Correlation functions of a two-dimensional gas with rational density, and a geometrical interpretation of the Laughlin quasiparticle operators

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Correlation functions are calculated for a two-dimensional gas of noninteracting charged particles in a magnetic field for rational values of the filling factor $\nu = q/p$. Using a basis of functions with definite angular momentum, we investigate the charge density distribution and calculate the excess charge Q which is present near the quantization axis. We note a difference in the magnitude of this charge for even and odd values of p: for even p, Q is never zero. We also study the operators which transform degenerate vacuum states into one another. A geometrical interpretation of the Laughlin quasiparticle creation operators is proposed; it is shown that they are the generators of infinitesimal complex magnetic translations.

It is now considered well-established that the fractional quantization of the Hall conductivity in two-dimensional ($2D$) systems[1] is a consequence of fundamental properties of the $2D$ electron gas in a strong magnetic field, i.e., the degeneracy of the ground-state ("vacuum") system. This assertion, attributed to Anderson,[2] received persuasive confirmation in the article by Niu, Thouless and Wu.[3] The numerical calculations of Su[4] also argue in favor of such a degeneracy. The structure of the vacuum states still remains a mystery; meanwhile, an intriguing hypothesis has been proposed: that the reason why plateaus are absent for values of the filling factor $\nu = q/p$ with even p is concealed within the transformational properties of the vacuum states.

A theory was advanced by Anderson,[2] according to which the $1/p$ states, described by determinants built out of single-particle functions for noninteracting particles, were "parent states," by which he meant that they could be transformed into exact states of the interacting system by adiabatically switching on the interaction. This attractive hypothesis makes it advisable to study the properties of such determinants; in section 1, we will calculate correlation functions associated with them.

The Laughlin approach[6] plays an important role in the theory of the fractional quantum Hall effect; this approach is based on the use of trial functions. Apparently, the method of introducing elementary excitations is the most delicate element of his theory. In section 2 we establish the connection between this method and the action of complex magnetic translations.

1. CALCULATION OF THE CORRELATION FUNCTIONS

In what follows we investigate a $2D$ gas of noninteracting particles with charge $e > 0$ in a magnetic field $\mathbf{B}$ in which all the particles are in the lowest Landau level. Let us choose the axial gauge for the vector potential

$$\mathbf{A} = 1/2\mathbf{B}\times\mathbf{r}$$

and use $l_B = (c\hbar/eB)^{1/2}$ as a unit of length. Then the wave function for a single-particle state with angular momentum projection m is

$$\psi_m(z) = (2\pi 2^m m!)^{-1/2} z^m \exp\{-|z|^2/4\}, \quad z = x+iy, \tag{1}$$

while the N-particle wave function with $m_1 < m_2 \ldots < m_N$ is then

$$\Psi_N(\{m_j\}\,|\,z_1\ldots z_N) = \left[N!\,(2\pi)^N \prod_{k=1}^{N} 2^{m_k} m_k!\right]^{-1/2} \times \begin{bmatrix} z_1^{m_1} & \ldots & z_N^{m_1} \\ \cdot & \cdots & \cdot \\ z_1^{m_N} & \ldots & z_N^{m_N} \end{bmatrix} \exp\left\{-\frac{1}{4}\sum_{j=1}^{N}|z_j|^2\right\}. \tag{2}$$

According to the usual definition, the s-particle correlation function is

$$\rho_s(\{m_j\}\,|\,z_1\ldots z_s) = \int dz_{s+1}\ldots\int dz_N\,|\Psi_N(\{m_j\}\,|\,z_1\ldots z_N)|^2. \tag{3}$$

Expanding the determinant appearing in the functions Ψ and Ψ^*, and performing all the integrations, we obtain after a series of algebraic transformations

$$\rho_s(\{m_j\}\,|\,z_1\ldots z_s) = \frac{(N-s)!}{(2\pi)^s N!}\exp\left(-\frac{1}{2}\sum_{k=1}^{s}|z_k|^2\right) \times \sum_{\{n_s\}}\left(\prod_{k=1}^{s} 2^{n_k} n_k!\right)^{-1} P\begin{Bmatrix} n_1\ldots n_s \\ n_1'\ldots n_s' \end{Bmatrix} z_1^{n_1}\ldots z_s^{n_s} z_1^{*n_1'}\ldots z_s^{*n_s'}. \tag{4}$$

