Electron temperature in a two-dimensional gas: Energy losses to optical phonons

S. E. Esipov and I. B Levinson

Solid State Physics Institute and Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 19 July 1985)
Zh. Eksp. Teor. Fiz. **90**, 330–344 (January 1986)

The distribution function is calculated for hot electrons in the ground state of a quantum well, in the energy region near or below the threshold for emission of optical phonons, under the assumption that below this threshold an electron temperature can be defined. The rate of energy loss (power) to optical phonons is also calculated.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The idea of an electron temperature T_e is a very popular one, providing as it does a convenient method of describing the behavior of nonequilibrium electron gases, including the two-dimensional electron gas.[1–4] It is meaningful to define an electron temperature T_e which differs from the lattice temperature T_L when the electron-electron scattering time τ_{ee} is smaller than the energy relaxation time to the lattice $\tilde{\tau}_L$. The electron temperature T_e is determined from the equation of energy balance $P = Q$, where P is the energy deposited in the electron gas by external fields or optical pumping and Q is the energy transferred to the lattice by phonon scattering. In the case where there is scattering by optical phonons of energy $\hbar\Omega_0$ and T_e, $T_L \ll \hbar\Omega_0$, calculation of Q meets with special difficulties. This is true because in the energy region below threshold ($\varepsilon < \hbar\Omega_0$), where emission of phonons is impossible, a temperature T_e can be defined even for low electron densities $n \gg n_c^-$, when $\tau_{ee} \ll \tilde{\tau}_A$ and $\tilde{\tau}_A$ is the energy relaxation time due to acoustic phonons. Meanwhile, above threshold ($\varepsilon > \hbar\Omega_0$) the temperature T_e can be defined only for high electron densities $n \gg n_c^+$, when $\tau_{ee} \ll \tau_O$ and $\tau_0(\varepsilon)$ is the emission time for optical phonons of energy $\hbar\Omega_0$. Therefore, in order to calculate Q when $n_c^- \ll n \lesssim n_c^+$ it is necessary to find the energy distribution function $f(\varepsilon)$ above threshold, where it deviates strongly from a Maxwellian distribution which would define T_e. This problem first received attention in Ref. 5 (see also Refs. 6 and 7). For the three-dimensional electron gas, Q was calculated in Ref. 8; in the present work, this problem is solved for a two-dimensional gas in a quantum well. In the two-dimensional case the problem appears to be more complicated, because, as we will show below, the integral for *e-e* scattering cannot be reduced to the Landau differential form, so that in place of a differential equation we must solve an integral equation.

An additional complexity in the equation of energy balance arises in the case when the electrons are heated optically; absorption of a photon can project them into the conduction band near a point ε_0 close to threshold, $|\varepsilon_0 - \hbar\Omega_0| \ll \hbar\Omega_0$. In the three-dimensional case this problem was studied in Ref. 9.

We study here a layer of narrow-gap semiconductor (of thickness d) inserted between two semi-infinite wide-gap semiconductors. In the layer d the electrons reside in a so-called quantum well.[10] Let us assume that the energy spacing between the two lowest levels of the well $E_2 - E_1 = 3\pi^2\hbar^2/2md^2 \gg \hbar\Omega_0$. When $T_e \ll \hbar\Omega_0$, we can neglect all levels in the well except the lowest.

Let us neglect any mismatch in the properties of the narrow- and wide-gap semiconductor lattices. This allows us to treat the phonon spectrum and electron-phonon interaction operator U_{eL} in the same way as in the usual three-dimensional case. Neglecting also disparities in the dielectric permittivity $\varkappa$, we will use an *e-e* interaction operator of the form $U_{ee} = e^2/\varkappa R$, where $\mathbf{R}$ is the three-dimensional distance between electrons. The matrix elements U_{eL} and U_{ee} are calculated using the electronic wave functions

$$\psi(r,\ z) = 1/L\chi(z)\,e^{i\mathbf{kr}}, \tag{1}$$

where $\mathbf{r}$, $\mathbf{k}$ are two-dimensional vectors in the plane of the layer; $\chi(z)$ is a function describing the quantized motion of an electron transverse to the layer, localized in the well and normalized to 1; L^2 is the area of the layer.

We will assume that if there were no interactions with phonons all the electrons would be distributed according to a Maxwellian:

$$f(\varepsilon) = (4\pi N/p_{T_e}^2)\,e^{-\varepsilon/T_e} \equiv f_{T_e}(\varepsilon),$$
$$\varepsilon = \varepsilon_k = \hbar^2k^2/2m, \quad \hbar^2 p_{T_e}^2 = 2mT_e, \tag{2}$$

where N is the number of electrons in 1 cm^2 of the layer.

When we take into account emission of phonons of energy $\hbar\Omega_0$, the distribution deviates from Maxwellian. If $T_e \ll \hbar\Omega_0$, then the deviation is weak in an "integrated" sense, because only the region near and above threshold—where there are few electrons—will be affected by the phonon emission, although in this region the distribution may be strongly modified. Meanwhile, in the region above threshold, the $f(\varepsilon)$ in question is determined by Q, which is connected to optical phonon emission.

The goal of this paper is to find $f(\varepsilon)$ near and above threshold, while also calculating Q, which is related to optical phonon scattering.

2. ELECTRON-ELECTRON SCATTERING PROBABILITY

If all the electrons are found in the lowest level of the quantum well, and have kinetic energy $\varepsilon = \hbar^2k^2/2m$ much smaller than the spacing between consecutive well levels, i.e., $k \ll d^{-1}$, then the Coulomb interaction matrix element for the function (1) does not depend on $\chi(z)$:

$$\left\langle \mathbf{k'p'} \left| \frac{e^2}{\varkappa R} \right| \mathbf{kp} \right\rangle = \frac{1}{L^2}\,\delta_{\mathbf{k'+p'-k-p}}\,\frac{2\pi e^2}{\varkappa q} \equiv M_{\mathbf{k'},\mathbf{p'};\,\mathbf{k},\mathbf{p}}, \tag{3}$$

where $\mathbf{q} = \mathbf{k'} - \mathbf{k} = \mathbf{p} - \mathbf{p'}$ is the two-dimensional momentum transfer.

 0038-5646/86/010191-09$04.00

Let us find the scattering probability

$$W_{\mathbf{k}\to\mathbf{k}'}=\sum_{p,p'} f_p \frac{2\pi}{\hbar}|M_{\mathbf{k}',\mathbf{p}';\,\mathbf{k},\mathbf{p}}|^2\,\delta(\varepsilon_{k'}+\varepsilon_{p'}-\varepsilon_k-\varepsilon_p) \qquad (4)$$

for a sample electron in a Maxwellian distribution of background electrons f_{T_e} (2).

