a few times larger than the Debye radius. Secondly, the electron nonlinearity vanishes for one-dimensional oscillations (see, for example, Ref. 8), and the caviton is flattened by a factor of 3–5, according to numerical calculations, and therefore the electron nonlinearity is suppressed in this case and is revealed only when the wave energy density is several times larger than n_0T. These two circumstances enable us to neglect the electron nonlinearity in situations where the initial energy density of trapped waves in the caviton is of the order of the threshold for their modulational instability (see Ref. 9). Generally, the question of the influence of electron nonlinearity on the dynamics of the collapse can not be avoided, since at a sufficiently high average energy density of the Langmuir turbulence, the initial energy density of trapped waves in the caviton can substantially exceed the threshold for the modulational instability (see Ref. 10, Section 9, and Ref. 11). Any initial excess is allowed, in principle, but, if the development of the electron nonlinearity in the process of adiabatic collapse is of interest, we have to require that the growth rate of the modulational instability $\gamma \sim \omega_{pi}(W/n_0T)^{1/2}$ be small compared to the dispersive frequency increase $\Delta\omega \sim \omega_{pe}(r_D^2/a^2)$ (ω_{pe} and ω_{pi} being the electron and ion plasma frequencies respectively) for the trapped waves in the caviton. In particular, the caviton size a_e for which W reaches the value of n_0T must be bounded from above by the inequality

$$g(a_e) \equiv (a_e^2/r_D^2)(m_e/m_i)^{1/2} \lesssim 1. \tag{1.1}$$

More accurate numerical estimates practically do not change condition (1.1). Since the domain of values of the parameter a_e/r_D defined by this condition is somewhat broader than the domain of strong Landau damping of Langmuir waves for a Maxwellian particle distribution, the electron nonlinearity may play a significant role in the initial stage of formation of the so-called accelerated electron tails. The models of strong Langmuir turbulence[12,9,4] describing this process obviously should be improved, if the electron nonlinearity either halts the collapse or gives rise to a significant leak of plasmons from the caviton. The first possibility is ruled out by rough estimates of the lower limit for the decrease rate of the caviton size. The possibility of violating the adiabaticity of the collapse requires more refined analysis. The definiteness of the sign of the Hamiltonian H_e of the electron nonlinearity is the most important question in this case. If the sign is not definite, we may expect pure electron collapse to develop.[1)] For positive definite H_e, the time of the collapse probably depends on the ion mass, but the collapse is not obviously adiabatic even in this case, which is more

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favorable for assuring adiabaticity. Thus, the collapse of nonlinear Langmuir waves require an extensive investigation.

2. BASIC EQUATIONS

The electron nonlinearity of Langmuir waves leads to an occurrence of the fundamental and the higher harmonics of the plasma frequency ω_{pe} in oscillations of the electrostatic potential φ:

$$\varphi=\tilde{\varphi}+[\psi\exp(-i\omega_{pe}t)+\text{c.c.}]+[\varphi_2\exp(-2i\omega_{pe}t)+\text{c.c.}]+\dots. \tag{2.1}$$

The envelopes of the harmonics ψ, $\tilde{\varphi}$, φ_2, etc., vary slowly on the time ω_{pe}^{-1}, $\tilde{\varphi}$ and φ_2 are quadratic functions of ψ, and the envelopes of the other harmonics are higher-order quantities. The equation derived in Ref. 2 for the fundamental envelope ψ is quite cumbersome, and therefore, instead of writing it explicitly, we find it convenient to employ the variational formulation

$$\frac{\partial}{\partial t}\Delta\psi=2\pi i\omega_{pe}\frac{\delta H}{\delta\psi^*}. \tag{2.2}$$

The Hamiltonian H can be conveniently written as a sum of two terms, the first of which represents the supersonic collapse of ordinary Langmuir waves, while the second describes the electron nonlinearity:

$$H=H_0+H_e,$$

$$H_0=\int d^3r\left[\frac{n_0(\nabla\Phi)^2}{2m_i}+\frac{1}{4\pi}\left(\frac{n}{n_0}|\nabla\psi|^2+3r_D^2|\Delta\psi|^2\right)\right], \tag{2.3}$$

$$H_e=(32\pi n_0m_e\omega_{pe}^2)^{-1}\int d^3rU;$$

$$U=\tfrac{2}{3}|\nabla\psi_2|^2-|\nabla\tilde{\psi}|^2-\tfrac{1}{2}|\nabla(\nabla\psi)^2|^2-(\nabla|\nabla\psi|^2)^2. \tag{2.4}$$

Here, $\tilde{\psi}$ and ψ_2 are proportional to the envelopes of the fundamental and second harmonics of the electron velocity potential, and are quadratic functionals of ψ:

$$\Delta\tilde{\psi}=i\nabla(\Delta\psi\nabla\psi^*-\Delta\psi^*\nabla\psi), \tag{2.5}$$

$$\Delta\psi_2=\nabla(\nabla\psi\Delta\psi)+\Delta(\nabla\psi)^2. \tag{2.6}$$

The ion density perturbation n and their momentum potential Φ obey the equations

$$\partial n/\partial t=\delta H/\delta\Phi,\qquad \partial\Phi/\partial t=-\delta H/\delta n,$$

which can be easily reduced to a single equation

$$\frac{\partial^2 n}{\partial t^2}=\frac{1}{4\pi m_i}\Delta|\nabla\psi|^2, \tag{2.7}$$

describing the evolution of the caviton in the supersonic collapse. The dispersion of the sound wave could be taken into account by adding the term

$$H_s=\int d^3r\left(\frac{m_ic_s^2n^2}{2n_0}-\frac{|\nabla\eta|^2}{8\pi}\right),$$

to the Hamiltonian. Here η is the solution of Poisson's equation

$$\Delta\eta=\nabla\left(\frac{n}{n_0}\nabla\psi\right),$$

and $c_s=(T/m_i)^{1/2}$ is the sound speed.[2)] Due to the structural difference between the sonic and the main part of the Hamiltonian, the smallness of H_s by itself is insufficient for neglecting it. When the electronic nonlinearity is absent, i.e., for $W\ll n_0T$, the correction H_s, despite its smallness, is important in the supersonic collapse regime, since the short-wavelength boundary of the domain of modulationally unstable perturbations is defined exactly by the sound dispersion (see, for example, the Figure in Ref. 10). As the wave energy grows, this boundary moves towards still lower scales k_M^{-1}, i.e., $k_Mr_D\sim(W/n_0T)$. For $W\gtrsim n_0T$ the sonic correction is small for modulational perturbations of all scales exceeding the Debye radius and thus can be neglected. The short-wavelength boundary of the modulational instability is defined, in this case, by dissipative processes. It is interesting that for $W\gg n_0T$, in contrast to the case $W\ll n_0T$, the sonic correction is destabilizing, i.e., $H_s<0$.

3. POSITIVE DEFINITENESS OF THE HAMILTONIAN

The study of the Hamiltonian H_e of the electron nonlinearity is complicated because this functional is of high (fourth) order with respect to ψ, and because the integrand in (2.4) is nonlocal. At present it has been established only that the functional H_e either vanishes for one-dimensional fields[3),8] or is positive for radially symmetric fields.[2] The important question regarding the definiteness of the sign of H_e on the set of all localized fields ψ has not been cleared up. It seems, at first, that this question can be easily resolved by a counter-example, i.e., by constructing a localized field $\psi\neq0$ such that $\psi_2=0$ on it and, obviously, $H_e<0$. Nevertheless, this and other more elaborate attempts to construct a counter-example have failed because the Hamiltonian H_e is positive definite on the ensemble of all localized fields ψ. The proof of this fact is given below.