In (4), the summation is over $\{n_s\}$, i.e., every set of s numbers which can be chosen from the sequence $m_1,\ldots,m_N$. The set of numbers $n_1',\ldots,n_s'$ differs from the set $n,\ldots,n_s$ only in a permutation in the order of appearance of these numbers; the sum is taken over these permutations also. The symbol $P = \pm 1$ is the sign function, whose value is determined by the parity of the permutation $\{n_s'\}$ relative to $\{n_s\}$. Formula (4) can be presented in the form of a determinant:

$$\rho_s(\{m_j\}\,|\,z_1\ldots z_s) = \frac{(N-s)!}{(2\pi)^s N!}\exp\left(-\frac{1}{2}\sum_{k=1}^{s}|z_k|^2\right) \times \mathrm{Det}\left\|\sum_n \frac{1}{n!}\left(\frac{z_i z_j^*}{2}\right)^n\right\|, \tag{5}$$

in which the summation on n in (5) is taken over n belonging to the set $\{m_j\}$.

Formula (5) entirely resolves the question of the correlation functions for an arbitrary choice of the set $\{m_j\}$. Previously, the correlation functions were well-known[7] only for

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the filling factor $\nu = 1$, where $\{m_s\} = 0,1,2,3...$, in which case the determinant in (2) reduces to the Vandermonde determinant W.

We now consider special cases:

a. Single-particle functions $\rho_1(z)$. It is convenient to replace ρ_1 by the filling factor $\nu = 2\pi N\rho_1$ an study ν, which differs from ρ_1 by the dimensionless area $S = 2\pi N$:

$$\nu(z)=\exp\left(-\frac{1}{2}|z|^2\right)\sum_n \frac{1}{n!}\left(\frac{|z|^2}{2}\right)^n, \quad n\subset\{m_j\}. \quad (6)$$

For $n = 0,1,..., N-1$[8]

$$\nu(z)=\frac{1}{(N-1)!}\Gamma\left(N,\frac{|z|^2}{2}\right)=\frac{1}{(N-1)!}\int_{|z|^2/2}^{\infty} e^{-t}t^{(N-1)}\,dt. \quad (7)$$

In the macroscopic limit $N\gg 1$, the integral (7) has a saddle point for $t_0\approx N$, so

$$\nu(z)\approx\frac{1}{(2\pi N)^{1/2}}\int_{|z|^2/2}^{\infty}\exp\left\{-\frac{(t-N)^2}{2N}\right\}dt. \quad (8)$$

In the internal region, where $z\sim 0$, we have $N-|z|^2 2\ll N$, or $(2N)^{1/2}-|z|\ll 1$, so that the filling factor satisfies $\nu(z) = 1$ with exponential accuracy. If $|z|\to\infty$, then $\nu(z)$ is exponentially small.

A hole at the origin of the system can be described, according to Laughlin,[6] by multiplying the function (2) by

$$A_0=z_1z_2\ldots z_N, \quad (9)$$

to which we attach the meaning of a particle annihilation operator. This is equivalent to changing the limits of the sum in (6) from $(0,N-1)$ to $(1,N)$. It is evident that in the entire interior region we now have

$$\nu(z)=1-\exp(-|z|^2/2) \quad (10)$$

and the total deficit of electron charge [$-Q$, see (13)] equals $+e$. For N applications of the operator A_0 with $N_1\gg 1$, formula (7) is transformed into

$$\nu(z)\approx\frac{1}{(N+N_1)!}\int_{|z|^2/2}^{\infty}e^{-t}t^{(N+N_1)}\,dt-\frac{1}{N_1!}\int_{|z|^2/2}^{\infty}e^{-t}t^{N_1}\,dt$$

$$\approx\theta\{[2(N+N_1)]^{1/2}-|z|\}\theta\{|z|-(2N_1)^{1/2}\}, \quad (11)$$

where $\theta(x) = 1$ or 0 for $x>0$ or $x<0$ respectively. Consequently, there arises a ring of almost constant fill factor $\nu\approx 1$. So, the operator A_0 acts to repel electrons from the origin $z=0$. Incidently, it is clear from (11) that there is no universal connection between filling the factor ν and the total system angular momentum $M=\Sigma m_j$: for spatially inhomogeneous filling, these quantities are independent.