The probability $W_{\mathbf{k}\to\mathbf{k}'}$ is conveniently calculated for $\varepsilon_k>\varepsilon_{k'}$; to calculate $W_{\mathbf{k}\to\mathbf{k}'}$ for $\varepsilon_k<\varepsilon_{k'}$, we use the principle of detailed balance.[7] As a result, we obtain

$$W_{\mathbf{k}\to\mathbf{k}'}=\pi^{1/2}W_0\frac{p_{T_e}^3}{q^3}\exp\left\{-\frac{1}{4p_{T_e}^2}\left[q-\frac{k^2-k'^2}{q}\right]^2\right\}, \qquad (5)$$

where

$$W_0=8\pi^2\frac{N}{L^2p_{T_e}^4}\frac{E_B}{\hbar}, \qquad E_B=\frac{me^4}{2\varkappa^2\hbar^2}. \qquad (6)$$

For isotropic or nearly isotropic electron distributions, the average of (5) over directions, $\mathbf{k}$, $\mathbf{k}'$ enters into the calculation:

$$W(\varepsilon\to\varepsilon')=\frac{1}{\pi}\int_0^\pi d\chi\, W_{\mathbf{k}\to\mathbf{k}'}, \qquad \varepsilon\equiv\frac{\hbar^2k^2}{2m}, \qquad (7)$$

where χ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. So as to perform the integration over χ, let us transform (for $\varepsilon>\varepsilon'$) to the variable $u=q/(k^2-k'^2)^{1/2}$ and introduce the static and dynamic quasi-elasticity parameters[7] ω and γ according to

$$\omega=\frac{\varepsilon-\varepsilon'}{T_e}, \qquad \gamma=\left|\frac{k-k'}{k+k'}\right|^{1/2}<1. \qquad (8)$$

Again, using the principle of detailed balance, we obtain for all ω

$$W(\varepsilon\to\varepsilon')=W_0|\omega|^{-1/2}e^{1/2\omega}\Psi(\gamma,\omega), \qquad (9)$$

where

$$\Psi(\gamma,\omega)=\frac{2}{\pi^{1/2}}\int_\gamma^{\gamma^{-1}}\frac{du}{u^2}\left[(u^2-\gamma^2)\left(\frac{1}{\gamma^2}-u^2\right)\right]^{-1/2}\times\exp\left\{-\frac{|\omega|}{4}\left(u^2+\frac{1}{u^2}\right)\right\}. \qquad (10)$$

Let us consider in detail the scattering probability for superthermal electrons $\varepsilon\gg T_e$. We note that in this case

$$\gamma^2/|\omega|=p_{T_e}^2/(k+k')^2\ll 1. \qquad (11)$$

We first examine the case of small γ. It can be verified that the conditions $\gamma\ll 1$ and $\gamma^2/|\omega|\ll 1$ allow us to set $\gamma=0$ in the integral (10), which gives

$$\Psi(\gamma,\omega)=2\pi^{-1/2}\gamma K_1(|\omega|/2), \qquad (12)$$

where K_1 is the modified Bessel function of the second kind. From this it follows that in the region of dynamic quasi-elasticity the scattering of superthermal electrons is

$$W(\varepsilon\to\varepsilon')=W_0K(\omega)(\varepsilon/T_e)^{-1/2}, \quad \varepsilon\gg T_e, \quad |\varepsilon-\varepsilon'|\ll\varepsilon, \qquad (13)$$

where

$$K(\omega)\equiv\pi^{-1/2}e^{1/2\omega}|\omega|^{-1}K_1(|\omega|/2). \qquad (14)$$

For static quasi-elastic scattering, the probability is symmetric relative to energy transfer

$$K(\omega)=2\pi^{-1/2}|\omega|^{-2}, \quad |\omega|\ll 1, \qquad (15)$$

while for static inelastic scattering it is strongly asymmetric

$$K(\omega)=\begin{cases}\omega^{-3/2}; & \omega>0, \quad \omega\gg 1,\\ |\omega|^{-3/2}e^{-|\omega|}; & \omega<0, \quad |\omega|\gg 1.\end{cases} \qquad (16)$$

We now turn to the case of large $|\omega|$. When $|\omega|\gg 1$, most of the integral (10) comes from a small region around the saddle point $u=1$. If $|\omega|^{-1}\ll 1-\gamma$, the exponent can be expanded around $u=1$. Then we obtain

$$\Psi(\gamma,\omega)=e^{-1/2|\omega|}|\omega|^{-1/2}(\gamma^{-1}-\gamma)^{-1}\times\{\mathrm{erf}\,[|\omega|^{1/2}(1-\gamma)/\gamma]+\mathrm{erf}\,[|\omega|^{1/2}(1-\gamma)]\}. \qquad (17)$$

From this it follows that

$$\Psi(\gamma,\omega)=\begin{cases}2|\omega|^{-1/2}e^{-1/2|\omega|}(\gamma^{-1}-\gamma)^{-1}, & 1-\gamma\gg|\omega|^{-1/2},\\ 2\pi^{-1/2}e^{-1/2|\omega|}, & |\omega|^{-1}\ll 1-\gamma\ll|\omega|^{-1/2}.\end{cases} \qquad (18)$$

These expressions for Ψ allow us to find the probability for dynamic inelastic scattering of superthermal electrons. For the transitions downward we have

$$W(\varepsilon-\varepsilon')=\begin{cases}\dfrac{1}{L^2}\dfrac{2\pi^2\hbar^3E_BN}{m^2}\dfrac{1}{\varepsilon'^{1/2}(\varepsilon-\varepsilon')^{3/2}}, & \varepsilon,\varepsilon',\varepsilon-\varepsilon'\gg T_e;\\ \dfrac{1}{L^2}\dfrac{4\pi^{3/2}\hbar^3E_BN}{m^2}\dfrac{1}{T_e^{1/2}\varepsilon^{3/2}}, & \varepsilon\gg T_e, \quad \varepsilon'\ll T_e.\end{cases} \qquad (19)$$

The probability expression given in the top line of (19) does not depend on T_e, and, as is easy to verify, coincides with the scattering probability for $T_e=0$. The upward scattering probability is obtained from the principle of detailed balance

$$W(\varepsilon\to\varepsilon')=\frac{1}{L^2}\frac{2\pi^2\hbar^3E_BN}{m^2}\frac{1}{\varepsilon^{1/2}(\varepsilon'-\varepsilon)^{3/2}}e^{-(\varepsilon'-\varepsilon)/T_e}; \quad \varepsilon\gg T_e, \quad \varepsilon'-\varepsilon\gg T_e. \qquad (20)$$

The dependence of W on ε' for fixed $\varepsilon\gg T_e$ is shown in Fig. 1.

As regards the scattering probability for thermal electrons $\varepsilon\sim T_e$, a relatively simple formula can be obtained only when $|\omega|\ll 1$. Noting that in this case $\gamma^2=(T_e/4\varepsilon)|\omega|\ll 1$, we can see that the integral (10) is concentrated around $u\sim\gamma$. Neglecting u compared to $1/\gamma$ or $1/u$, we obtain

$$\Psi(\gamma,\omega)=2|\omega|^{-1/2}e^{-\varepsilon/T_e}\times\mathrm{erfi}\,[(\varepsilon/T_e)^{1/2}],$$
$$\mathrm{erfi}(x)=\frac{2}{\pi^{1/2}}\int_0^x dt\,e^{t^2}. \qquad (21)$$

From this, it is clear that the probability $W(\varepsilon\to\varepsilon')$ for thermal electrons as $\omega\to 0$ has the same Coulomb singularity

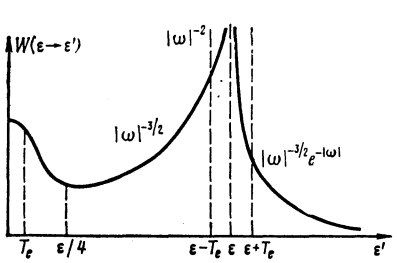

FIG. 1. The probability $W(\varepsilon\to\varepsilon')$ for an electronic transition from energy ε to energy ε' due to interelectron scattering in a 2D gas with a Maxwellian distribution with temperature T_e, $\varepsilon\gg T_e$.

$|\omega|^{-2}$ as does the superthermal electron W.

