The path group and the interaction of quantum strings

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(Submitted 1 August 1985)
Zh. Eksp. Teor. Fiz. **90**, 416–428 (February 1986)

If one constructs the theory of quantum strings on the basis of the path group by analogy with the previously proposed theory of gauge-charged particles, the carrier of the interaction of the strings must be a nonlocal field which is itself a string.[1)]

1. INTRODUCTION

Over the last few years the theory of quantum strings has aroused great interest. This theory is closely related to the loop approach to gauge theory, in particular to quantum chromodynamics.[1–15] In a number of papers it was assumed that the strings are extended elementary objects which manifest themselves phenomenologically like ordinary localized fields and particles. In the majority of cases one considers strings which appear as singular configurations of ordinary matter, for example, as narrow bundles of flux lines of the gauge field (gluon field). For the description of the interaction of quantum strings one usually considers processes in which they undergo a topological restructuring[4]: fusion, division, closing up of open strings, etc. Sometimes it is assumed that the interaction of the strings is carried by fields, for example by the field of a second-rank antisymmetric tensor.[5]

In order to study the behavior of quantum strings and having in mind the development of the loop approach to quantum chromodynamics, geometric methods have been proposed[6–11] based on the concept of a connection in the space of loops. Here, however, the loops appear as a method of description of an ordinary gauge field, as a result of which the connections on the loop space have vanishing curvature.[8–10]

In previous papers of the author[12–14] an algebraic method was used for the description of the gauge field, a method which made use of induced representations of the so-called path group.[12] In Refs. 14 and 15 it was proposed to make sure of the path group on the space of paths for the description of strings. In order to distinguish the latter from the ordinary path group (for instance, the path group of Minkowski space), the path group in the space of paths was called the group of 2-paths. Its elements are surfaces in the original space (e.g., Minkowski space), parametrized in a special way. The group of 2-paths allow one to construct the theory of strings in complete analogy with the theory of gauge-charged particles. The theory predicts that the interaction of strings is transmitted by a nonlocal field described by a functional of the path, and is therefore itself some kind of string. This nonlocal field was named the 2-gauge field.[14,15] It was shown[15] that there can exist 2-gauge fields of topological origin which should lead to interference effects for the strings of the type of the Aharonov-Bohm effect for ordinary particles.

In the present paper we shall describe some details of this approach to the theory of strings. In particular, we shall introduce the notion of generalized gauge transformations (2-gauge transformations) and generalized covariant derivatives. The algebraic approach based on the path group is closely related to the geometric approach, so that the geometric methods are a working tool and adequate language for the former. In particular, the 2-gauge field introduced as a representation of the group of 2-loops can be considered as a connection in the space of paths. However, in distinction from the case when this connection is generated by a gauge field[7–10], the curvature for a nonlocal 2-gauge field does not vanish. Its vanishing would be a sign that the 2-gauge field effectively reduces to a gauge field. Another radical distinction of the proposed algebraic approach is the presence of a group (the path group or the group of 2-paths), which allows one to be guided at every step by invariance requirements.

The paper is organized as follows: Section 2 contains a brief exposition of the algebraic approach to gauge theory based on the path group. The group of 2-paths is defined in Section 3 and the basic principles for the description of strings in terms of representations of this group are formulated. Sections 4 and 5 contain more detailed characterizations of the corresponding representations, which means a more detailed description of the 2-gauge field and its action on the string. Section 6 is devoted to an investigation of the generalized gauge transformations and Section 7 considers the special case of a 2-gauge field which reduces to a local field. Section 8 contains a brief summary of the obtained results.

2. AN ALTERNATIVE APPROACH TO GAUGE THEORY

An alternative approach to gauge theory is based on the path group[12] $P = P(M)$ in Minkowski space. An element of this group is a path $p = [x] \in P$, defined as an equivalence class of continuous curves $\{x(\tau) \in M \,|\, 0 \leqslant \tau \leqslant 1\}$ in Minkowski space. Two curves belong to the same class if they differ by parametrization $x'(\tau) = x(f(\tau))$, by an overall translation $x(\tau)$, by attachment to one of the curves of an "appendix" (a segment traversed first in one direction and then back), and finally, by any combination of these operations. The inverse path $p^{-1} = [x]^{-1}$ is defined by the same curve but traversed in the opposite direction: $x^{-1}(\tau) = x(1-\tau)$. The identity element is defined as the constant path $x(\tau) = \text{const}$. For

 0038-5646/86/020239-07$04.00

each path one defines the 4-vector

$$\Delta p = \Delta[x] = x(1) - x(0),$$

called the shift vector along the given path. The mapping $\Delta: P \to M$ is a homomorphism of the path group into the group of vectors (or the translation group) of Minkowski space. In this sense the path group generalizes the translation group. This generalization allows one to consider not only free particles, but also particles in an external field. The kernel of the homomorphism is the subgroup of closed paths or the group of loops, $L = L(M) = \{l \in P \mid \Delta l = 0\}$. This subgroup plays a key role for gauge theories over Minkowski space.