For $\nu = 1/p$ (p odd), it is convenient to choose the set $\{m_j\} = (p-1)/2,\ (p-1)/2+p$ as the vacuum state. Applying this to the important case $p=3$ gives $\{m_j\} = 1,4,7,...$. For calculating the sum (6) with this set it is convenient to take advantage of the identity

$$\sum_{k=0}^{\infty}\frac{x^{km}}{(km)!}=\frac{1}{m}\sum_{n=0}^{\infty}\frac{x^n}{n!}\left[1+\exp\left\{\frac{2\pi i}{m}n\right\}+\ldots+\exp\left\{\frac{2\pi i(m-1)}{m}n\right\}\right].$$

For $\nu = 1/3$, this leads to

$$\nu(z)=\frac{1}{3}+\frac{2}{3}\exp\left(-\frac{3}{4}|z|^2\right)\sin\left(\frac{3^{1/2}}{4}|z|^2-\frac{\pi}{6}\right). \quad (12)$$

It is worth noting that the oscillatory part of $\nu(z)$ decays rapidly, and $\nu(z)\approx 1/3$ for $|z|^2\gg 1$. This occurs despite the fact that $\nu(z)$ is constructed out of the set of functions $\{m_j\}$, which corresponds to charged layers with finite thickness. The cause of this "flatness" in the density lies in the fact that according to (1) the thickness of every layer is of order 1, while the spacing between sequences of layers is of order $m^{-1/2}$, that is small for $m\gg 1$.[(1)] For $|z|\lesssim 1$, the function $\nu(z)$ is noticeably different from 1/3, but the total excess charge

$$Q=e\int[\nu(z)-1/3]\,|z|\,d|z|=0, \quad (13)$$

One can also convince oneself of this by direct calculation.

Two other vacuum states orthogonal to the one under study here can be obtained by shifting the set $\{m_j\}$ by 1 from above and below on the ladder of natural numbers. For the first operation we need a supplementary condition that the filled state is $m=0$. These transformations correspond to the $T_{1/3}$ translations of Anderson.[2] These very states can be obtained via the action of the Laughlin operator corresponding to A_0 on the determinant function:

$$A_0^+=\prod_j d/dz_j; \quad (14)$$

A_0^+ can be interpreted as a particle creation operator. The result of this is the filling factor.

$$\nu_-(z)=e^{-\xi}\int_0^{\xi}d\xi e^{\xi}\nu(z), \quad \nu_+(z)=e^{-\xi}\frac{d}{d\xi}\{e^{\xi}\nu(z)\}, \quad (15)$$

where $\xi = |z|^2/2$. Here, the indices " $-$ " and " $+$ " correspond to states obtained by the action of A_0 and A_0^+. The functions ν_- and ν_+ can be obtained from (12) by changing the phase $\pi/6$ to $5\pi/6$ and $-\pi/6$, respectively. For $|z|\to\infty$, the functions $\nu\pm(z) = 1/3$. However, near $z=0$ there is an uncompensated charge $Q\pm = \pm e/3$. Thus, the $T_{1/3}$ operators give rise to inter-vacuum-state transformations, and simultaneously generate fractional charges which, in agreement with the concepts presented in Ref. 2.

A similar construction for $\nu = 2/3$ leads to states with $Q=0$, if $\{m_j\} = 0,2,3,5,6,...$. With the help of the $T_{1/3}$ operator, we can obtain two other states with $Q=\pm e/3$ respectively for $\{m_j\} = 0,1,3,4,...$and $\{m_j\} = 1,2,4,5,...$. Application of the A_0 operator generates the state $\{m_j\} = 1,3,4,6,7,...$with $Q=-2e/3$, which differs from the state with $Q=e/3$ only in its behavior in the vicinity of $z=0$. Thus, for $|z|\gg 1$ we again obtain three vacuum states which are mutually orthogonal and spatially homogeneous.

Analogous constructions can also be implemented for even p. An important difference arises here, however, in that in this case all the defects which arise at $z=0$ are charged ($Q=0$). For example, for $p=1/2$ the smallest charged defect has $Q=\pm e/4$. If we assume that in the interacting system the generation and aggregation of such defects is advantageous as a consequence of their Coulomb interactions, then this could lead to the destruction of the homogeneous correlated liquid phase, and could explain the absence of plateaus in σ_{xy} for even p. For the argument presented here to hold, we must make an important assumption: that by virtue of the incompressibility of the liquid, the charge Q

which is present in the noninteracting system is conserved when the interactions are switched on, and therefore that the charges Q/e connected with the defects are topological charges.

b. Two-particle correlation function $\rho_z(z_1,z_2)$. According to (5), this is determined by a second-rank determinant. The sums, which consist of various terms from the determinant, were already evaluated above in determining ρ_1. For example, for $\nu = 1/3$ (in the case $|z_1|$, $|z_2| \gg 1$) we can take advantage of formula (12), retaining only the first term which equals ν. Then

$$\rho_2(z_1,z_2)=\frac{\nu^2}{(2\pi N)^2}\exp\left[-\frac{1}{2}(|z_1|^2+|z_2|^2)\right]$$
$$\times\begin{vmatrix}\exp(|z_1|^2/2) & \exp(z_1z_2^*/2)\\ \exp(z_1^*z_2/2) & \exp(|z_2|^2/2)\end{vmatrix}$$
$$=\frac{\nu^2}{(2\pi N)^2}\left\{1-\exp\left(-\frac{1}{2}|z_1-z_2|^2\right)\right\}. \qquad (16)$$