The first two moments of the transition probability are determined by integrating:

$$Q_{ee}(\varepsilon)=\int_{-\varepsilon}^{\infty} d\tilde{\omega}\,\tilde{\omega} g(\varepsilon+\tilde{\omega}) W(\varepsilon\to\varepsilon+\tilde{\omega}), \quad \tilde{\omega}=-\omega T_e, \tag{22}$$

$$D_{ee}(\varepsilon)=\frac{1}{2}\int_{-\varepsilon}^{\infty} d\tilde{\omega}\,\tilde{\omega}^2 g(\varepsilon+\tilde{\omega}) W(\varepsilon\to\varepsilon+\tilde{\omega}), \tag{23}$$

where $g(\varepsilon)$ is the density of states. In the two-dimensional case, $g(\varepsilon)=(m/2\pi\hbar^2)L^2$. The convergence of these integrals in the two-dimensional gas is determined by the convergence of the integrals

$$\int d\omega \frac{\omega}{|\omega|^2}=\int\frac{d\omega}{\omega}, \quad \int d\omega \frac{\omega^2}{|\omega|^2}=\int d\omega,$$

where the first is understood in the sense of a principal value (this becomes clear if we introduce a small cutoff around the $|\omega|^{-2}$ singularity). The integrals for Q_{ee} and D_{ee} are convergent. When we calculate them, it is clear that a characteristic energy transfer is $|\omega|\sim\varepsilon$ for a downward transition ($\omega<0$ and $|\omega|\sim T_e$ for an upward ($\omega>0$). From these arguments, it is clear that a Fokker-Planck approximation for the *e-e* scattering integral in the two-dimensional gas, analogous to the one obtained by Landau[11] for the three-dimensional case, cannot be derived here. We remind the reader that in the three-dimensional case the singularity of $W(\varepsilon\to\varepsilon')$ is $|\omega|^{-3}$, the moments Q_{ee} and D_{ee} diverge, and even after cutoff there still remain singularities proportional to a large Coulomb logarithm, whereas the higher moments are finite and do not contain this logarithm.

The coefficient of dynamic viscosity is

$$A_{ee}(\varepsilon)=Q_{ee}(\varepsilon)|_{T_e=0}=\pi^2\hbar E_B N/m. \tag{24}$$

If it is cast in the form

$$A_{ee}(\varepsilon)=\frac{\varepsilon}{\tau_{ee}(\varepsilon)}, \quad \frac{1}{\tau_{ee}(\varepsilon)}=\frac{\pi^2\hbar E_B N}{m\varepsilon}=2\pi^2\frac{E_B}{\hbar}\frac{N}{k^2}, \tag{25}$$

then τ_{ee} will be the *e-e* scattering time, determining the rate of energy exchange between electrons. In order to estimate this time, we can allow ε to equal T_e. Then τ_{ee} is the time it takes to establish an electron temperature.

Further on we will need the probability for scattering near the threshold $\varepsilon=\hbar\Omega_0$, when $|\varepsilon-\hbar\Omega_0|$, $|\varepsilon'-\hbar\Omega_0|\ll\hbar\Omega^0$. From (13) we have

$$W(\varepsilon\to\varepsilon')=\frac{1}{L^2}\frac{1}{\bar{\tau}_{ee}}\frac{4}{p_0^2}\left(\frac{T_e}{\hbar\Omega_0}\right)^{-1/2}K(\omega),$$
$$\bar{\tau}_{ee}=\tau_{ee}(\varepsilon)|_{\varepsilon=\hbar\Omega_0}, \quad \hbar^2p_0^2=2m\hbar\Omega_0. \tag{26}$$

3. KINETIC EQUATION

Electron-electron scattering is described by a collision integral

$$C_{ee}(\varepsilon)=-\frac{1}{g}\frac{d}{d\varepsilon}[gJ_{ee}(\varepsilon)]$$
$$=\int_0^{\infty} d\varepsilon' g[f(\varepsilon')W(\varepsilon'\to\varepsilon)-f(\varepsilon)W(\varepsilon\to\varepsilon')], \tag{27}$$

where W is the *e-e* scattering probability and J_{ee} the current which arises from it along the energy axis

$$-gJ_{ee}(\varepsilon)=\sum_{\substack{\mathbf{k}',\mathbf{k}''\\ \varepsilon_{k''}>\varepsilon,\ \varepsilon_{k'}<\varepsilon}}[f_{\mathbf{k}''}W_{\mathbf{k}''\to\mathbf{k}'}-f_{\mathbf{k}'}W_{\mathbf{k}'\to\mathbf{k}''}]$$
$$=\int_{\varepsilon}^{\infty} d\varepsilon'' g\int_0^{\varepsilon} d\varepsilon' g[f(\varepsilon'')W(\varepsilon''\to\varepsilon')-f(\varepsilon')W(\varepsilon'\to\varepsilon'')]. \tag{28}$$

Emission of optical phonons corresponds to collision terms

$$S(\varepsilon)=-\frac{f(\varepsilon)}{\tau_0(\varepsilon)}+B(\varepsilon), \quad B(\varepsilon)=\frac{f(\varepsilon+\hbar\Omega_0)}{\tau_0(\varepsilon+\hbar\Omega_0)}, \tag{29}$$

where $\tau_0(\varepsilon)$ is the emission time; absorption proportional to $e^{-\hbar\Omega_0/T_e}$ is neglected.

The probability W is a functional of the distribution $f(\varepsilon)$; however, since $f(\varepsilon)$ differs (in an "integrated" sense) only slightly from a Maxwellian (2), for W near threshold we can use expression (26). Let us put the distribution in the form

$$f(\varepsilon)=A[e^{-t}-\varphi(t)], \quad A=f_{T_e}(\varepsilon)|_{\varepsilon=\hbar\Omega_0}, \quad t=(\varepsilon-\hbar\Omega_0)/T_e. \tag{30}$$

For physical reasons, above threshold ($t>0$)

$$0<\varphi(t)<e^{-t}. \tag{31}$$

As regards corrections below threshold ($t<0$), we assume (and this is confirmed by solving the kinetic equation) that for $t\to-\infty$ the function $\varphi(t)$ decreases to zero on a characteristic scale $|t|\sim 1$, i.e.,

$$\lim_{t\to-\infty}\varphi(t)=0, \tag{32}$$

$$\varphi(t)/\varphi(0)\ll 1, \quad |t|\gg 1. \tag{33}$$

Substituting (30) into (27) with W from (26), we obtain

$$C_{ee}(\varepsilon)=\frac{1}{\pi}\frac{1}{\bar{\tau}_{ee}}\left(\frac{T_e}{\hbar\Omega_0}\right)^{-1/2}A\hat{K}\varphi(t), \tag{34}$$

where the integral operator is of the form

$$\hat{K}\varphi(t)=\int_{-\infty}^{\infty} dt'[\varphi(t)K(t-t')-\varphi(t')K(t'-t)]. \tag{35}$$

Using the properties of φ and K, namely (31), (33) and (14), we can verify that in the integral (35) the significant times satisfy $|t'|<1$. Therefore we can replace the lower limit $-\hbar\Omega_0/T_e$ by $-\infty$.