There exists an infinite set of equivalent representations of the density U of the Hamiltonian H_e, which can be constructed from each other by integrating by parts and redefining functions nonlocally related to ψ. It is desirable to find a representation of U in the form of a sum of squares of absolute values of several quantities. It is clear that if such a representation exists, each of its terms vanishes on one-dimensional fields ψ. The function ψ_2 does not possess this property, and thus should be replaced by the function

$$F=\psi_2-\tfrac{3}{2}(\nabla\psi)^2,\qquad \Delta F=\nabla(\nabla\psi\Delta\psi)-\tfrac{1}{2}\Delta(\nabla\psi)^2. \tag{3.1}$$

After the transition from ψ_2 to F, the density of the Hamiltonian of the electron nonlinearity becomes

$$U=\frac{2}{3}|\nabla F|^2+2\,\mathrm{Re}[\nabla F^*\nabla(\nabla\psi)^2]+|\nabla(\nabla\psi)^2|^2 - (\nabla|\nabla\psi|^2)^2-|\nabla\tilde{\psi}|^2. \tag{3.2}$$

For real (up to a phase factor independent of position) fields ψ, only the first two terms remain in Eq. (3.2). We seek to represent these terms as a sum of the squares of absolute values of several quantities and a residue which vanishes on real fields. The following sequence of transformations achieves this goal. Firstly, Eq. (3.1) is rewritten in the form

$$\Delta F=\frac{\partial^2}{\partial x_\alpha\partial x_\beta}(\Pi_{\alpha\beta}-\delta_{\alpha\beta}\Pi_{\gamma\gamma}),\qquad \Pi_{\alpha\beta}=\frac{\partial\psi}{\partial x_\alpha}\frac{\partial\psi}{\partial x_\beta}. \tag{3.3}$$

Integration of Eq. (3.3) yields

$$\frac{\partial F}{\partial x_\alpha}=\frac{\partial}{\partial x_\beta}(\Pi_{\alpha\beta}-\delta_{\alpha\beta}\Pi_{\gamma\gamma})+e_{\alpha\beta\gamma}\frac{\partial F_\gamma}{\partial x_\beta}. \tag{3.4}$$

Here the vector function F_γ is defined to within an additive term in the form of the gradient of an arbitrary scalar function. The latter can always be chosen to guarantee that the field F_γ is divergenceless:

$$\partial F_\gamma/\partial x_\gamma=0. \tag{3.5}$$

Then the equation for F_γ is obtained from the condition that Eq. (3.4) for F be soluble, and, on using Eq. (3.5), has the form

$$\Delta F_\alpha=e_{\alpha\beta\gamma}\frac{\partial\Pi_{\gamma\sigma}}{\partial x_\beta\,\partial x_\sigma}. \tag{3.6}$$

It is not difficult to show, by using relations (3.3)–(3.6), that

$$\int d^3\mathbf{r}\frac{\partial\Pi_{\alpha\beta}}{\partial x_\beta}\frac{\partial\Pi_{\alpha\gamma}^*}{\partial x_\gamma}=\int d^3\mathbf{r}\{|\nabla[F+(\nabla\psi)^2]|^2+\nabla F_\alpha\nabla F_\alpha^*\}. \tag{3.7}$$

The integration of Eq. (3.6) yields

$$\frac{\partial F_\alpha}{\partial x_\sigma}=e_{\alpha\beta\gamma}\frac{\partial\Pi_{\gamma\sigma}}{\partial x_\beta}+e_{\sigma\beta\gamma}\frac{\partial F_{\alpha\gamma}}{\partial x_\beta}. \tag{3.8}$$

Without loss of generality the condition

$$\partial F_{\alpha\gamma}/\partial x_\gamma=0, \tag{3.9}$$

can be imposed on the tensor function $F_{\alpha\gamma}$. Then this function obeys the equation

$$\Delta F_{\alpha\alpha_1}=e_{\alpha\beta\gamma}e_{\alpha_1\beta_1\gamma_1}\frac{\partial^2\Pi_{\gamma\gamma_1}}{\partial x_\beta\,\partial x_{\beta_1}}. \tag{3.10}$$

Equations (3.8)–(3.10) can be used to show that

$$\int d^3\mathbf{r}(\nabla F_\alpha^*\nabla F_\alpha+\nabla F_{\alpha\beta}^*\nabla F_{\alpha\beta})=\int d^3\mathbf{r}\frac{\partial\Pi_{\alpha\beta}^*}{\partial x_\gamma}\left(\frac{\partial\Pi_{\alpha\beta}}{\partial x_\gamma}-\frac{\partial\Pi_{\alpha\gamma}}{\partial x_\beta}\right). \tag{3.11}$$

The trace of the tensor $F_{\alpha\beta}$ is simply related to the original function F:

$$F_{\alpha\alpha}=-F. \tag{3.12}$$

Separating the traceless part of this tensor

$$A_{\alpha\beta}=F_{\alpha\beta}+{}^1/_3\delta_{\alpha\beta}F \tag{3.13}$$

allows us to strengthen the result of the previous transformations, because

$$\int d^3\mathbf{r}\,\nabla F_{\alpha\beta}^*\nabla F_{\alpha\beta}=\int d^3\mathbf{r}\left(\nabla A_{\alpha\beta}^*\nabla A_{\alpha\beta}+\frac{1}{3}|\nabla F|^2\right). \tag{3.14}$$

Combining Eqs. (3.7), (3.11) and (3.14), we have

$$\int d^3\mathbf{r}\left\{\frac{2}{3}|\nabla F|^2+2\,\mathrm{Re}[\nabla F^*\nabla(\nabla\psi)^2]+|\nabla(\nabla\psi)^2|^2\right\}$$
$$=\int d^3\mathbf{r}\left(\nabla A_{\alpha\beta}^*\nabla A_{\alpha\beta}+2\frac{\partial\Pi_{\alpha\gamma}^*}{\partial x_\beta}\frac{\partial\Pi_{\alpha\beta}}{\partial x_\gamma}-\nabla\Pi_{\alpha\beta}^*\nabla\Pi_{\alpha\beta}\right).$$

By employing this inequality, we can obtain the following representation of the density of the Hamiltonian of the electron nonlinearity Hamilton, equivalent to Eq. (3.2):

$$U=\nabla A_{\alpha\beta}^*\nabla A_{\alpha\beta}+|f|^2-|\nabla\tilde\psi|^2, \tag{3.15}$$

$$f=2\,\mathrm{Im}\left(\frac{\partial\psi^*}{\partial x_\alpha}\nabla\frac{\partial\psi}{\partial x_\alpha}\right), \tag{3.16}$$

which proves the positive definiteness of the functional H_e on an ensemble of real fields ψ. In order to expand the proof to an ensemble of all complex fields, it is sufficient to observe that Eq. (2.5) can be presented in the form

$$\Delta\tilde\psi=-\mathrm{div}\,\mathbf{f}.$$

To complete the proof, we decompose the field $\mathbf{f}$ into potential and solenoidal parts

$$\mathbf{f}=-\nabla\tilde\psi+\mathrm{rot}\,\mathbf{A} \tag{3.17}$$

and use the obvious equality

$$\int d^3\mathbf{r}|f|^2=\int d^3\mathbf{r}(|\nabla\tilde\psi|^2+|\mathrm{rot}\,\mathbf{A}|^2). \tag{3.18}$$

The representation

$$U=\nabla A_{\alpha\beta}^*\nabla A_{\alpha\beta}+|\mathrm{rot}\,\mathbf{A}|^2 \tag{3.19}$$

resulting from (3.15) and (3.18) proves the positive definiteness of the Hamiltonian of the electron nonlinearity on the set of all localized complex fields ψ.