A gauge theory arises when one requires that an elementary particle should exhibit two properties: it should be local (with localization in Minkowski space M), and covariant with respect to the path group $P = P(M)$. The first requirement means that the states of a particle can be described by functions defined on Minkowski space, $\psi(x)$, $x \in M$. The second means that these functions transform according to a representation of the group P. It follows[12] from the general theory of local properties of covariant systems[16–18] that the localized properties of particles form a vector space $\mathscr{H}_\alpha$ which carries a representation induced from the subgroup of loops, $U_\alpha(P) = \alpha(L) \uparrow P$. The representation $\alpha(L)$ of the subgroup of loops L which is the source of the induction, may be arbitrary. This is the only element remaining arbitrary in the theory of particle localization. We note that the states which make up the space $\mathscr{H}_\alpha$ are virtual states. They are states which are created and annihilated in the process of local particle interactions. Real states of particles which one can observe form a space of generalized vectors in $\mathscr{H}_\alpha$. We shall have no explicit need for these states.

It turns out that the representation $\alpha(L)$ of the loop group is adequate for the description of some configuration of a classical gauge field. This representation is determined by the vector potential of a gauge field by means of the path-ordered exponential

$$\alpha(l) = P\exp\left\{ i \int_{l_O} A_\mu(x)\, dx^\mu \right\}, \tag{1}$$

where the integration is along a loop l which starts at the origin O of Minkowski space (one could have chosen any other point as well). The representation $U_\alpha(P)$ which describes a particle is generated by the covariant derivative $\nabla_\mu = \partial_\mu - iA_\mu(x)$, i.e.,

$$U_\alpha(p) = P\exp\left\{ -\int_p dx^\mu\, \nabla_\mu \right\}. \tag{2}$$

Thus, the main ingredients of a gauge theory–the vector potential and the covariant derivative–arise naturally within the group-theoretic method based on the loop group. It is not necessary to introduce requirement of gauge invariance. If the representation $\alpha(L)$ has dimension n one effectively obtains a gauge theory with gauge group $U(n)$. It is important for the sequel that the group-theoretic considerations based on the loop group determine the form of the covariant derivative ∇_α, i.e., the interaction between the particle and the gauge field. In the construction of a complete theory of particles this has the consequence that the gauge field will mediate the interaction between the particles. The purpose of the present paper is to derive in an analogous manner, from group-theory considerations, the form of the interaction between strings and the field which mediates the interaction among the strings (the 2-gauge field).

The representation $\alpha(L)$ describes the gauge field as a physical object, i.e., is gauge-independent. At the same time, in its description in terms of Eq. (1) there appears the vector potential, which is known to depend on the gauge choice. In order to explain the meaning of this we also introduce, in addition to the path group P, the path groupoid $\hat{P}$ and consider its representations. An element of the groupoid $\hat{P}$ is a path with fixed ends, $\hat{p} = p_x^{x'} = [x]_x^{x'}$, defined as the equivalence class of curves $\{x(\tau) \in M \mid 0 \leqslant \tau \leqslant 1\}$, differing by parametrization and the attachment of appendices, but not by shifts (translations). The beginning and the end of such a curve is fixed $x(0) = x$, $x(1) = x'$. Each path as a class of curves is subdivided into subclasses which are paths with fixed ends $\hat{p}$. Therefore, prescribing a path $p \in P$ and the point x uniquely defines a fixed path $p_x = p_x^{x'}$, where $x' = x + \Delta p$; conversely, the prescription of a fixed path $\hat{p}$ uniquely determines the (free) path p to which it belongs as a subclass. The set of fixed paths forms a groupoid in the sense that not all fixed paths can be multiplied (composed). The elements $\hat{p}$ and $\hat{p}'$ of the groupoid can be multiplied and one obtains $\hat{p}\hat{p}' = \hat{p}''$ only if $\hat{p} = p_y^x$ and $\hat{p}' = p'^y_z$, i.e., if the end of the first path coincides with the beginning of the second. In this case $\hat{p}'' = p''^x_z$. The multiplication in the groupoid $\hat{P}$ is associative (when defined). To each point $x \in M$ corresponds a unit $1_x \in \hat{P}$, such that $1_x p_y^x = p_y^x$, $p_y^x 1_y = p_y^x$. In the same manner, to each point $x \in M$ corresponds its own subgroup of loops $L_x = L_x(M)$. It consists of paths starting and ending at the same point. All groups L_x, $x \in M$ are isomorphic to each other and to the "free" loop group L.