We turn now to S. Near threshold *we have* $B=0$; as regards $\tau_0(\varepsilon)$, near and above threshold $\tau_0(\varepsilon)$ does not depend on ε (since the density of final states does not depend on $\varepsilon-\hbar\Omega_0$). For deformation *DO* scattering with $\hbar\Omega_0\ll E_2-E_1$, we have near threshold that

$$\tau_0^{-1}=a(\pi/p_0d)\bar{\tau}_{DO}^{-1}, \tag{36}$$

where a is a numerical factor depending on the well shape. If we use the dimensionless quantities

$$\chi(z)=d^{-1/2}\varphi(\zeta), \quad \zeta=z/d, \quad \int d\zeta|\varphi(\zeta)|^2=1, \tag{37}$$

then

$$a=\int d\zeta|\varphi(\zeta)|^4. \tag{38}$$

For a square well with infinitely high walls, $a = 3/2$. For polarization *PO*-scattering there is no dependence on the shape of the well[12]:

$$\tau_0^{-1} = (\pi/2)\,\bar{\tau}_{PO}^{-1}. \tag{39}$$

In (36) and (39), $\bar{\tau}_{DO}$ and $\bar{\tau}_{PO}$ are nominal scattering times.[7]

Thus, near threshold,

$$S(\varepsilon) = A\theta(t)\,[e^{-t} - \varphi(t)]/\tau_0, \tag{40}$$

where θ is the Heaviside step-function.

As a result, the equation for φ is

$$\lambda\pi^{-1/2}\hat{K}\varphi(t) = \theta(t)\,[e^{-t} - \varphi(t)], \tag{41}$$

where the parameter which describes the competition between *ee* scattering and optical phonon emission near threshold is

$$\lambda = \frac{1}{\pi^{1/2}}\frac{\tau_0}{\bar{\tau}_{ee}}\left(\frac{T_e}{\hbar\Omega_0}\right)^{-1/2} = \frac{N}{N_c^+}. \tag{42}$$

This relation determines the critical density N_c^+ above which, as we will show, the distribution in the region $\varepsilon > \hbar\Omega_0$ differs only slightly from a Maxwellian. Now, to determine N_c^-, we must know the energy relaxation time for acoustic phonons $\bar{\tau}_A(\varepsilon)$. In order to find $\tilde{\tau}_A$, it is sufficient to calculate the power loss Q_A for $T_L = 0$ and cast it in the form[7]

$$Q_A|_{T_L=0} = \frac{\varepsilon}{\bar{\tau}_A(\varepsilon)}. \tag{43}$$

By direct calculation we show that for $\varepsilon \ll E_2 - E_1$,

$$\frac{1}{\bar{\tau}_{DA}(\varepsilon)} = \frac{\pi b}{4}\frac{1}{\bar{\tau}_{DA}}\frac{2ms^2}{\varepsilon}\frac{1}{(p_0 d)^3}, \tag{44}$$

$$\frac{1}{\bar{\tau}_{PA}(\varepsilon)} = \frac{\pi a}{2}\frac{1}{\bar{\tau}_{PA}}\frac{2ms^2}{\varepsilon}\frac{1}{(p_0 d)}, \tag{45}$$

where s is the mean sound velocity, $\bar{\tau}_{DA}$ and $\bar{\tau}_{PA}$ are nominal times for deformation and piezoelectric scattering by acoustic phonons,[7] the coefficient a is given by (38), and

$$b = \int d\zeta\,[d/d\zeta\,|\varphi(\zeta)|^2]^2. \tag{46}$$

For the square well, $b = 2\pi^2$.

The power loss to optical phonons $\hbar\Omega_0$, which is needed for the detailed balance equation, is calculated for a single electron to be

$$Q = \frac{\hbar\Omega_0}{\tau_0}e^{-\hbar\Omega_0/T_e}\Phi(\lambda), \tag{47}$$

where

$$\Phi(\lambda) = 1 - \int_0^\infty dt\,\varphi(t). \tag{48}$$

The integral depends only on λ because this is the only parameter which enters into the dimensionless kinetic equation (41).

4. SOLUTION OF THE KINETIC EQUATION

We solve equation (41) by the Wiener-Hopf technique.[13] To begin with, we set

$$\varphi(t) = \theta(t)\varphi^+(t) + \theta(-t)\varphi^-(t) \tag{49}$$

and perform a Fourier transform

$$\chi(t) \to \bar{\chi}(k) = \int_{-\infty}^{\infty} dt\,e^{-ikt}\chi(t). \tag{50}$$

Then

$$\hat{K}\chi(t) \to -\bar{K}(k)\bar{\chi}(k),$$

$$\bar{K}(k) = \int_{-\infty}^{\infty} dt\,K(t)\,(e^{ikt} - 1). \tag{51}$$

For the function $K(t)$ in expression (14) we find

$$\bar{K}(k) = -2\pi^{1/2}[k(k-i)]^{1/2}. \tag{52}$$

From the fact that $K(t) > 0$ it follows that the correct sign of the branch in (52) is determined from the condition

$$\operatorname{Re}\bar{K}(k) < 0, \quad \operatorname{Im} k = 0. \tag{53}$$

From (31) and (32) it follows that φ^+ has no singularities for $\operatorname{Im} k < 1$, while φ^- has none for $\operatorname{Im} k > 0$. After Fourier transforming Eq. (41) we obtain the form

$$-\lambda\pi^{-1/2}\bar{K}(k)\,[\bar{\varphi}^+(k) + \bar{\varphi}^-(k)] = (ik+1)^{-1} - \bar{\varphi}^+(k). \tag{54}$$

It is more convenient to go to a more symmetrical representation: let $z = x + iy$ where z is defined by $k = i(1-z)/2$. Then $\psi^+(z) = \bar{\varphi}^+(k)$ has no singularities for $x > -1$ while $\psi^-(z) = \bar{\varphi}^-(k)$ has none for $x < 1$. Equation (54) takes the form

$$\psi^+(z)H_\lambda(z) - \lambda^{-1}G(z) = -\psi^-(z), \tag{55}$$

where

$$H_\lambda(z) = 1 + \frac{1}{\lambda(1-z^2)^{1/2}}, \quad G(z) = \frac{2}{1+z}\frac{1}{(1-z^2)^{1/2}}. \tag{56}$$

The functions H_λ and G are determined in the plane with cuts $R^\pm$ as shown in Fig. 2. The following choice of arguments corresponds to condition (53):

$$|\arg(1 \pm z)| \leqslant \pi,$$
$$\operatorname{signarg}(1 \pm z) = \pm\operatorname{sign} y. \tag{57}$$

Following the Wiener-Hopf method, we write in the analyticity strip $|x| < 1$

$$H_\lambda(z) = H_\lambda^+(z)/H_\lambda^-(z), \tag{58}$$

where

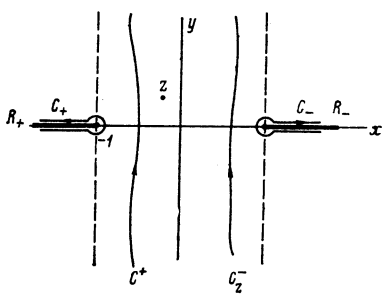

FIG. 2. Contour in the complex ζ-plane.

$$\ln H_\lambda^{\pm}(z) = -\frac{1}{2\pi i}\int\limits_{C_z^{\pm}} \frac{d\zeta}{\zeta - z}\ln H_\lambda(\zeta). \tag{59}$$

Here $C_z^{\pm}$ is a contour in the strip $|x| < 1$, going from $-i\infty$ to $+i\infty$ and passing the point z on the right (left) (see Fig. 2). Equation (55) now becomes

$$\psi^+(z)H_\lambda^+(z) + \psi^-(z)H_\lambda^-(z) = \lambda^{-1}G(z)H_\lambda^-(z). \tag{60}$$

It solution is

$$\psi^{\pm}(z) = F_\lambda^{\pm}(z)/H_\lambda^{\pm}(z), \tag{61}$$

where

$$F_\lambda^{\pm}(z) = \mp\frac{1}{2\pi i\lambda}\int\limits_{C_z^{\pm}} \frac{d\zeta}{\zeta - z}G(\zeta)H_\lambda^-(\zeta). \tag{62}$$

The functions $\psi^{\pm}(z)$ are determined from Eq. (55) up to an arbitrary polynomial $P(z)$; however, as will be seen, the choice of solution is dictated by the requirement that $\varphi(t)$ satisfy the physical conditions (31) and (32).