Without loss of generality, the vector field $\mathbf{A}$ can be assumed to satisfy the constraint

$$\mathrm{div}\,\mathbf{A}=0, \tag{3.20}$$

as a result of which the condition that Eq. (3.17) for ψ be soluble reduces to a Poisson equation for $\mathbf{A}$:

$$\Delta\mathbf{A}=-\mathrm{rot}\,\mathbf{f}. \tag{3.21}$$

4. THE ADIABATIC APPROXIMATION

The positive definiteness of the electron nonlinearity Hamiltonian shows that the collapse cannot develop without continually expelling ions from the localization region of the Langmuir waves. Due to the large ions inertia, we can assume that the caviton deepens slowly, and attempt to develop an adiabatic description of trapped nonlinear Langmuir waves. As in the familiar linear theory, we express the function ψ in the form

$$\psi(\mathbf{r},t)=\psi_\lambda(\mathbf{r},t)\exp\left\{i\int^t\lambda(t_1)\,dt_1\right\}, \tag{4.1}$$

where $\psi_\lambda(\mathbf{r},t)$ depends on the scale of the caviton time variation. The equation for ψ_λ can be easily obtained from Eq. (2.2):

$$\frac{\partial}{\partial t}\Delta\psi_\lambda=2\pi i\omega_{pe}\frac{\delta H}{\delta\psi_\lambda^*},$$
$$\tilde H=H+\lambda I,\quad I[\psi_\lambda]=\frac{1}{2\pi\omega_{pe}}\int|\nabla\psi_\lambda|^2\,d^3\mathbf{r}. \tag{4.2}$$

The functional $I[\psi_\lambda]$ has the meaning of the total number of plasmons (the wave action) and is the exact integral of Eq. (4.2). For a given slowly varying time-dependence of the caviton, the function ψ_λ can be found by successive approximations. In lowest order we neglect the time variation of the caviton, i.e., the left-hand side of Eq. (4.2), and find the function $\psi_\lambda^{(0)}$ for which the functional $H[\psi_\lambda]$ has a maximum at a given value of the functional $I[\psi_\lambda]$.

$$\delta H/\delta\psi_\lambda^{(0)*}=0, \tag{4.3}$$

$$I[\psi_\lambda^{(0)}]=I. \tag{4.4}$$

At least one such a function certainly exists, since for a given value of the wave action, the Hamiltonian is bounded from below. In a linear problem, the Lagrangian multiplier $\lambda^{(0)}$ is derived from the condition that Eq. (4.3) be soluble and relation (4.4) is employed in normalizing the function $\psi_\lambda^{(0)}$. In the nonlinear problem $\lambda^{(0)}$ depends on the value of the wave action and is determined by relation (4.4). By using Euler's formula for homogeneous functionals, one easily obtains the following equality

$$\lambda^{(0)} I + H_{0e}[\psi_\lambda^{(0)}] + 2H_e[\psi_\lambda^{(0)}] = 0, \tag{4.5}$$

where H_{0e} denotes the electron contribution in H_0 [the last two terms in Eq. (2.3)]. As can be seen in Eq. (4.5), the quantity $\lambda^{(0)}$ is real. For a deep enough caviton, this quantity becomes positive and can be interpreted as the plasmon binding energy.

In the first-order approximation we substitute the implicitly time-dependent function $\psi_\lambda^{(0)}$ into the left-hand side of Eq. (4.2) and solve the resulting equation for $\psi_\lambda \simeq \psi_\lambda^{(0)} + \psi_\lambda^{(1)}$. By assuming that the correction $\psi_\lambda^{(1)}$ is small, we can limit ourselves to evaluating $\tilde{H}[\psi_\lambda]$ up to second-order terms with respect to $\psi_\lambda^{(1)}$

$$\tilde{H}[\psi_\lambda] \approx \tilde{H}[\psi_\lambda^{(0)}] + \tilde{H}^{(2)}[\psi_\lambda^{(0)}, \psi_\lambda^{(1)}] + \lambda^{(1)} I^{(1)}[\psi_\lambda^{(0)}, \psi_\lambda^{(1)}],$$

$$I^{(1)}[\psi_\lambda^{(0)}, \psi_\lambda^{(1)}] = \frac{1}{2\pi\omega_{pe}} \int d^3\mathbf{r} (\psi_\lambda^{(0)*} \psi_\lambda^{(1)} + \text{c.c.}), \tag{4.6}$$

$$\tilde{H}^{(2)}[\psi_\lambda^{(0)}, \psi_\lambda^{(1)}] = \langle \chi | \hat{L} | \chi \rangle.$$

Here, $\chi = (\operatorname{Re}\psi_\lambda^{(1)}, \operatorname{Im}\psi_\lambda^{(1)})$ is a two-component function, $\hat{L}$ is a 2×2 matrix self-adjoint operator depending on $\psi_\lambda^{(0)}$, and the angle brackets denote the standard scalar product. The first-order equation in this notation becomes

$$2\pi\omega_{pe}\hat{L} \begin{vmatrix} \operatorname{Re}\psi_\lambda^{(1)} \\ \operatorname{Im}\psi_\lambda^{(1)} \end{vmatrix} = \frac{\partial}{\partial t}\Delta \begin{vmatrix} \operatorname{Im}\psi_\lambda^{(0)} \\ -\operatorname{Re}\psi_\lambda^{(0)} \end{vmatrix} + \lambda^{(1)}\Delta \begin{vmatrix} \operatorname{Re}\psi_\lambda^{(0)} \\ \operatorname{Im}\psi_\lambda^{(0)} \end{vmatrix}. \tag{4.7}$$

The condition that Eq. (4.7) be soluble, is given by the orthogonality of the right-hand side to the null space (kernel) of the operator $\hat{L}$. It is assumed in the following that $\psi_\lambda^{(0)}$ is the function corresponding to the smallest of the possible values of the Hamiltonian H for a given number of plasmons. Obviously, the function $\psi_\lambda^{(0)}$ is associated with a nonnegative operator $\hat{L}$. There is no positive definiteness, since the invariance of H and I with respect to the one-parameter transformation group $\psi \to e^{i\alpha}\psi$, where α is a real number, yields the eigenfunction $\chi_0 = (-\operatorname{Im}\psi_\lambda^{(0)}, \operatorname{Re}\psi_\lambda^{(0)})$, corresponding to the zero eigenvalue of the operator $\hat{L}$. This function is orthogonal to the right-hand side of Eq. (4.7) if

$$\frac{d}{dt} I[\psi_\lambda^{(0)}] = 0. \tag{4.8}$$

The meaning of the condition (4.8) is obvious: the inequality (4.4) should not be violated during the evolution process. If the zero eigenvalue of the operator is nondegenerate, then condition (4.8) guarantees the solubility of Eq. (4.7). The constant $\lambda^{(1)}$ in Eq. (4.7) is chosen to eliminate the first-order correction to the number of plasmons.