The vector potential $A_\mu(x)$ (i.e., a vector field whose values are operators in a space $\mathscr{L}$) determines a representation of the groupoid $\hat{P}$ according to the formula

$$\hat{\alpha}(\hat{p}) = P\exp\left\{ i \int_{\hat{p}} A_\mu(x)\, dx^\mu \right\}. \tag{3}$$

This means that the following identity holds

$$\hat{\alpha}(\hat{p}\hat{p}') = \hat{\alpha}(\hat{p})\hat{\alpha}(\hat{p}'). \tag{4}$$

We select an arbitrary point in Minkowski space, for example the origin $O \in M$, and consider the subgroup of loops based on that point, L_0. A restriction of the representation of the groupoid to this subgroup yields a representation $\hat{\alpha}(L_0)$ of this subgroup. But on account of the isomorphism $L_0 = L$, this also determines a representation of the group of free loops and yields the formula $\alpha(l) = \hat{\alpha}(l_0)$, in agreement with the previous definition (1). Thus, prescribing a representation $\hat{\alpha}(\hat{P})$ defines a representation $\alpha(L)$, i.e., a gauge field. The converse is obviously not true. One can show[12,14] that in the case of a simply connected space M the representation $\alpha(L)$ is completely determined by the field strength tensor (curvature)

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu, A_\nu]. \quad (5)$$

In a multiply connected space the representation $\alpha(L)$ can describe also gauge fields of topological origin[14] corresponding to vanishing field strength, but leading to physical effects of the type of the Aharonov-Bohm effect. In terms of the representation $\alpha(\hat{P})$ of the groupoid one can rewrite the formula (2) for the induced representations of the path group in the form:

$$(U_\alpha(p)\psi)(x)=\hat{\alpha}(p_{x'^x})\psi(x'). \quad (6)$$

In this equation we have used the notation $x' = x - \Delta p$.

Thus, in the group-theory approach, to the vector potential A_μ corresponds the representation $\hat{\alpha}$ of the groupoid of fixed paths $\hat{P}$, whereas to the field strength $F_{\mu\nu}(x)$ corresponds the representation of the loop group $\alpha(L)$. A change of the vector potential $A_\mu \to A'_\mu$ always changes the representation $\hat{\alpha}(\hat{P})$. But in some cases the new representation $\hat{\alpha}'(\hat{P})$ may engender the same representation of the loop group (gauge field) $\alpha'(L) = \alpha(L)$. Here the equality sign is to be understood in the sense of equivalence of representations: $\alpha'(l) = Q\alpha(l)Q^{-1}$. If this occurs, the transformation $A \to A'$ of the potential is called a gauge transformation. One must subject at the same time the wave function ψ describing the state of a particle to the transformation

$$\psi'(x)=V(x)\psi(x), \quad (7)$$

where the "phase function" V has the expression

$$V(x)=\hat{\alpha}'(p_0)Q[\hat{\alpha}(p_0)]^{-1} \quad (8)$$

in terms of an arbitrary path p_0 leading from the origin to the point x, i.e., a path such that $x = \Delta p + O$. The transformation of the potential can be expressed in terms of the phase function in the following manner:

$$A_\mu'=VA_\mu V^{-1}-i(\partial_\mu V)V^{-1}. \quad (9)$$

Under a gauge transformation the description of the field and of the state of the particles in terms of functions depending on the point (local functions) are changed. However, the fundamental objects of the theory, i.e., the representations $\alpha(L)$ and $U_\alpha(P)$, remain invariant. If one desires, one may use an invariant language from the beginning to the end, without introducing gauge-dependent quantities. For this the states of the particles are no longer described by the local function ψ: $M \to \mathscr{L}$, but by a function(al) of the path Ψ: $P \to \mathscr{L}$, subject to the supplementary condition, the so-called structural condition,

$$\Psi(pl)=\alpha(l^{-1})\Psi(p). \quad (10)$$

Under the action of a representation such functions transform by a left translation:

$$(U_\alpha(p)\Psi)(p')=\Psi(p^{-1}p'). \quad (11)$$

Thus, the group-theory approach leads to the path-dependent Mandelstam formalism.[19,20] If one uses the latter neither gauge-dependent quantities nor gauge transformations ever appear in a gauge theory within the framework of the alternative algebraic approach. In practice it is, of course, convenient to introduce such quantities in order to simplify the formalism (a point is simpler than a path). In a certain sense they are even necessary, but from a standpoint of principle they are secondary.

Now it is our problem to carry through the same reasoning but starting from a different path group, and as a result to obtain a description of strings and fields acting on them.

3. THE GROUP OF 2-PATHS AND THE QUANTUM STRING

The key moment allowing one to construct a theory of strings by analogy with gauge theory is the circumstance that a path group can be constructed not only in Minkowski space or in general in a Euclidean space, but also on an arbitrary group. The author has proposed to make use of the path group $P(P(M))$ on the group of paths in order to formulate requirements to be imposed on a theory of extended objects of the type of strings.