Since in $H_\lambda^{\pm}(z) \to 0$ for $z \to \infty$, the contour for $H_\lambda^{\pm}$ in (59) can be deformed arbitrarily as long as it does not cross the point z or the cuts. Deforming the contour for H_λ^+ into contour C_+, which surrounds the cut R^+ (Fig. 2), we define a function $H_\lambda^+(z)$ which is analytic in the entire z-plane with a cut R^+. Analogously, we determine a function $H_\lambda^-(z)$ analytic in the z-plane with cut R^-. After this continuation of the functions $H_\lambda^{\pm}$ from the strip $|x| < 1$, we can also deform the contours for $F_\lambda^{\pm}$ in (62) to the edges of the strip.

By carrying out a change in the variable of integration $\zeta \to -\zeta$ and $\zeta \to -\zeta^*$ in the integral representation (59) with the contours $C_\pm$, and noting that by doing this we interchange the contours C_+, C_-, it is easy to verify that

$$H_\lambda^-(z)H_\lambda^+(-z) = 1, \quad H_\lambda^-(z)H_\lambda^+(-z^*)^* = 1, \tag{63}$$

from which it follows that

$$H_\lambda^{\pm}(z^*) = H_\lambda^{\pm}(z)^*. \tag{64}$$

Because of property (63), we now have enough information to determine the function H_λ^+. To do this, it is convenient to study the function

$$h_\lambda(z) = H_\lambda^+(z)/H_\lambda^+(1). \tag{65}$$

In (59), we pick the integration contour C_+ for H_λ^+; taking into account condition (57), we find that

$$\ln h_\lambda(z) = \frac{1-z}{\pi}\int\limits_1^\infty \frac{ds}{(s+z)(s+1)}\operatorname{arctg}\frac{1}{\lambda(s^2-1)^{1/2}}. \tag{66}$$

It is easy to calculate

$$h_0(z) = \left(\frac{1+z}{2}\right)^{-1/2}, \tag{67}$$

where the argument is determined according to (57). Now it is natural to represent

$$H_\lambda^+(z) = H_\lambda^+(1)h_0(z)e^{Q_\lambda(z)}, \tag{68}$$

where

$$Q_\lambda(z) = \ln\frac{h_\lambda(z)}{h_0(z)} = \frac{z-1}{\pi}\int\limits_1^\infty \frac{ds}{(s+z)(s+1)}\operatorname{arctg}\lambda(s^2-1)^{1/2}. \tag{69}$$

Substituting into (62) the explicit form for G in (56), expressing $H_\lambda^-(z)$ as $H_\lambda^+(-z)$ with the help of (63) and representing H_λ^+ in the form (68), we obtain

$$F_\lambda^{\pm}(z) = \mp\frac{1}{2^{1/2}\pi\lambda i}\frac{1}{H_\lambda^+(1)}\int\limits_{C_z^{\pm}} \frac{d\zeta}{\zeta - z}\frac{1}{(1+\zeta)^{1/2}}e^{-Q_\lambda(-\zeta)}. \tag{70}$$

Here, $C_z^{\pm}$ is an arbitrary contour running from $-i\infty$ to $+i\infty$ and passing the point z on the right (left). Then the solution is explicit in terms of $H_\lambda^+(1)$ and $Q_\lambda(\zeta)$.

The loss power function (48) is

$$\Phi(\lambda) = 1 - \psi^+(1). \tag{71}$$

In order to calculate $\psi^+(z)$, it is convenient to use the following procedure. We note that

$$\int\limits_{C_z^+} \frac{d\zeta}{\zeta - z}\frac{1}{(1+\zeta)^{1/2}} = -\frac{2\pi i}{(1+z)^{1/2}}, \tag{72}$$

because this integral reduces to the residue at the point $\zeta = z$. Subtracting the integral (72) from the integral (70) for F_λ^+, we obtain

$$F_\lambda^+(z) = \frac{2^{1/2}}{\lambda H_\lambda^+(1)}\left\{\frac{1}{(1+z)^{1/2}} - \frac{1}{2\pi i}\int\limits_{C_z^+} \frac{d\zeta}{\zeta - z}\frac{1}{(1+\zeta)^{1/2}} \times [e^{-Q_\lambda(-\zeta)} - 1]\right\}. \tag{73}$$

Now we can deform C_z^+ into C_+, since the nonintegrable singularity $(1+\zeta)^{-3/2}$ at $\zeta = -1$ is eliminated by the vanishing of the quantity inside the square brackets. After transforming to the contour C_+, where $\zeta = -\xi \pm i0$,

$$F_\lambda^+(z) = \frac{2^{1/2}}{\lambda}\frac{1}{H_\lambda^+(1)}\left\{\frac{1}{(1+z)^{1/2}} + \frac{1}{\pi}\int\limits_1^\infty \frac{d\xi}{\xi + z}\frac{1}{(\xi-1)^{1/2}} \times [1 - e^{-Q_\lambda(\xi)}]\right\}. \tag{74}$$

Using this representation for $F_\lambda^+(1)$, we find $\psi^+(1)$ and the power loss function

$$\Phi(\lambda) = 1 - \frac{2^{1/2}}{\lambda}\frac{1}{H_\lambda^+(1)^2}\left\{\frac{1}{2^{1/2}} + \frac{1}{\pi}\int\limits_1^\infty \frac{d\xi}{(\xi+1)(\xi-1)^{1/2}} \times [1 - e^{-Q_\lambda(\xi)}]\right\}. \tag{75}$$

5. EVALUATION OF $H_\lambda^+(1)$

In order to evaluate $H_\lambda^+(1)$, we choose in (59) to integrate along the imaginary axis, which then gives us

$$\ln H_\lambda^+(1) = \frac{1}{\pi}\int\limits_0^\infty \frac{d\eta}{1+\eta^2}\ln\frac{1+\lambda(1+\eta^2)^{1/2}}{\lambda(1+\eta^2)^{1/2}} = J(\lambda) - \frac{1}{2}\ln 2\lambda, \tag{76}$$

$$J(\lambda) = \frac{1}{\pi}\int\limits_0^\infty \frac{d\eta}{1+\eta^2}\ln[1+\lambda(1+\eta^2)^{1/2}].$$

By differentiating $J(\lambda)$ with respect to λ, we can perform the

η-integral. Re-integrating this result with respect to λ and using the fact that $J(0) = 0$, we find

$$J(\lambda)=\frac{1}{\pi}\int_0^{\arcsin\lambda} dt\,\ln\operatorname{ctg}\frac{t}{2}, \quad \lambda\leqslant 1$$
$$J(\lambda)=\frac{2G}{\pi}+\frac{2}{\pi}\int_0^{\operatorname{Arch}\lambda} dt\,\operatorname{arctg}\operatorname{th}\frac{t}{2}, \quad \lambda>1, \tag{77}$$

where G is Catalan's constant. It is now easy to find the behavior of $H_\lambda^+(1)$ for small and large λ:

$$H_\lambda^+(1)=\frac{1}{(2\lambda)^{1/2}}\left(1+\frac{\lambda}{\pi}\ln\frac{2e}{\lambda}+\ldots\right), \quad \lambda\ll 1,$$
$$H_\lambda^+(1)=1+\frac{1}{\pi\lambda}+\ldots, \quad \lambda\gg 1. \tag{78}$$