$$I^{(1)}[\psi_\lambda^{(0)}, \psi_\lambda^{(1)}] = 0.$$

The smallness of $\psi_\lambda^{(1)}$ relative to $\psi_\lambda^{(0)}$ is guaranteed if inequality

$$\gamma \ll \Delta\lambda, \tag{4.9}$$

is satisfied. Here γ^{-1} is the characteristic time of variation of the caviton, and $\Delta\lambda$ is the smallest eigenvalue of $\hat{L}$ on the set of functions orthogonal to χ_0. The series of successive approximations can easily be continued. It is known that in the linear theory, condition (4.9) guarantees the smallness of the corrections at each iteration step compared to the preceeding approximation. In the nonlinear problem there exists an additional limitation associated with the presence of nonlinear terms in the higher-order equations. For example, in the equation for $\psi_\lambda^{(1)}$, in addition to $i\partial\Delta\psi_\lambda^{(1)}/\partial t$, there exists a forcing term which is quadratic with respect to $\psi_\lambda^{(1)}$. If we introduce an equivalent frequency shift $\delta\lambda$ so that this forcing term would be of order $\delta\lambda\Delta\psi_\lambda^{(1)}$, then the second condition on the smallness of $\psi_\lambda^{(2)}$ relative to $\psi_\lambda^{(1)}$ assumes the form

$$\delta\lambda \ll \Delta\lambda. \tag{4.10}$$

These conditions for the applicability of perturbation theory are not always necessary. For example, when the electron nonlinearity is weak, condition (4.9) is not satisfied and perturbation theory cannot be employed. Indeed, when the electron nonlinearity is absent, the operator $\hat{L}$ is equal to the product of the unit 2×2 matrix with the scalar operator:

$$\hat{L}_0 = \frac{1}{4\pi} \nabla \left(3r_D^2 \Delta - \frac{n}{n_0} - \frac{2\lambda^{(0)}}{\omega_{pe}} \right) \nabla.$$

Next, the zero eigenvalue of operator $\hat{L}_0$ is assumed to be nondegenerate and located far from the rest of the eigenvalues of $\hat{L}_0$. Then we can apply the conventional theory of adiabatic perturbations to Eq. (4.2) and without loss of generality the function $\psi_\lambda^{(0)}$ can be assumed to be real. In the absence of the electron nonlinearity, the zero eigenvalue of the operator $\hat{L}$ is doubly degenerate, i.e., in addition to $\chi_0 = (0, \psi_\lambda^{(0)})$, the function $\tilde{\chi}_0 = (\psi_\lambda^{(0)}, 0)$ also belongs to the kernel. Therefore, $\Delta\lambda = 0$ and the condition (4.9) is not satisfied. The necessary condition for the solubility of (4.7), i.e., the orthogonality of $\tilde{\chi}_0$ to the right-hand side of this equation, yields $\lambda^{(1)} = 0$. A weak nonlinearity splits the initially degenerate zero eigenvalues of operator $\hat{L}$ into two eigenvalues, one of which is still zero and the other, $\Delta\lambda$, is proportional to the nonlinearity. Due to the nonlinearity, the projections of the right-hand sides of (4.7) and higher-order equations on an eigenfunction of the operator $\hat{L}$ close to $\tilde{\chi}_0$ no longer vanish, but remain proportional to the nonlinearity like $\Delta\lambda$, thus justifying the use of the perturbation theory.

It is instructive to observe that an initially real function $\psi_\lambda^{(1)}$ of the first-order approximation remains real in the process of the adiabatic collapse, despite a possibility that an evolved electron nonlinearity can have complex solutions of Eq. (4.3). The matrix $\hat{L}$ is diagonal on the real field $\psi_\lambda^{(0)}$:

$$\hat{L} = \begin{vmatrix} \hat{L}_1 & 0 \\ 0 & \hat{L}_2 \end{vmatrix}.$$

Here $\hat{L}_1$ and $\hat{L}_2$ are scalar self-adjoint operators. In the absence of an electron nonlinearity they are equal to $\hat{L}_0$. As a result of the nonlinearity the operators $\hat{L}_1$ and $\hat{L}_2$ are different. With the above assumptions, the operator $\hat{L}_1$ is positive definite, $\hat{L}_2$ is nonnegative and a single eigenfunction $\psi_\lambda^{(0)}$ corresponds to the zero eigenvalue of $\hat{L}_2$. Due to the invariance of Eq. (4.2) under the time reversal and complex conjugation for a real field $\psi_\lambda^{(0)}$, corrections to the real and imaginary parts of ψ_λ arise only in even and odd orders, respectively. In particular, the equations in the first two orders have the form

$$2\pi\omega_{pe}\hat{L}_2\psi_\lambda^{(1)} = i\frac{\partial}{\partial t}\Delta\psi_\lambda^{(0)}, \quad \lambda^{(1)} = 0; \tag{4.11}$$

$$2\pi\omega_{pe}\hat{L}_1\psi_\lambda^{(2)} = \Delta\left(\frac{\partial}{\partial t}\operatorname{Im}\psi_\lambda^{(1)} + \lambda^{(2)}\psi_\lambda^{(0)}\right) - 2\pi\omega_{pe}\frac{\delta}{\delta\operatorname{Re}\psi_\lambda}H_e^{(2)}[\operatorname{Re}\psi_\lambda, \operatorname{Im}\psi_\lambda]. \tag{4.12}$$

The variation of the functional $H_e^{(2)}$ in Eq. (4.12) is evaluated at the point

$$\operatorname{Re}\psi_\lambda = \psi_\lambda^{(0)}, \quad \operatorname{Im}\psi_\lambda = -i\psi_\lambda^{(1)}.$$

The functional $H_e^{(2)}$ is the part of H_e which is quadratic in $\operatorname{Im}\psi_\lambda$ and $\operatorname{Re}\psi_\lambda$. The quantity $\lambda^{(2)}$ is defined by the condition

$$I^{(2)} \equiv \int d^3\mathbf{r}(2\nabla\psi_\lambda^{(0)}\nabla\psi_\lambda^{(2)} + |\nabla\psi_\lambda^{(1)}|^2) = 0.$$

When the series of successive approximations has this structure, condition (4.10) is redundant for two reasons. Firstly, the characteristic value ω_1^{-1} of the operator $\Delta(\omega_{pe}\hat{L}_+)^{-1}$ associated with the last term in (4.12) can be much less than $(\Delta\lambda)^{-1}$. Secondly, the limitation $\psi_\lambda^{(2)} \ll \psi_\lambda^{(1)}$ imposed above is unnecessary now for justifying the iterative procedure and can be replaced by a weaker limitation $\psi_\lambda^{(2)} \ll \psi_\lambda^{(0)}$. This enables us to replace (4.10) by the condition

$$\omega_1 \gg \delta\lambda_1, \tag{4.13}$$

where $\delta\lambda_1$ is the equivalent frequency shift, defined so that the last term in Eq. (4.12) is on the order of $\delta\lambda_1\Delta\psi_\lambda^{(0)}$.

5. THE GROWTH RATE OF THE SMALL-SCALE INSTABILITY

The deepening of the caviton is caused by the modulational instability. As was mentioned above, at the onset of the electron nonlinearity $W \sim n_0T$ and the short-wavelength boundary of the domain of modulationally unstable perturbations reaches the Debye radius. This conclusion is based on the well known behavior of the growth rate of the modulational instability at $W \ll n_0T$, when the electron nonlinearity is negligible. For $W \gg n_0T$ the dependence of the growth rate on the wavelength of the perturbation has not been adequately investigated. Nevertheless, the question of the character of this dependence is important, since if the perturbations with wavelength short compared to the size of the caviton develop faster than the other perturbations, then the caviton will break up. Otherwise, the caviton as a whole can collapse. Reference 2 has shown that the electron nonlinearity does not affect the modulational instability of a uniform field. Therefore, the contribution of the electron nonlinearity to the short-wavelength instability is suppressed by a factor characterizing the quasiclassicality of the perturbation, and equal to the ratio between the wavelength of the perturbation and the spatial scale of variation of the field. On the other hand, the contribution of the electron nonlinearity is enhanced by a factor $W/n_0T \gg 1$, and thus can be dominant even for short-wavelength perturbations. Therefore, the growth rate of the modulational instability of a localized condensate of nonlinear Langmuir waves is unknown and has to be evaluated in order to understand the dynamics of the collapse.