A path $[g]$ on a group G is defined as an equivalence class $\{g(\tau)\in G\,|0\leqslant\tau\leqslant1\}$ of continuous curves on the group. A class contains curves differing by reparametrization $g'(\tau) = g(f(\tau))$ (with f a monotonically increasing continuous map of $[0,1]$ onto itself), a common right translation $g'(\tau) = g(\tau)\bar{g}$ or the attachment of appendices (segments which are traversed forward and backward). As is easily seen, such equivalence classes form a group which is naturally denoted by $P(G)$. For each path $[g]\in P(G)$ one defines a group element in G denoted by $\Delta[g] = g(1)(g(0))^{-1}$ and called the shift along the given path. The mapping Δ:$P(G)\to G$ is a group homomorphism. The kernel of this homomorphism $L(G) = \Delta^{-1}(1)$ consists of those paths for which the beginning and the end coincide, i.e., it is the subgroup of loops in G. The group G can therefore be characterized as the quotient of the path group with respect to the loop group: $G = P(G)/L(G)$. If one considers the result of this quotienting as a homogeneous space, one is defining an action of the path group on the original group G. Under this action the path $[g]$ maps the point g' into the point $[g]g' = (\Delta[g])g'$. We express this by saying that "the path $[g]$ leads to the point g' " (without specifying the point from which it starts out), but it is understood that the starting point is the identity element of G, i.e., $g' = \Delta[g]$. This is, of course, related to the selection of the group identity as the origin of the homogeneous space G.

If one proceeds in a manner analogous to the preceding section, then the group $P(G)$ allows one to construct the theory of an object for which the configuration space is the group G. In other words, it is an object localized in G. In particular, if $G = P(M)$ is the path group of another space (e.g., Minkowski space, or three-dimensional Euclidean space), then one can obtain the theory of an object localized in path space. Such an object could be called a 2-particle or a string. The group $P^{(2)} = P(P(M))$ in this case could be called the group of 2-paths on the space M. Thus, a 2-path on M is a path on $P(M)$.

If one chooses $G = L(M)$ then the position of an extended object is characterized by a loop in the space M, so that this object can be interpreted as a closed string.

For simplicity we shall denote the group of 2-paths by

$S=P^{(2)}=P(P(M))$, and its loop subgroup by $K=L^{(2)}=L(P(M))$. Then by analogy with the theory of particles one should assume that a 2-particle or a string must be described by an induced representation $U_\chi(S)=\chi(K)\uparrow S$, and the representation $\chi(S)$ from which the induction starts must describe a generalized gauge field acting on the string. There remains only the task to describe in more detail these mathematical objects, in order to obtain a detailed description of the 2-gauge field and its action on the string.

4. DESCRIPTION OF THE 2-GAUGE FIELD

We have seen in Section 2 that before constructing a representation of the loop group it is convenient to construct first a representation of the groupoid of paths with fixed ends. We shall proceed in the same manner here also. Our first task will be to construct a representation $\chi(K)$ of the group of 2-loops. But first we construct a representation $\hat{\chi}(\hat{S})$ of the groupoid of fixed-end 2-paths, $\hat{S}=\hat{P}(P(M))$. By analogy with Eq. (3) we write this representation in the form

$$\hat{\chi}(\hat{s})=P\exp\left\{i\int_{\hat{s}}h\right\}, \tag{12}$$

where h must be a 1-form in the space $P(M)$ similar to the way in which the 1-form $A=A_\mu dx^\mu$ in M appears in Eq. (3). Here $\hat{s}$ denotes a fixed-end path which is uniquely determined if one defines a family of paths $\hat{s}=\{p(\tau)\in P(M)|0\leqslant\tau\leqslant 1\}$. If we construct the representation (12) then the representation of 2-loops we are looking for can be constructed according to the formula

$$\chi(k)=\chi(k_1), \tag{13}$$

where k is a closed 2-path and k_1 is the fixed closed 2-path determined by it and starting out from the point $1\in P(M)$. Making use of the expression (12) one can rewrite the formula which represents 2-loops in the form

$$\chi(k)=P\exp\left\{i\int_{k_1}h\right\}. \tag{14}$$

In order to define the 1-form in path space we introduce a parametrization (coordinates) for it. Each path will be defined by a curve in the space M so that the $p=(x(\sigma)\in M|0\leqslant\sigma\leqslant 1)$. In order to remove partially the ambiguity inherent in the parametrization we fix the beginning of these curves, say, by requiring that $x(0)=0$. This still leaves some arbitrariness, but we construct the 1-form in such a manner that it does not depend on this arbitrariness. In the adopted coordinatization, a path p is determined by the set of numbers $x^\nu(\sigma)$, $0\leqslant\sigma\leqslant 1$. Consequently, the pair $a=(\nu,\sigma)$ consisting of the discrete index ν and the continuous index σ can be considered as the multi-index labeling the coordinates in the space of paths. Therefore a 1-form in the space of paths has the form