6. THE FUNCTION $Q_\lambda(z)$

The integral (69) determines the function $Q_\lambda(z)$ in the z-plane with the cut $(-\infty, -1)$, across which the jump is

$$Q_\lambda(x+i0)-Q_\lambda(x-i0)=2i\operatorname{arctg}\lambda(x^2-1), \quad x<-1. \tag{79}$$

From this determination it is clear that

$$Q_0(z)=0, \quad Q_\lambda(1)=0.$$

Furthermore, the function $Q_\lambda(z)$ will be needed only for small λ, where it has the following representation, correct for all z:

$$Q_\lambda(z)=\frac{z-1}{z}\left\{q(\lambda z)-\frac{\lambda z}{\pi}[\Lambda(z)-\ln 2z]\right\}. \tag{80}$$

Here, the function

$$\Lambda(z)=(1+z)\int_0^\infty\frac{dt}{z+\operatorname{ch} t}=\left(\frac{z+1}{z-1}\right)^{1/2}\ln[z+(z^2-1)^{1/2}] \tag{81}$$

is determined in the z-plane with the cut $(-\infty, -1)$, so that for $z = x > -1$ it is positive. The behavior of this function is clear from the following asymptotic expansions

$$\Lambda(z)=\frac{\pi}{2}+\left(\frac{\pi}{2}-1\right)z, \quad |z|\ll 1,$$
$$\Lambda(z)=\left(1+\frac{2}{z}\right)\ln 2z, \quad |z|\gg 1,$$
$$\Lambda(z)=\frac{\pi}{2^{1/2}}(1+z)^{1/2}, \quad |z+1|\ll 1, \tag{82}$$
$$\Lambda(1)=2.$$

The function

$$q(z)=\frac{1}{\pi}\int_0^\infty\frac{dt}{1+t^2}\ln(1+zt) \tag{83}$$

is defined in the z-plane with a cut $(-\infty, 0)$, and is positive for $z = x > 0$. Computing $q'(z)$ from (83), we can do the t-integral, and then integrate the result with respect to z, using $q(0) = 0$. This gives

$$q(z)=\frac{1}{4}\ln(1+z^2)-\frac{1}{\pi}\int_0^z\frac{dt}{1+t^2}\ln t. \tag{84}$$

From this it is easily seen that

$$q(z)=\frac{z}{\pi}(1-\ln z), \quad |z|\ll 1,$$
$$q(z)=\frac{1}{2}\ln z, \quad |z|\gg 1. \tag{85}$$

Using the properties of q as a function of Λ, we can verify that

$$Q_\lambda(z)=q(\lambda z), \quad |z|\gg 1,$$
$$Q_\lambda(z)=\frac{\lambda}{\pi}(z-1)\left[\ln\frac{2e}{\lambda}-\Lambda(z)\right], \quad |z|\ll 1. \tag{86}$$

The representation (80) is obtained in the following way: differentiating (69), we find

$$\frac{\partial}{\partial\lambda}Q_\lambda(z)=\frac{z-1}{\pi}I(\lambda), \tag{87}$$

where

$$I(\lambda)=\int_1^\infty ds\,f_\lambda(s)\left(\frac{s-1}{s+1}\right)^{1/2},$$
$$f_\lambda(s)=\frac{1}{(s+z)[1+\lambda^2(s^2-1)]}. \tag{88}$$

To improve convergence, we write

$$I=I_1-I_2, \tag{89}$$

where

$$I_1(\lambda)=\int_1^\infty ds\,f_\lambda(s), \quad I_2(\lambda)=\int_1^\infty ds\,f_\lambda(s)\left[1-\left(\frac{s-1}{s+1}\right)^{1/2}\right]. \tag{90}$$

The integral I_2 is convergent for $\lambda = 0$, because (neglecting corrections which are small for arbitrary z) we have

$$I_2(\lambda)=I_2(0)=\Lambda(z)-\ln 2(1+z). \tag{91}$$

The integral I_1 can be evaluated explicitly. For small λ and any z, we find, keeping in mind (84), that

$$I_1(\lambda)=\frac{1}{1+\lambda^2z^2}\left[-\ln\lambda(z+1)+\frac{\pi}{2}\lambda z\right]$$
$$=\pi q'(\lambda z)-\frac{1}{1+\lambda^2z^2}\ln\frac{z+1}{z}. \tag{92}$$

Substituting (91) and (92) into (89) and then into (87), we integrate with respect to λ, taking into account (69). This gives

$$Q_\lambda(z)=\frac{z-1}{z}\left\{q(\lambda z)-\frac{1}{\pi}\operatorname{arctg}(\lambda z)\ln\frac{z+1}{z}\right.$$
$$\left.-\frac{\lambda z}{\pi}[\Lambda(z)-\ln 2(1+z)]\right\}. \tag{93}$$

Comparing the separate terms for various z, we can verify that the term with the arctangent is important only when $\lambda|z|\ll 1$. Therefore, arctan λz can be replaced by λz, after which (80) is obtained.

7. ENERGY DISTRIBUTION FOR SMALL λ

We first find $\psi^-(z)$ for $|z| \ll \lambda^{-1}$. To do this, the expression (70) for F_λ^- we deform C_z^- into C_-, while we derive H_λ^- in (61) from H_λ^+ and make use of (68) and (67). Then we obtain

$$\psi^-(z) = \frac{1}{\pi\lambda i}\frac{1}{(1-z)^{1/2}} e^{Q_\lambda(-z)} \int_1^\infty \frac{d\xi}{\xi - z}\frac{1}{(1+\xi)^{1/2}} \times [e^{-Q_\lambda(-\xi+i0)} - e^{-Q_\lambda(-\xi-i0)}]. \quad (94)$$

As will be clear in the subsequent calculation of the integral over ξ, for $|z| \ll \lambda^{-1}$ we can set $\xi \sim |z|$ if $|z| \gg 1$ and $\xi \sim 1$ if $|z| \lesssim 1$. Using (86) and (85), it can be shown that for such choices of ξ and z,

$$Q_\lambda(-z) \ll 1, \quad Q_\lambda(-\xi \pm i0) \ll 1. \quad (95)$$

Let us expand the exponent in the integrand and then use (79), replacing the arctangent by its argument. We can replace the exponent throughout the integral by unity; as a result, we obtain for $|z| \ll \lambda^{-1}$

$$\psi^-(z) = -\frac{2}{\pi}\frac{1}{(1-z)^{1/2}} \int_1^\infty \frac{d\xi}{\xi - z}\frac{(\xi-1)^{1/2}}{\xi+1} = -\frac{2}{1+z}\left[1 - \left(\frac{1-z}{2}\right)^{-1/2}\right]. \quad (96)$$

The asymptotic expansion for $\psi^+(z)$ for $|z| \ll \lambda^{-1}$ can be found once we substitute (96) into equation (55), which gives

$$\psi^+(z) = \frac{2}{1+z}\left[1 - 2\lambda\left(\frac{1+z}{2}\right)^{1/2}\right]^{-1}. \quad (97)$$

Approximations (96) and (97) allow us to find the asymptotic form of $\varphi^\pm(t)$ for $|t| \gg \lambda$. Performing the inverse Fourier transform, we find that

$$\varphi^+(t) = e^{-t}(1 - 2\lambda\pi^{-1/2}t^{-1/2}), \quad t \gg \lambda, \qquad \varphi^-(t) = e^{-t}\operatorname{erfc}|t|^{1/2}, \quad |t| \gg \lambda. \quad (98)$$

Let us now study the functions $\varphi^\pm(t)$ in the immediate vicinity of threshold, i.e., for $|t| \ll \lambda$. According to the general properties of Fourier transforms,

$$\varphi^\pm(0) = \pm 1/2 \lim_{z\to\infty} z\psi^\pm(z). \quad (99)$$

From Eq. (55) and the behavior of $H(z)$ and $G(z)$ at infinity, it is clear that for $z \to \infty$ we have $\psi^+(z) = -\psi^-(z)$. Therefore, in (99) it follows that $\varphi(t)$ is continuous at the point $t = 0$.