In the approximation under discussion here, ν enters into ρ_2 only as a multiplying factor. For small $|z_1 - z_2| \ll 1$, formula (16) implies that $\rho_2 \sim |z_1 - z_2|^2$. This behavior is the same as in the case of the interacting electron gas (in the limit $B \to \infty$ treated here), because it coincides with the exact solution to the two-dimensional problem corresponding to the smallest angular momentum.[9] If for $\nu = 1/3$ we include terms of order W^3 in the dominant contribution to the wave function,[6] then we can anticipate the appearance an intermediate asymptotic form $\rho_2 \sim |z_1 - z_2|$.[6] Because of this, we also expect to find a difference between the exact ρ_2 and (16) for $\nu > 1/3$. It is interesting to note, however, that for $\nu = 1/p$ formula (16) gives rise, when the interaction is turned on, to the same p-dependence $\Delta_p \sim e^2/p^2 l_B \varepsilon$ of the gaps Δ_p in the energy spectrum as for the Laughlin theory (for p odd)[6]: here, ε is the dielectric constant. It is clear that for finite B the behavior of ρ_2 as $|z_1 - z_2| \to 0$ is controlled by the Coulomb interactions. Therefore, the asymptotic behavior of ρ_2 must be determined by the well-known Sommerfeld factor which appears in the wave function for the problem of Coulomb repulsion.

Two numerical calculations of the correlation function are known to us for interacting electrons.[10,11] Judging from the conclusions of Ref. 11, these results are still preliminary in character.

2. COMPLEX MAGNETIC TRANSLATIONS

In certain cases, it was demonstrated above that the action of the Laughlin operators A_0 and A_0^+ on the determinant function (2) causes a shift in the sets $\{m_j\}$, which has as a consequence a change (increase or decrease) in the density ν near $z = 0$. Such a change occurs because the correlated liquid is displaced towards or away from its center $z = 0$. We show in this section that with the operators A_0 and A_0^+ we can associate other operators which have a rigorous geometrical meaning (independent of the determinantal function). However, to what extent and precision these operators can be described as quasiparticle creation operators remains open.

The magnetic translation operator, in analogy with the vector-potential axial gauge, is given by the formula

$$\hat{T}_{\mathbf{a}}\psi(\mathbf{r})=\psi(\mathbf{r}+\mathbf{a})\exp\left(\frac{i}{2}\mathbf{r}[\mathbf{ab}]\right), \qquad (17)$$

where $\mathbf{b}$ is a unit vector parallel to $\mathbf{B}$. For an infinitesimal translation,

$$\hat{T}_{\mathbf{a}}\approx\hat{1}+\mathbf{a}\hat{\mathbf{t}}, \quad \hat{\mathbf{t}}=\nabla+\frac{i}{2}[\mathbf{br}]. \qquad (18)$$

Therefore, the complex operators for infinitesimal translations are equal to

$$\hat{t}_\pm=\hat{t}_x\pm i\hat{t}_y=(\partial_x\pm i\partial_y)\mp\frac{1}{2}(x\pm iy). \qquad (19)$$

Applying these operators to the functions ψ_m gives

$$\hat{t}_+\{z^m\exp[-|z|^2/4]\}=-\exp[-|z|^2/4]z^{m+1},$$
$$\hat{t}_-\{z^m\exp[-|z|^2/4]\}=\exp[-|z|^2/4]\left(2\frac{d}{dz}z^m\right). \qquad (20)$$

Comparing (20) with (9) and (14) shows that A_0 and A_0^+ have the significance of generators of infinitesimal magnetic translations.

The energies of states which are obtainable from each other by symmetry transformations must coincide. In particular, this must hold for the three states with $p = 3$ and $q = 1$, which were derived in section 1. The energy densities near $z = 0$ in these states differ because of the noncoincidence of the charges: $Q/e = 0$, $\pm 1/3$, but this difference for finite N is compensated by the change in boundary energy for $|z| \approx (2N)^{1/2}$. In every intermediate region, the three vacuum states have identical energy density. Therefore, the contradiction between the work of Anderson[2] and Laughlin[6] discussed in Ref. 13, from our point of view, amounts to pure terminology, if we understand the variational function of Laughlin as a wave function of one of three vacuum states.

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[1]These arguments show that, using determinants, we can construct states not only with rational ν but with arbitrary ν. To accomplish this, we need a set of random numbers which function as the set $\{m_j\}$ and which have an average density ν^{-1} on the numerical axis.

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