$$h(p)=\int_0^1 d\sigma\, h_\nu(p,\sigma)\,\delta x^\nu(\sigma). \tag{15}$$

This expression is the long-hand version of the short-hand expression $h(p)=h_a\delta p^a$. In order that the quantity $h(p)$ should not change under a reparametrization of the path, $\sigma\to f(\sigma)$, it is necessary that

$$h_\nu(p,\sigma)\,d\sigma=h_{\mu\nu}(p,x(\sigma))\,\dot{x}^\mu(\sigma)\,d\sigma,$$

so that Eq. (15) can be replaced by

$$h(p)=\int_0^1 d\sigma\,\dot{x}^\mu(\sigma)\,h_{\mu\nu}(p,x(\sigma))\,\delta x^\nu(\sigma). \tag{15'}$$

In going from (15) to (15') we have partially taken into account the arbitariness of the coordinatization of the space of paths. It is easy to show that this arbitrariness leads to the antisymmetry of the quantity $h_{\mu\nu}(p,x)$ with respect to its indices. Indeed, the increment $\delta x^\nu(\sigma)$ describes in general a transformation from the path p to a neighboring path $p+\delta p$. But in the case if this increment is proportional to the tangent vector $\dot{x}^\nu(\sigma)$, the shifted curve $\{x(\sigma)+\delta x(\sigma)\}$ describes in effect the same path (the curve "slides along itself"). Since in this case $\delta p=0$, the form $h(p)$ must vanish for such an increment. But this is guaranteed by antisymmetry in the indices μ,ν.

Thus a 1-form in the space of paths is determined according to Eq. (15) by the functional $h_{\mu\nu}(p,x)$ which depends on the path p and a point x on that path. The representation $\hat{\chi}(\hat{S})$ is determined by such a 1-form according to Eq. (12). If the 2-path is defined by a curve in the space of paths, $\hat{s}=\{p(\tau)|0\leqslant\tau\leqslant 1\}$, and each path entering into this family is in turn determined by a curve in the space M, so that

$$p(\tau)=\{x(\sigma,\tau)\in M|0\leqslant\sigma\leqslant 1\},$$

then Eq. (12) yields

$$\hat{\chi}(s)=P_\tau\exp\left\{i\int_0^1 d\tau\int_0^1 d\sigma\, h_{\mu\nu}(p(\tau),x(\sigma,\tau))\times\frac{\partial x^\mu(\sigma,\tau)}{\partial\sigma}\frac{\partial x^\nu(\sigma,\tau)}{\partial\tau}\right\}. \tag{16}$$

According to Eq. (13) this yields a representation of the group K. Formally that expression also has the form (16), but in place of the 2-path $\hat{s}$ one must substitute k_1—the 2-loop which begins and ends at the point $1\in P$ (the origin in path space). This determines a 2-gauge field for which the functional $h_{\mu\nu}(p,x)$ plays the role of vector potential. Consequently, this is a nonlocal field, i.e., is in itself some kind of a string (more precisely, a string with a distinguished point). If we replace the functional $h_{\mu\nu}$ by another one $h'_{\mu\nu}$ such that the representation $\chi(K)$ remains unchanged, then from a physical point of view nothing should have changed. Such a replacement represents a generalized gauge transformation which will be discussed in Section 6.

5. THE ACTION OF A 2-GAUGE FIELD ON A STRING

The string on which a 2-gauge field $\chi(K)$ acts must be described by an induced representation of the group of 2-paths, $U_\chi(S)=\chi(K)\uparrow S$. This follows from the general theory of local systems[16–18] by requiring that the string should be covariant with respect to the group S and local in the space of paths. Indeed, covariance signifies that the space of states of the string is subject to transformations according to a representation of the group S. Locality means that this representation is imprimitive and has the space P of paths as a basis of imprimitivity. Finally, the imprimitivity theorem asserts that an imprimitive representation with an imprimitivity basis $P = S/K$ is equivalent to the representation induced from the subgroup K. This leads to the assertion we have made about the form of the representation. As far as the representation $\chi(K)$ is concerned, the locality requirement does not impose any restrictions on its form. But in the preceding section it has been shown that the most general form of such a representation is determined by a functional $h_{\mu\nu}(p,x)$ according to Eq. (16). By analogy with gauge theory we have called such a representation a 2-gauge field. It remains to construct the representation $\chi(K)\uparrow S$, in order to describe its action on the string.