Let λ, n, and ψ_λ be solutions of Eqs. (2.7), (4.2) and let

$$\delta n = ue^{i\mathbf{kr}} + \text{c.c.}, \quad \delta\psi_\lambda = \chi_+ e^{i\mathbf{kr}} + \chi_- e^{-i\mathbf{kr}} \tag{5.1}$$

be a small perturbation of this solution. For small-scale perturbations, considered here, the exponentials in (5.1) are rapidly oscillating functions of the spatial coordinates. To the lowest significant order to the WKB approximation, the linearized equations (2.7) and (4.2) are

$$\frac{\partial^2 u}{\partial t^2} = -\frac{ik^2}{4\pi m_i}[(\mathbf{k}\nabla\psi_\lambda^*)\chi_+ + (\mathbf{k}\nabla\psi_\lambda)\chi_-^*], \tag{5.2}$$

$$is(\mathbf{k}\nabla\psi_\lambda)\frac{u_s}{2n_0} = k^2\left[\frac{3}{2}k^2r_D^2 + \frac{n}{2n_0} + \frac{1}{\omega_{pe}}\left(\lambda - i\frac{\partial}{\partial t}\right)\right]\chi_s + 2\pi\frac{\delta H_e^{(2)}}{\delta\nu_s^*}. \tag{5.3}$$

where $s = \pm$, $u_+ = u$, $u_- = u^*$. In evaluating the quadratic perturbation $H_e^{(2)}$ of H_e, we proceed from the representation (3.19), since otherwise we initially encounter terms proportional to high powers of k, which later cancel each other. The terms associated with the real and imaginary parts of the field ψ_λ may have different structures, and in particular contain different powers of k. In order to take into account the possible competition between the adiabaticity parameter (in terms of which the imaginary part of ψ_λ is small) and other parameters, we have to evaluate $H_e^{(2)}$ on complex fields ψ_λ. According to (3.19),

$$H_e^{(2)} = (32\pi n_0 m_e\omega_{pe}^2)^{-1} \times \int d^3\mathbf{r}[\nabla A_{\alpha\beta}^{(1)*}\nabla A_{\alpha\beta}^{(1)} + 2\operatorname{Re}(\nabla A_{\alpha\beta}^{(0)*}\nabla A_{\alpha\beta}^{(2)}) + |\operatorname{rot}\mathbf{A}^{(1)}|^2 + 2\operatorname{rot}\mathbf{A}^{(0)}\operatorname{rot}\mathbf{A}^{(2)}]. \tag{5.4}$$

Here, $\mathbf{A}_{\alpha\beta}^{(i)}$, $\mathbf{A}^{(i)}$ $(i = 0, 1, 2)$ are the ith order terms in the expansions of fields $A_{\alpha\beta}$, $\mathbf{A}$ in terms of the amplitude $\delta\psi_\lambda$ of the perturbation. By using Eqs. (3.3), (3.10), (3.12), (3.16) and (3.21) it is easy to find that the first three terms in (5.4) are quadratic and the last term is cubic in k. On the other hand, the last term vanishes on a real field ψ_λ and hence is small in terms of the adiabaticity parameter and can be found by ignoring the quadratic corrections in k. The third term in (5.4), which also vanishes for real ψ_λ and $\delta\psi_\lambda$, differs structurally from the corrections given above. In particular, it does not vanish for imaginary perturbations $\delta\psi_\lambda$ of a real field ψ_λ. Thus, we have to take into account all four terms in Eq. (5.4). The final result for $H_e^{(2)}$ is presented in the following form:

$$H_e^{(2)}=(8\pi n_0 m_e\omega_{pe}^2)^{-1}k^2\int d^3\mathbf{r}\{(P_1+P_3)(|\chi_+|^2+|\chi_-|^2)$$
$$-2\,\mathrm{Re}[(P_2+\bar{P}_3)\chi_+^*\chi_-^*]+P_4(|\chi_+|^2-|\chi_-|^2)\}. \quad (5.5)$$

The indexes of the coefficients P_i $(i=1, 2, 3, 4)$ denote the number of the corresponding term in Eq. (5.4). Equations for the coefficients P_i simplify in the coordinate system with one of the axes (say the z-axis) in the direction of the wave vector $\mathbf{k}$:

$$P_1=\frac{1}{3}\left(\left|\frac{\partial^2\psi_\lambda}{\partial x^2}\right|^2+\left|\frac{\partial^2\psi_\lambda}{\partial y^2}\right|^2+\left|\frac{\partial^2\psi_\lambda}{\partial x^2}-\frac{\partial^2\psi_\lambda}{\partial y^2}\right|^2\right)+2\left|\frac{\partial^2\psi_\lambda}{\partial x\,\partial y}\right|^2,$$
$$P_2=\frac{1}{2}\frac{\partial^2 A_{xx}}{\partial y^2}+\frac{1}{2}\frac{\partial^2 A_{yy}}{\partial x^2}-\frac{\partial^2 A_{xx}}{\partial x\,\partial y}, \quad (5.6)$$
$$P_3=2\left(\left|\frac{\partial^2\psi_\lambda}{\partial x\,\partial z}\right|^2+\left|\frac{\partial^2\psi_\lambda}{\partial y\,\partial z}\right|^2\right),$$
$$\bar{P}_3=2\left[\left(\frac{\partial^2\psi_\lambda}{\partial x\,\partial z}\right)^2+\left(\frac{\partial^2\psi_\lambda}{\partial y\,\partial z}\right)^2\right],\quad P_4=-\mathbf{k}\,\mathrm{rot}\,\mathbf{A}.$$

By substituting Eq. (5.5) into Eq. (5.3) and neglecting the term with $\partial\chi_s/\partial t$, which is small with respect to the adiabaticity parameter, we can express χ_s via u:

$$\chi_s=\frac{2ism_e\omega_{pe}^2}{k^2}\frac{(P+P_1+P_3-sP_4)k\nabla\psi_\lambda+(P_2+\bar{P}_3)\mathbf{k}\nabla\psi_\lambda^*}{(P+P_1+P_3)^2-P_4^2-|P_2+\bar{P}_3|^2}u_s. \quad (5.7)$$

Here, we have introduced the notation

$$P=4n_0m_e\omega_{pe}^2\left(\frac{3}{2}k^2r_D^2+\frac{n}{2n_0}+\frac{\lambda}{\omega_{pe}}\right). \quad (5.8)$$

By taking Eq. (5.7) into account, we can rewrite Eq. (5.2) in the form

$$\frac{\partial^2}{\partial t^2}u=R_{\mathbf{k}}(\mathbf{r},t)u, \quad (5.9)$$

$$R_{\mathbf{k}}(\mathbf{r},t)=\frac{\omega_{pi}^2}{\pi}\frac{(P+P_1+P_3)|\mathbf{k}\nabla\psi_\lambda|^2+\mathrm{Re}[(P_2+\bar{P}_3)(\mathbf{k}\nabla\psi_\lambda^*)^2]}{(P+P_1+P_3)^2-P_4^2-|P_2+\bar{P}_3|^2}. \quad (5.10)$$

Expression (5.10) is simplified significantly for a real field

$$R_{\mathbf{k}}(\mathbf{r},t)=\frac{\omega_{pi}^2}{\pi}\frac{(\mathbf{k}\nabla\psi_\lambda)^2}{P+P_1-P_2}. \quad (5.11)$$

The condition

$$P_4^2+(\mathrm{Im}\,\bar{P}_3)^2+|P_2+P_3||P_3-\mathrm{Re}\,\bar{P}_3|$$
$$\ll|P+P_1-P_2||P+P_1+P_2+2P_3| \quad (5.12)$$

must be satisfied for the real-field approximation to hold.