Let us now find the behavior of $\psi^+(z)$ as $z \to \infty$. Noting that $H_\lambda^+(\infty) = 1$, we have $\psi^+(z) = F_\lambda^+(z)$ for $z \to \infty$. For $F_\lambda^+(z)$ we use (74), replacing $\xi + z$ by z under the integral sign and discarding the first term in the curly brackets. Finally, substituting everything into (99), we obtain

$$\varphi(0) = \varphi^+(0) = a_\lambda/2^{1/2}\pi\lambda H_\lambda^+(1), \qquad a_\lambda = \int_1^\infty \frac{ds}{(s-1)^{1/2}}[1 - e^{-Q_\lambda(s)}]. \quad (100)$$

To lowest order, for λ small the integral a_λ is concentrated around $s \sim \lambda^{-1} \gg 1$. Using (86), we obtain[1)]

$$a_\lambda = \lambda^{1/2}\int_0^\infty \frac{dt}{t^{3/2}}[1 - e^{-q(t)}] = \pi\lambda^{1/2}. \quad (101)$$

Substituting (101) and (78) into (100), to lowest order in λ we find $\varphi(0) = 1$. In order to find the next order, we write

$$a_\lambda - \pi\lambda^{1/2} = \int_1^\infty \frac{ds}{(s-1)^{3/2}} \exp\{-q(\lambda(s-1))\} \times [1 - \exp\{-[Q_\lambda(s) - q(\lambda(s-1))]\}]. \quad (102)$$

This integral is concentrated around $\xi \sim 1$. Making use of (86), we find that

$$Q_\lambda(s) - q[\lambda(s-1)] = -(s-1)\frac{\lambda}{\pi}[\Lambda(s) - \ln 2(s-1)], \quad (103)$$

which allows us to expand the exponent in the remainder. From (85), it is clear that we can replace the exponent in front of the square brackets by 1. In the integral obtained after this replacement, we make the change of variable $s = \operatorname{ch}\xi$. As a result, we find

$$a_\lambda = \pi\lambda^{1/2} - \pi 2^{1/2}\lambda. \quad (104)$$

Substituting (104) and (78) in (100), we obtain

$$\varphi(0) = 1 - (2\lambda)^{1/2}. \quad (105)$$

The calculation of $\varphi(0)$ we have just performed shows that the integral in (100) is concentrated at $s \sim \lambda^{-1}$ and $s \sim 1$, corresponding to the two terms in (105). Threfore our assumption $|z| \gg s$, which we used to go from (74) to (100), is fulfilled when $|z| \gg \lambda^{-1}$. This implies that the expansion (100) for $\psi^+(z)$ is correct for $|z| \gg \lambda^{-1}$, i.e., $\varphi(t) = \varphi(0)$ for $|t| \ll \lambda$. As the final conclusion of this section, we have shown that from (97) we can obtain

$$\psi^+(1) = 1 - 2\lambda.$$

8. THE ENERGY DISTRIBUTION FOR LARGE λ

When $\lambda \gg 1$, we can assume to lowest order in λ^{-1} that $H_\lambda(z) = 1$, which gives $H_\lambda^\pm(z) = 1$ and the integral (62) for $F_\lambda^\pm(z)$ can be evaluated by elementary means. Therefore it follows that the approximation we have made is correct for the functions $\psi^\pm(z)$ if z is not too close to the beginning of the respective cuts, i.e., when $|z \pm 1| \gg \lambda^{-2}$. We do not write out the results obtained in following this procedure to find the functions $\psi^\pm(z)$, because they are easily transcribed to give for $|t| \ll \lambda^2$

$$\varphi^\pm(t) = \frac{1}{\lambda}e^{-t/2}|t|[K_1(1/2|t|) \pm K_0(1/2|t|)], \quad (106)$$

where $K_{1,0}$ are modified Bessel functions of the second kind. We will later need the values

$$\psi^+(1) = 2\pi^{-1}\lambda^{-1}, \quad \psi^-(-1) = -(2/3\pi)\lambda^{-1}. \quad (107)$$

In order to calculate $\varphi^\pm(z)$ for $|t| \gg \lambda^2$, we turn to Eq. (55) and find $\varphi^\pm(z)$ from it near $z = \mp 1$. When $|z+1| \ll 1$, we can, in view of (107), discard $\psi^-(z) \approx \psi^-(-1)$ in comparison to $\lambda^{-1}G(z)$, while in $H_\lambda(z)$ we discard terms of

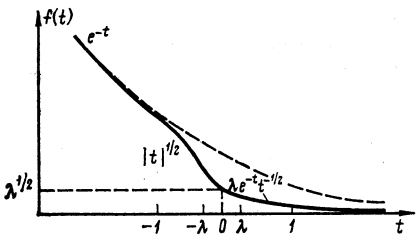

FIG. 3. Electron distribution function near threshold, for the case $\lambda \ll 1$. The strong depletion of the distribution is apparent; see formula (110).

order unity. Turning to the function $\psi^+(z)$ obtained in this way, we find that the asymptotic form of $\varphi^+(t)$ for $t \gg \lambda^2$ coincides with (98). When $|z-1| \ll 1$, by again using (107) we can neglect $\psi^+(z)H_\lambda(z) \approx \psi^+(1)H_\lambda(z)$ compared to $\lambda^{-1}G(z)$. Likewise, for $\psi^-(z)$ obtained in this way we observe that $\varphi^-(t)$ coincides with the asymptotic form of $\varphi^-(t)$ for $t \gg 1$. In other words, (106) is a correct form for $\varphi^-(t)$ for all t.

9. RESULTS AND DISCUSSION

The lower critical density N_c^-, above which the distribution in the "passive" region $\varepsilon < \hbar\Omega_0$ is close to $f_{T_e}(\varepsilon)$, is determined by comparing the time $\tilde{\tau}_A(\varepsilon)$ from (44) and (45) with the time $\tau_{ee}(\varepsilon)$ from (25) for $\varepsilon = T_e$. This density N_c^- does not depend on T_e; however, it does depend (through $\tilde{\tau}_A$) on the dimensions and shape of the well. For the square well, we have

$$N_c^- = \frac{1}{2\pi^2} p_0^2 \frac{\hbar/E_B}{\bar{\tau}_A} \frac{2ms^2}{\hbar\Omega_0}, \qquad \frac{1}{\bar{\tau}_A} = \frac{\pi b}{4} \frac{1}{(p_0 d)^3} \frac{1}{\bar{\tau}_{DA}} + \frac{\pi a}{2} \frac{1}{p_0 d} \frac{1}{\bar{\tau}_{PA}}. \tag{108}$$

From Section 8 it is clear that for $\lambda \gg 1$ corrections to the distribution $f_{T_e}(\varepsilon)$ are small for all $\varepsilon - \hbar\Omega_0 \ll T_e\lambda^2$. Therefore the condition $\lambda = 1$ determines the upper critical density of electrons

$$N_c^+ = \frac{1}{2\pi^{1/2}} p_0^2 \frac{\hbar/E_B}{\tau_0} \left(\frac{T_e}{\hbar\Omega_0}\right)^{1/2}, \tag{109}$$

above which the distribution is close to Maxwellian even in the activated region $\varepsilon > \hbar\Omega_0$.