We assume that for each choice of (p,x) the quantity $h_{\mu\nu}(p,x)$ is an operator (matrix) in some linear space $\mathscr{L}$. Then $\chi(K)$ is a representation by means of operators on $\mathscr{L}$. In this case the carrier space of the induced representation[18,21] consists of functions Ψ defined on the group S, taking values in the space $\mathscr{L}$, and subjected to the additional structural condition

$$\Psi(sk)=\chi(k^{-1})\Psi(s), \qquad (17)$$

which must be satisfied for any $s\in S$, $k\in K$. The induced representation $U_\chi(S)$ acts on such functions by left translations

$$(U_\chi(s)\Psi)(s')=\Psi(s^{-1}s'). \qquad (18)$$

Thus, we have obtained a covariant method of description for the string. Since the representation $\chi(K)$ is not changed by a change of gauge (see the end of the preceding section), the elements of the induced representation appearing in Eqs. (17) and (18) will also not be subject to change. More precisely speaking, the representation χ may be replaced by an equivalent one. Then the operators $\chi(k)$ are replaced by $\chi'(k) = Q\chi(k)Q^{-1}$, and the function Ψ goes into $\Psi'(s) = Q\Psi(s)$. However, such transformations by an operator Q, which does not depend on the argument of the function, are trivial. In essence (as far as its functional dependence is concerned), the function Ψ which describes the state of the particle does not change. Therefore in a formalism which uses such functions there is no gauge leeway left. It is a gauge-invariant formalism. It is however convenient to go over to another description of the states, a description which is gauge-dependent. The advantage of this approach is that the states in it are described not by functions of 2-paths, but by functions of paths. With such a description the localization property in path space becomes explicit.

In order to go over to path-dependent functions we set

$$\psi(p)=\hat{\chi}(s_1{}^p)\Psi(s), \qquad (19)$$

where $s\in S$ is an arbitrary 2-path leading to the point $p\in P$ (starting from the group identity), i.e., a path satisfying the condition $\Delta s = p$. On account of the structure condition (17) and the properties of representations of the path groupoid

$$\hat{\chi}((sk)_1{}^p)=\hat{\chi}(s_1{}^p)\chi(k)$$

the right-hand side of the equation (19) does not depend on the choice of 2-path s within the indicated equivalence class. Therefore the definition of the function ψ given by Eq. (19) is correct. Making use of Eq. (18), and as the necessity arises, of the structural condition (17), and the groupoid representation property $\hat{\chi}(\hat{s}\hat{s}') = \hat{\chi}(\hat{s})\hat{\chi}(\hat{s}')$, it is easy to derive the transformation law for the function ψ:

$$(U_\chi(s)\psi)(p)=\hat{\chi}(s_{p'}{}^p)\psi(p'), \qquad (20)$$

where $p' = (\Delta s)^{-1}p$. Here we have denoted (not quite correctly from a mathematical point of view) the new realization of the induced representation by the same letter as the preceding realization (18). This does not lead to misunderstandings, since the choice of realization is obvious from which function the operator acts on.

We derive yet another form of the representation U_χ which makes use of the analog of the covariant derivative. For this we substitute into Eq. (20) in place of s a "short" smooth 2-path leading from the point $p' = \{x(\sigma) - \delta x(\sigma)\}$ to the point $p = \{x(\sigma)\}$. Then from (16) we obtain for $\hat{\chi}(s_{p'}^p)$ the approximate expression

$$\hat{\chi}(s_{p'}{}^p)\approx 1+i\int_0^1 d\sigma\, \dot{x}^\mu(\sigma)\, h_{\mu\nu}(p,x(\sigma))\,\delta x^\nu(\sigma).$$

Substituting this into Eq. (20) and retaining only first-order terms we obtain

$$U_\chi(s)=\exp\left\{-\int_0^1 d\sigma\,\delta x^\nu(\sigma)\,\nabla_\nu(\sigma)\right\},$$

where we have introduced the "2-gauge derivative"

$$\nabla_\nu(\sigma)\psi(\{x(\sigma)\})=\left(\frac{\delta}{\delta x^\nu(\sigma)}-i\dot{x}^\mu h_{\mu\nu}(p,x(\sigma))\right)\psi(\{x(\sigma)\}). \qquad (21)$$

The corresponding equation for a finite 2-path is easily obtained if one represents it as a product of a large number of "short" 2-paths, makes use for each of these of the expression for the 2-covariant derivative, and takes into account the properties of the representation. This yields

$$U_\chi(s)=P_\tau\exp\left\{-\int_0^1 d\tau\int_0^1 d\sigma\frac{\partial x^\nu(\sigma,\tau)}{\partial\tau}\nabla_\nu(\sigma)\right\}. \qquad (22)$$

This expression is completely analogous to Eq. (2) which figured in gauge theory.