Equation (5.10) allows us to calculate the growth rate of the modulational instability in the entire region $a^{-1}\ll k\ll r_D^{-1}$, where a is the smallest characteristic dimension of the caviton. In the long-wavelength limit, this equation yields an estimate for the characteristic time γ^{-1} for the caviton to deepen. The necessary stability condition of the caviton as a whole is given by the inequality $\gamma_k\lesssim\gamma$. Simple analysis of the dependence of the right-hand side of Eq. (5.10) on k shows that this inequality is satisfied only when the linear dispersion term $k^2r_D^2$ is significant even for $k\sim a^{-1}$. In this case, the electron nonlinearity does not influence the development of fluctuations with $k\gg a^{-1}$, and their growth rate does not depend on k and is on the order of γ:

$$\gamma_k\sim\gamma\sim\omega_{pi}(W/n_0T)^{1/2}. \quad (5.13)$$

When the growth rate γ_k of the small-scale instability is less than γ, the convective character of the instability must be taken into account. Any point $\mathbf{r}$, except the point $\mathbf{r}_s$ to which the caviton is collapsing, sooner or later lies outside the limit of the main energy condensate, and thus, also, beyond the strong-instability region. Hence for $\gamma_k\sim\gamma$ the threat to the stability of the caviton comes only from small-scale perturbations, localized in its central region. The condition

$$\lim_{t\to t_s}R_{\mathbf{k}}^{-1}(\mathbf{r}_s,t)n^{-1}(\mathbf{r}_s,t)\frac{\partial^2 n(\mathbf{r}_s,t)}{\partial t^2}\geqslant 1 \quad (5.14)$$

is necessary and sufficient to prevent the growth of the relative magnitude of these perturbations. Here, t_s is the instant at which the singularity is created. In the absence of the electron nonlinearity, this condition reduces to that obtained in Ref. 10.

6. DYNAMICS OF THE COLLAPSE (ESTIMATES)

The simplest assumption on the form of the caviton is to characterize it by a single length scale a. For a single-length-scale caviton, the linear dispersion term in Eqs. (4.2) and (5.10) is less by a factor W/n_0T than the nonlinear term. As has been found above, in this case the short-wavelength instability develops much faster than the instability on the main scale. Hence the estimates related to the one-dimensional caviton (see Ref. 10, Sect. 10), give only the lower limit on the velocity of the collapse. The development of the short-wavelength instability must give rise to a second length scale in the caviton. The simplest possibility of this kind is that the caviton flattens in a certain direction. Let a be the characteristic thickness of the flattened caviton and $b\gg a$ be its diameter. In estimating the energy of the electron nonlinearity, it is useful to note that, like an electrostatic potential, the function $A_{\alpha\beta}$ obeys Poisson's equation [see Eqs. (3.10) and (3.13)]. The "charge density" in this Poisson equation is on the order of W/b^2 and is concentrated in the caviton region. The field $\nabla A_{\alpha\beta}$ in the caviton region is of the order of $(W/b^2)a$ and the density of the electron nonlinearity is of the order

$$W_e\sim W^2a^2/n_0m_e\omega_{pe}^2b^4. \quad (6.1)$$

If the terms associated with the plasma density decrease and the linear and nonlinear dispersions are equally important in Eq. (4.2), the corresponding energy densities must be of the same order. This condition yields the following estimates

$$n/n_0\sim Wa^2/n_0m_e\omega_{pe}^2b^4\sim r_D^2/a^2. \quad (6.2)$$

In closing the system of the equations we must determine the relation between the wave energy density and the dimensions of the caviton. The condition $Wab^2=\mathrm{const}$ which suggest itself turns out to be wrong. The reason is that the integral along the flattened direction of the caviton (say the

z-axis) of the "charge density" in the equation for $A_{\alpha\beta}$ does not vanish automatically, and the terms of the form $\partial^2 J/\partial x^2$, $\partial^2 J/\partial y^2$, $\partial^2 J/\partial x \partial y$ are left, where

$$J=\int_{-\infty}^{\infty} dz(\partial\psi_\lambda/\partial z)^2. \tag{6.3}$$

If we estimate these terms roughly by $\partial^2 J/\partial x^2 \sim Wa/b^2$, we should conclude that the field $\nabla A_{\alpha\beta}$ is of the order of Wa/b^2 not only in the caviton, but also in the entire interior of a sphere of radius of order b. Due to the equality between the different types of the energy densities, the total energy of the electron nonlinearity H_e would be larger by a factor b/a than the contribution to the Hamiltonian associated with the density decrease, and could not be compensated for by this negative contribution. Conservation of H would forbid the growth of H_e and the development of the collapse.[4] Therefore, the collapse can develop only in such a way that the field $\nabla A_{\alpha\beta}$ does not "fall out" from the caviton. This is achievable if the quantity J, defined in Eq. (6.3), does not depend on x and y at distances $r \lesssim b$ from the axis of the caviton. Let $c \gg b$ be the spatial scale on which the function $J(x,y)$ decreases and $W(r)$, $n(r)$ and $a(r)$ be the characteristic values of the energy density of the waves, the depth and the width of the caviton, respectively, at distance $r \gtrsim b$ from its axis. From arguments like those used in reducing Eq. (6.2), we can obtain the estimates

$$n(r)/n_0 \sim W(r)a^2(r)/n_0 m_e \omega_{pe}^2 r^4 \sim r_D^2/a^2(r). \tag{6.4}$$

Noting that quantity

$$J \sim W(r)a(r), \tag{6.5}$$

does not depend on r, we can easily establish that

$$a(r) \sim a(r/b)^{4/3}, \quad W(r) \sim W(b/r)^{4/3}, \quad n(r) \sim n(b/r)^{8/3}. \tag{6.6}$$

The prohibition against the r-dependence of J is removed at the distance $r \sim c$ from the axis, such that the width of the caviton is comparable to r:

$$c \sim b^4/a^3. \tag{6.7}$$

From the conservation of the total number of trapped waves in the process of the adiabatic collapse, we have the following relation:

$$Jc^2 = \text{const}. \tag{6.8}$$

As a result of Eqs. (6.2), (6.5) and (6.7)

$$J \sim Wa \propto b^4/a^3 \sim c. \tag{6.9}$$

The comparison of this estimate with (6.8) shows that the quantities J and c do not vary during the collapse. The time-dependence of other quantities is found by using the estimates given in this section and the estimate (5.13) for the growth rate of the modulational instability,

$$W \propto \tilde{t}^{-2}, \quad a \propto \tilde{t}^2, \quad b \propto \tilde{t}^{3/2}, \quad n \propto \tilde{t}^{-4} \quad (\tilde{t}=t_s-t). \tag{6.10}$$

The coefficients in Eqs. (6.10) are easily reconstructed from the known values of all the quantities at the onset of the electron nonlinearity. At the onset

$$W \sim n_0 T, \quad a \sim b \sim c \sim a_e, \quad n \sim n_0 (r_D/a_e)^2.$$

It is useful to notice that the time-dependence disappears in Eqs. (6.6):

$$a(r) \sim a_e \left(\frac{r}{a_e}\right)^{4/3}, \quad W(r) \sim n_0 T \left(\frac{a_e}{r}\right)^{4/3},$$

$$n(r) \sim n_0 \frac{r_D^2}{a_e^2}\left(\frac{a_e}{r}\right)^{8/3},$$

since the caviton parameters in the region $r \gg b$ do not vary much while the central part of the caviton collapses.