For densities $N_c^- \ll N \lesssim N_c^+$, the distribution deviates strongly from $f_{Te}(\varepsilon)$ near and above threshold. From (98) and (105), we find for $\lambda \ll 1$ that

$$f(\varepsilon) = \begin{cases} 2^{1/2} A\lambda^{1/2}, & |t| \ll \lambda; \\ 2\pi^{-1/2} A\lambda t^{-1/2} e^{-t}, & t > 0, \quad t \gg \lambda; \\ Ae^{-t} \operatorname{erf} |t|^{1/2}, & t < 0, \quad |t| \gg \lambda. \end{cases} \tag{110}$$

At threshold and in the immediate vicinity of threshold for $|t| \ll \lambda$ the distribution is smaller than f_{T_e} by a factor of $\lambda^{1/2}$. Above threshold $f(\varepsilon)$ falls off within an energy interval T_e, as is true with $f_{T_e}(\varepsilon)$, but it is smaller in amplitude. In the three-dimensional case $f(\varepsilon)$ above threshold is also smaller in amplitude than $f_{T_e}(\varepsilon)$, but, in addition to this, it falls off within an energy interval smaller than T_e. The larger penetration depth of electrons into the active region in the two-dimensional case is connected with the fact that the deposition of electrons in this region does not come about by diffusion over the threshold, but rather by "projection" out of the passive region by virtue of "large" energy transfers $(\varepsilon' - \varepsilon) \sim T$.

The error function in (110) differs from 1 only for $|t| \lesssim 1$. In this region,

$$f(\varepsilon) = A|t|^{1/2}, \quad t < 0, \quad \lambda \ll |t| \ll 1. \tag{111}$$

This means that $f(\varepsilon)$ is much smaller compared to $f_{T_e}(\varepsilon)$ below threshold down to a depth T_e. For larger depths, corrections to f_{T_e} are small, but nonetheless they can be written down

$$f(\varepsilon) - f_{T_e}(\varepsilon) = -A\pi^{-1/2}|t|^{-1/2}, \quad 1 \ll |t|. \tag{112}$$

Their physical meaning is very simple: by calculating the current (28) with this distribution and the probability (25), one can easily convince oneself that

$$J_{ee}(\varepsilon) = \text{const} > 0, \tag{113}$$

that is, this small correction to $f_{T_e}(\varepsilon)$ ensures a particle current from above along the energy axis, which compensates for the decrease in particle density in the active region because of emission of optical phonons.

The power loss function (75) has a simple asymptotic form, which can be obtained if we use the calculated value of $\psi^+(1)$ in (97) and (107):

$$\Phi(\lambda) = \begin{cases} 2\lambda, & \lambda \ll 1, \\ 1 - 2(\pi\lambda)^{-1}, & \lambda \gg 1. \end{cases} \tag{114}$$

As in the three-dimensional case,[5] for $\lambda \ll 1$ we can get τ_0 from Q, while for $\lambda \gg 1$, if we include only the important term in (114), we determine τ_{ee}.

In the result derived above we used the probability $W(\varepsilon \to \varepsilon')$ near threshold calculated under the assumption that $\varepsilon, \varepsilon' \ll E_2 - E_1$, that is in practice that $\hbar\Omega_0 \ll E_2 - E_1$. In reality, we can dispose of this criterion and replace it with a weaker one: $E_2 - (E_1 + \hbar\Omega_0) \gg T_e$; in this case we can neglect the level E_2. As for the probability $W(\varepsilon \to \varepsilon')$, it then can be calculated by dispensing with the assumption $\varepsilon, \varepsilon' \ll E_2 - E_1$. For arbitrary ε and ε' relative to the level E_1, the matrix element (21) is multiplied by the integral

$$\int dz_1 \int dz_2 |\chi(z_1)|^2 |\chi(z_2)|^2 \exp\{-q|z_1 - z_2|\}. \tag{115}$$

Near threshold, as was clear from the solution of the kinetic equation, essentially we have $|\varepsilon - \varepsilon'| \sim T_e$, that is $q \sim mT_e/p_0$. If $E_2 - E_1 \sim \hbar\Omega_0$, then $d^{-1} \sim p_0$ and $qd \sim T_e/\hbar\Omega_0 \ll 1$. Therefore, in (115) the exponent can be replaced by 1 after which every integral turns out to equal 1.

In conclusion, we present estimates for a quantum well in GaAs whose parameters were given in Ref. 7, for $d = 150$ Å where $(E_2 - E_1)/\hbar\Omega_0 \approx 2$. Including the fact that in GaAs *PO* scattering dominates, for $T_e = 20$ °K we find N_c^+ to be 1.7×10^{11} cm^{-2}. In fact, for $N \gg N_c^+$ and $T_e \ll \hbar\Omega_0$, the electron gas is already degenerate, since $\varepsilon_F/\hbar\Omega_0 = 4\pi N/p_0 \approx N/10^{12}$ cm^{-2}. For *PO* scattering, N_c^+ does not depend on d. For the d given here we have $p_0 d = 3.8$, $\tilde{\tau}_A = 7$ ps and

$N_c^- /d = 2\times 10^{12}\,\mathrm{cm}^{-3}$, which is close to the critical density for bulk GaAs.[5,9]

[1)]Our statement that the integral in (101) is equal to π is based on numerical calculations only (the agreement is good to 14 decimal places).

[1]C. V. Shank, R. L. Fork, R. Yan *et al.*, Solid State Commun. **47**, 981 (1983).
[2]J. F. Ryan, R. A. Taylor, A. Maciel *et al.*, Phys. Rev. Lett. **53**, 1841 (1984).
[3]J. Shah, A. Pinczuk, H. L. Stormer *et al.*, Appl. Phys. Lett. **44**, 322 (1984).
[4]J. Shah, A. Pinczuk, A. C. Gossard and W. Wiegmann, Phys. Rev. Lett. **54**, 2045 (1985).
[5]N. B. Levinson, G. E. Mazhuolite, Zh. Eksp. Teor. Fiz. **50**, 1048 (1966) [Sov. Phys. JETP **23**, 697 (1966)].
[6]J. P. Leburton, K. Hess, Phys. Lett. **99A**, 335 (1978).
[7]V. F. Gantmakher, N. B. Levinson, Rasseniya Nositelei Toka v Metallakh i Poluprovodnikakh (Scattering of Current Carriers in Metals and Semiconductors), Nauka, Moscow (1984).
[8]B. L. Gel'mont, R. I. Lyagushchenko, and I. N. Yassevich, Fiz. Tverd. Tela (Leningrad) **14**, 533 (1972) [Sov. Phys. Solid State **14**, 445 (1972)].
[9]S. E. Esipov and N. B. Levinson, Zh. Eksp. Teor. Fiz. **86**, 1915 (1984) [Sov. Phys. JETP **59**, 1113 (1984)].
[10]L. Esaki, J. de Phys. **45(C5)**, 3 (1984).
[11]L. D. Landau, Zh. Eksp. Teor. Fiz. **7**, 199 (1937).
[12]P. J. Price, Surface Sci. **113**, 199 (1983).
[13]Morse and Feshback, *Methods of Theoretical Physics*, Vol. 2 (McGraw-Hill, New York, 1953).

Translated by Frank J. Crowne