We have thus obtained for string theory a covariant derivative and have thus in effect figured out how the 2-gauge field acts on the string. In all the preceding reasonings use was made only of local properties of the string and its dynamics was completely ignored. If one now describes the dynamics in some manner, and then replaces all the usual (functional) derivatives by covariant ones, one obtains the dynamics of a string in an external 2-gauge field. We are consciously avoiding this, in order not to be tied down by any concrete model of the dynamics. It would be desirable to derive the dynamics of the string from group-theory considerations too. This could probably be done within the framework of the approach developed in Refs. 22, 23. But this is a separate problem.

6. THE GENERALIZED GAUGE TRANSFORMATION

We now return to the problem of generalized gauge transformations, already touched upon at the end of Section 4. Let $\hat{\chi}$ and $\hat{\chi}'$ be two representations of the groupoid $\hat{S}$ defined respectively by the functionals $h_{\mu\nu}$ and $h'_{\mu\nu}$. We assume that these representations lead to the same representation of the group of 2-loops, $\chi(K)$. This means that for any 2-loop $k\in K$

$$\chi'(k)=Q\chi(k)Q^{-1}. \tag{23}$$

In this case we are dealing with two different methods of describing the same configuration of the 2-gauge field. The transition from one method of description of the other is naturally called a generalized gauge transformation or a 2-gauge transformation. We now make use of the representations $\hat{\chi}$ and $\hat{\chi}'$ in Eq. (17). It is easy to see that the transition from $\hat{\chi}$ to $\hat{\chi}'$ (a 2-gauge transformation) must be accompanied by a transition from Ψ to Ψ', with

$$\Psi'(s)=Q\Psi(s). \tag{24}$$

The functions Ψ and Ψ' describe the same state of the string in two different gauges. In order to find the transformation law of the functions ψ which describe the same state in the local form, we substitute $\hat{\chi}$ and $\hat{\chi}'$ into Eq. (19). We then obtain

$$\psi'(p)=\hat{\chi}'(s_1{}^p)\Psi'(s).$$

Substituting the expression for Ψ' from Eq. (24) and taking into account Eq. (19) we obtain the transformation law of the function ψ in the form

$$\psi'(p)=V(p)\psi(p), \tag{25}$$

where we have used the notation

$$V(p)=\hat{\chi}'(s_1{}^p)Q[\hat{\chi}(s_1{}^p)]^{-1}. \tag{26}$$

The transformation from the potential $h_{\mu\nu}$ to the potential $h'_{\mu\nu}$ can be expressed in terms of the phase function $V(p)$. For this we replace in Eq. (26) the path p by a nearby path p'. Assume that the 2-path $\bar{s}$ takes p into p'. Then the 2-path $\bar{s}s$ will take the origin $1\in P$ into p'. Making use of this path in Eq. (26) we obtain

$$V(p')=\hat{\chi}'(\bar{s}_p{}^{p'}s_1{}^p)Q[\hat{\chi}(\bar{s}_p{}^{p'}s_1{}^p)]^{-1}.$$

Making use of the representation property exhibited both by χ and χ' and factorizing by means of this property we find the relation

$$V(p')=\hat{\chi}'(\bar{s}_p{}^{p'})V(p)[\hat{\chi}(\bar{s}_p{}^{p'})]^{-1}.$$

Since the paths p and p' are near to each other and $\bar{s}_p^{p'}$ is a "short" 2-path between them, one can apply Eq. (16) to the latter retaining only first-order terms. This yields

$$\hat{\chi}(\bar{s}_p{}^{p'})=1+i\int_0^1 d\sigma\,\dot{x}^\mu(\sigma)h_{\mu\nu}(p,x(\sigma))\,\delta x^\nu(\sigma)$$

and a similar expression for the primed quantities. Here we have used the notation $p=\{x(\sigma)\}$, $p'=\{x(\sigma)+\delta x(\sigma)\}$. Substituting the expressions so obtained into the preceding formula and taking into account the arbitrariness of the increment $\delta x^\nu(\sigma)$, we obtain

$$\frac{\delta V(p)}{\delta x^\nu(\sigma)}=i\dot{x}^\mu(\sigma)\,[h_{\mu\nu}'(p,x(\sigma))V(p)-V(p)h_{\mu\nu}(p,x(\sigma))],$$

which can be rewritten in the form

$$\dot{x}^\mu(\sigma)h_{\mu\nu}'(p,x(\sigma))$$

$$=V(p)\dot{x}^\mu(\sigma)h_{\mu\nu}(p,x(\sigma))V^{-1}(p)-i\frac{\delta V(p)}{\delta x^\nu(\sigma)}V^{-1}(p). \tag{27}$$

The analogy between this formula and the transformation law (9) of the potential of a gauge field is obvious.