In order to find the domain within which these estimates are applicable, we have to ascertain how long the collapse remains adiabatic when the electron nonlinearity develops. The "binding energy" λ of plasmons, by definition, does not depend on r and is determined by the depth of the caviton at $r \sim a_e$. Together with the depth, the quantity λ does not change its order of magnitude during the development of the electron nonlinearity and remains of order $\omega_{pe}(r_D/a_e)^2$. The growth rate $\gamma(a_e)$ of the modulational instability on the periphery of the caviton also does not vary and remains of order ω_{pi}, but at the center of the caviton it grows as $\gamma(a) \sim W^{1/2} \sim a^{-1/2}$ and, in principle, can become larger than λ.

For $\gamma(a) \gtrsim \lambda$, there is a danger that initially trapped particles in the caviton will undergo a transition into the states of the continuous spectrum of the operator $\hat{L}_2$ with frequencies $\omega \lesssim \gamma(a)$. Simple estimates show, however, that the main contribution in the convolution integral of this state with the right-hand side of Eq. (4.11) comes from the periphery of the caviton. Therefore, the condition (4.9) contains $\gamma(a)$ and remains practically unchanged while the electron nonlinearity develops. The imaginary correction $\psi_\lambda^{(1)}$ can still be estimated by

$$\psi_\lambda^{(1)} \sim g(a_e)\psi_\lambda^{(0)}. \tag{6.11}$$

The verification of the condition (4.3), requires more detailed consideration of the central region $r \lesssim b$ of the caviton, since this region gives the main contribution to the right-hand side of Eq. (4.12). The contributions of the electron nonlinearity to the operators $\hat{L}_1$ and $\hat{L}_2$ in the central region of the caviton can be estimated by

$$L_1 \sim W/n_0 m_e \omega_{pe}^2 b^4, \quad L_2 \sim (b/a)^2 L_1. \tag{6.12}$$

Remarkable difference between the "hardness" of the electron nonlinearity with respect to real and imaginary perturbations of a real field can be explained by the fact that the second term in Eq. (3.19), containing the largest number of derivatives in the direction of the flattening of the caviton, vanishes on real fields ψ. For an evolved electron nonlinearity, the quantity $\omega_{pe}L_2a^2$ (and also $\omega_{pe}L_1a^2$) significantly exceeds the binding energy of the plasmons and the growth rate of the modulational instability. As a result, the function $\psi_\lambda^{(1)}$ in the central region of the caviton is proportional to $i\psi_\lambda^{(0)}$, up to a small correction $\delta\psi_\lambda^{(1)}$ of order $[\gamma(a)/\omega_{pe}L_2a^2]\psi_\lambda^{(0)}$. When the exact proportionality $\psi_\lambda^{(1)} \sim i\psi_\lambda^{(0)}$ holds, the forcing term in Eq. (4.12) vanishes. Therefore, this term is proportional to $\delta\psi_\lambda^{(1)}$ and the equivalent frequency shift can be estimated by

$$\delta\lambda_1 \sim \omega_{pe} L_2 a^2 \frac{\psi_\lambda^{(1)}}{\psi_\lambda^{(0)}} \frac{\delta\psi_\lambda^{(1)}}{\psi_\lambda^{(0)}} \sim \frac{\gamma^2(a)}{\lambda}. \tag{6.13}$$

The quantity $\delta\lambda_1 \sim a^{-1}$ grows more slowly than $\omega_1 \sim \omega_{pe} L_1 a^2 \sim a^{-2}$ during the process of the collapse, and therefore the condition (4.13) is satisfied.

It can be easily shown that because of the proximity of $\psi_\lambda^{(1)}$ to $ig(a_e)\psi_\lambda^{(0)}$ in the central region of the caviton, the condition for the validity of the real-field approximation in calculating the growth rate of the small-scale modulational instability is also weakened, i.e., we can multiply the left-hand side of (5.12) by a small number $\delta\psi_\lambda^{(1)}/\psi_\lambda^{(1)}$. The corrected validity condition is satisfied.

It has been shown above that the field $\psi_\lambda^{(0)}$ is well localized in the direction of the flattening of the caviton, i.e., it decreases rapidly at distances $|z| \gg a(r)$ from the plane $z = 0$. This assumption must be justified, since the "binding energy" λ of the plasmons is small compared to the "depth of the potential well" $\omega_{pe}(n/n_0)$, and as in the theory of the linear Schrödinger equation, we may suspect a poor localization. The next section is devoted to the clarification of this question.

7. SELF-SIMILAR SOLUTION

According to the estimates of the previous section, the solutions of Eqs. (4.3) and (4.4) in the case of an evolved electron nonlinearity at distances $r \ll a_e$ from the axis of the caviton, can be written in the form

$$\begin{gathered} \psi_\lambda^{(0)}(\mathbf{r}, t) = (24 n_0 T)^{1/2} a_e \tau \chi(\zeta, \boldsymbol{\rho}), \\ n(\mathbf{r}, t) = 3 (r_D/a_e)^2 n_0 \tau^{-4} u(\zeta, \boldsymbol{\rho}), \\ \tau = (2/\pi)^{1/2} \omega_{pi} (t_s - t), \quad \zeta = \tau^{-2} z/a_e, \quad \boldsymbol{\rho} = \tau^{-3/2} \mathbf{r}_\perp / a_e, \\ \lambda \approx \text{const} \ll \omega_{pe} n/n_0. \end{gathered} \tag{7.1}$$

Here, the functions u and χ obey the equations

$$\left(5 + 2\zeta \frac{\partial}{\partial\zeta} + \frac{3}{2}\rho \frac{\partial}{\partial\rho}\right)\left(4 + 2\zeta \frac{\partial}{\partial\zeta} + \frac{3}{2}\rho \frac{\partial}{\partial\rho}\right) u = \frac{\partial^2}{\partial\zeta^2}\left(\frac{\partial\chi}{\partial\zeta}\right)^2, \tag{7.2}$$

$$\frac{\partial^4\chi}{\partial\zeta^4} - \frac{\partial}{\partial\zeta}\left(u \frac{\partial\chi}{\partial\zeta}\right) + \frac{1}{2}\frac{\delta\mathcal{H}}{\delta\chi} = 0, \tag{7.3}$$

$$\mathcal{H} = \int d\zeta\, d^2\rho \left\{ \frac{2}{3}\left(\frac{\partial F}{\partial\zeta}\right)^2 + 3\left(\frac{\partial\chi}{\partial\zeta}\right)^2 [\Delta(\nabla\chi)^2 - 2\nabla(\nabla\chi\,\Delta\chi)] \right\}, \tag{7.4}$$

$$\frac{\partial^2 F}{\partial\zeta^2} = 2\frac{\partial}{\partial\zeta}\left(\frac{\partial\chi}{\partial\zeta}\Delta\chi\right) - \Delta\left(\frac{\partial\chi}{\partial\zeta}\right)^2,$$

where ∇ denotes differentiation with respect to $\boldsymbol{\rho}$. The function u can be expressed via χ if we use Eq. (7.2). For this purpose let us introduce the variables R and θ:

$$\zeta = R\cos\theta, \quad \rho^{4/3} = R\sin\theta \tag{7.5}$$

and integrate the resulting equation with respect to R. As a result we obtain the following equation

$$u = \frac{1}{2R^2}\left\{ \int_0^R dR_1 R_1 G(R_1, \theta) - \frac{1}{R^{1/2}} \int_0^R dR_1 R_1^{3/2} G(R_1, \theta) \right\}, \tag{7.6}$$

$$G(R, \theta) = \frac{\partial^2}{\partial\zeta^2}\left(\frac{\partial\gamma}{\partial\zeta}\right)^2 \Big|_{\zeta = R\cos\theta,\ \rho = (R\sin\theta)^{3/4}}.$$

In the region $R \gg 1$, the function $G(R, \theta)$ decreases rapidly (see Section 6 and below), and asymptotically u has the form

$$u(R, \theta) \approx \frac{1}{2R^2} \int_0^\infty dR_1 R_1 G(R_1, \theta). \tag{7.7}$$

The condition $r \ll a_e$, for which the equations (7.1) are valid, corresponds to $R \ll R_M \sim \tau^{-2} \to \infty$, at $\tau \to 0$.

If we use the relations (7.4) and (7.6), then the only remaining unknown function is $\chi(\zeta, \boldsymbol{\rho})$, which satisfies the nonlinear integral equation (7.3). Even in the axisymmetric case, this equation is two-dimensional and too complicated for a complete investigation. Considerable simplification takes place in the region $\rho \gg 1$, where spatial self-similarity holds:

$$\begin{gathered} \chi(\zeta, \rho) = \rho^{2/3}\chi_1(\xi), \quad \xi = \zeta/\rho^{4/3}, \\ u(\zeta, \rho) = \rho^{-8/3} u_1(\xi), \quad F(\zeta, \rho) = \rho^{-2/3} F_1(\xi). \end{gathered} \tag{7.8}$$

According to Eq. (7.7),

$$u_1(\operatorname{ctg}\theta) = \frac{1}{2}\sin^2\theta \int_0^\infty dR\, RG(R, \theta). \tag{7.9}$$

The main contribution to the integral (7.9) comes from the region $R \lesssim 1$, and therefore the function $u_1(\xi)$ is not related directly to $\chi_1(\xi)$ and can be assumed known in the region $\rho \gg 1$. Equation (7.4) can be integrated once in the region $\rho \gg 1$ without violating its local nature

$$\frac{dF_1}{d\xi} = \frac{8}{9}\frac{d\chi_1}{d\xi}\left(\chi_1 - 2\xi\frac{d\chi_1}{d\xi}\right). \tag{7.10}$$

This is the result of the constancy of the integral (6.3). The function $\chi_1(\xi)$ obeys the following equation:

$$\frac{d^4\chi_1}{d\xi^4} - \frac{d}{d\xi}\left(u_1\frac{d\chi_1}{d\xi}\right) + \frac{16}{27}\left\{ \frac{4}{\xi}\frac{d}{d\xi}\left(\xi^2\frac{d\chi_1}{d\xi}\frac{dF_1}{d\xi}\right) - \frac{d^2}{d\xi}(\chi_1 F_1) + 3\left(2\xi\frac{d\chi_1}{d\xi} - \chi_1\right)\frac{d}{d\xi}\left(\frac{d\chi_1}{d\xi}\frac{d\xi\chi_1}{d\xi}\right)\right\} = 0. \tag{7.11}$$

At $\xi \to \pm\infty$ the function $\chi_1(\xi)$ must approach constant values and $F_1(\xi)$ converges to zero.[5)] By linearizing Eq. (7.11) with respect to the zero order solution $\chi_1 = C$ and $F_1 = 0$, it can be easily found out that, at $\xi \to \pm\infty$, the small perturbation has the form

$$\begin{gathered} \delta\chi_1 = C_1 + C_2\xi + C_3 e^{p\xi} + C_4 e^{-p\xi}, \\ \delta F_1 = {}^8/_9 C\delta\chi_1 + C_5, \quad p = {}^4/_9 (35/3)^{1/2} |C|. \end{gathered} \tag{7.12}$$

The solution of Eq. (7.11) which is regular at $\xi \to +\infty$

$$\chi_1 = C + C_4 e^{-p\xi}, \quad F_1 = {}^8/_9 C C_4 e^{-p\xi} \tag{7.13}$$

depends on two parameters C and C_4. These parameters must be chosen so that the values of function $\chi_1(\xi)$ and its second derivative vanish at $\xi = 0$:

$$\chi_1(0)=0, \quad \left.\frac{d^2\chi_1}{d\xi^2}\right|_{\xi=0}=0. \tag{7.14}$$

Then the function $\chi_1(\xi)$ is odd and $F_1(\xi)$ is even and the regularity conditions on the solution at $\xi\to-\infty$ will also be satisfied. This cannot be achieved without requiring that $\chi_1(\xi)$ have a specific parity, because there exist three regularity conditions at each of the infinities (for example, at $\xi\to+\infty$ we have $C_2=C_3=C_5=0$). An even function $\chi_1(\xi)$ is unacceptable, since the electric field at the center of the caviton vanishes in this case.

We notice that for arbitrary parameters C and C_4 the function χ_1 has a singularity of type

$$\chi_1(\xi)=\pm\frac{9}{4\xi_s}\left(\frac{3}{14}\right)^{1/2}\ln|\xi-\xi_s|, \quad \xi\to\xi_s \tag{7.15}$$

at a certain point ξ_s. The condition $\xi_s<0$ defines a region in the plane (C, C_4) where the solution of Eq. (7.11) can be analytically continued from the asymptotic form (7.13) to zero. Avoiding the solution which is singular at zero,

$$\chi_1(\xi)|_{\xi\approx0}=\pm\frac{3\sqrt{3}}{4\sqrt{2}}\frac{1}{\xi} \tag{7.16}$$

also does not require additional parameters, since in the five-dimensional space of small perturbations of the solution (7.16), this solution can only be approached along four directions and therefore occurs on distinct lines in the plane (C, C_4).

Thus, the investigation of the self-similar solution in the spatial similarity region $\delta\gg1$, provides evidence of the exponential localization of this solution in the caviton.

The caviton stability condition with respect to small-scale perturbations (5.14), has been already discussed above in connection with the estimates. A simple quantitative form of this condition, i.e., $R_k\ (r_s, t)<20(t_s-t)^{-2}$, or, equivalently,

$$\left(\frac{\partial\chi}{\partial\zeta}\right)^2\Big|_{\zeta=\rho=0}<10 \tag{7.17}$$

can easily be obtained, if we use the self-similar solution (7.1).

7. CONCLUSIONS

The present study shows that the electron nonlinearity not only does not halt the collapse of Langmuir waves, but maintains the explosive nature of this process. The most fundamental contribution to the theory of strong Langmuir turbulence consists in the discovery of a universal collapse law in the inertial region, by taking the electron nonlinearity into account and in particular, the discovery of the fact that the nonlinearity does not give rise to the escape of the plasmons from the caviton. This result allows us to construct the spectrum of strong Langmuir turbulence in a previously uninvestigated domain of the inertial region in immediate proximity to the dissipative region. Convenient representations of the Hamiltonian of the electron nonlinearity, obtained above, provide a basis for numerical modeling of the collapse of nonlinear Langmuir waves and for investigating its dynamics, including dissipative processes.

[1)]This circumstance was noticed by V. V. Yan'kov.
[2)]The ion temperature, for simplicity, is assumed to be less than the electron temperature.
[3)]The problem of the evolution of a one-dimensional Langmuir wave in the absence of electron thermal motion and ion density perturbations, as is well known, is conveniently solved in Lagrangian coordinates. The solution shows that the envelope of all the harmonics in (2.1) is time-independent. Therefore, the Hamiltonian of the electron nonlinearity vanishes exactly on one-dimensional fields (and not only to fourth order in the field amplitude).[13]
[4)]It is useful to notice that conservation of H_e, in the absence of a linear dispersion, does not halt the unlimited growth of the energy density of the waves.
[5)]The potential drop in the caviton takes place at $r\gtrsim a_e$, i.e., outside the region of interest. The requirement $F_1(\xi)\to0$ at $\xi\to\infty$ is necessary for the localization of the electron nonlinearity in the caviton.

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