7. THE LOCAL 2-GAUGE FIELD

We have seen that in the general case the 2-gauge field is described by an arbitrary functional $h_{\mu\nu}(p,x)$ which depends on the path p and on the point x on that path, $x\in p_0$. In general one cannot express such a field in terms of a function of a point, e.g., in terms of a tensor $H_{\mu\nu}(x)$. Thus, in general, the 2-gauge field is nonlocal. However, in terms of the antisymmetric field $H_{\mu\nu}(x)$ and the vector field $A_\mu(x)$ (gauge potential), one can construct a 2-gauge field of a special form which could be called a local 2-gauge field.

According to Eq. (3), the gauge potential $A=A_\mu dx^\mu$ determines a representation $\alpha_A(\hat{P})$ of the path groupoid. From it we construct the path-dependent second-rank tensor

$$\mathscr{H}_{\mu\nu}(p)=[\alpha_A(p_0)]^{-1}H_{\mu\nu}(\Delta p)\alpha_A(p_0). \tag{28}$$

Here Δp is in fact the point to which the path p leads from the coordinate origin O. By p_0 we denote, as always, the fixed-end path beginning at the point O and belonging to the equivalence class p. We now define the functional $h_{\mu\nu}(p,x)$ by setting

$$h_{\mu\nu}(p,x)=\mathscr{H}_{\mu\nu}(p_0(x)), \tag{29}$$

where $p_0(x)$ denotes the initial section of the path p_0 leading from O to the point x. Substituting the functional (29) into the general formula (16), we obtain the 2-gauge field of the special form:

$$\hat{\chi}_{H,A}(\hat{s})=P_\tau\exp\left\{i\int_0^1 d\tau\int_0^1 d\sigma\,\mathscr{H}_{\mu\nu}(p_\sigma(\tau))\frac{\partial x^\mu(\sigma,\tau)}{\partial\sigma}\frac{\partial x^\nu(\sigma,\tau)}{\partial\tau}\right\}. \tag{30}$$

In this equation $p_\sigma(\tau)$ denotes the initial portion of the path $p(\tau)$, defined by the curve $\{x(\sigma\sigma',\tau)\,|\,0\leqslant\sigma'\leqslant 1\}$.

The 2-gauge field $\hat{\chi}_{H,A}$ defined by the 1-form A and the 2-form H in the original space M plays a special role in the theory of particles and strings. From a mathematical point of view this manifests itself through the fact that it appears in the formulation of the nonabelian Stokes theorem.[12,14] This theorem allows one to interrelate 1-dimensional and 2-dimensional integrals (more precisely, ordered exponentials of integrals) involving nonabelian differential forms. If $k\in K$ is a 2-loop parametrized as $k=\{p(\tau)\in P\,|\,0\leqslant\tau\leqslant 1\}$, then it is natural to define as its boundary the loop swept out by the ends of the path $p(\tau)$ with their beginnings fixed. This means that $\partial K=\{\Delta p(\tau)\,|\,0\leqslant\tau\leqslant 1\}$. With these definitions the nonabelian Stokes theorem is formulated as the relation

$$\chi_{DA,A}(k)=\alpha_A(\partial k). \qquad (31)$$

In this equation DA denotes the covariant differential of the 1-form

$$A=A_\mu dx^\mu,$$

i.e., the 2-form

$$F={}^1/_2F_{\mu\nu}dx^\mu\wedge dx^\nu,$$

where the field-strength tensor $F_{\mu\nu}$ is defined by Eq. (5). This means, in effect, that the left-hand side of Eq. (31) contains the representation $\chi_{H,A}$ for $H_{\mu\nu}=(1/2)F_{\mu\nu}$. In other formulations the nonabelian Stokes theorem is discussed in Refs. 24, 25.

The forms A and DA define according to Eqs. (29) and (15') a 1-form in path space, i.e., a connection, relative to which the representation $\chi_{DA,D}$ plays the role of holonomy group. If we restrict our attention to the subspace of closed paths (loops), then in it any 2-loop will have a trivial boundary. Indeed, if

$$k=\{l(\tau)\in L(M)\,|\,0\leqslant\tau\leqslant 1\},$$

then on account of $\Delta l(\tau)=0$ we obtain $\partial k=1$. In this case Stokes' theorem (31) yields $\chi_{DA,A}(k)=1$. The holonomy group turns out to be trivial, i.e., the curvature of the corresponding connection vanishes. This special case was considered in Refs. 7–10.

8. CONCLUDING REMARKS

Let us summarize the results obtained. Introducing the group of 2-paths and constructing its representations by analogy with the path group of gauge theories we have obtained 2-dimensional analogues of concepts which were characteristic for a gauge theory: the string (or 2-particle) takes the place of the usual local particle and the 2-gauge field takes the place of the ordinary gauge field. We convinced ourselves that the interaction between the strings is carried by the 2-gauge field. By defining the 2-dimensional analogue of the covariant derivative we derived a rule according to which, knowing the dynamics of the free string one can find the form of its interaction with the 2-gauge field.

The state of a string is described by a path-dependent wave function $\psi(p)$ and the state of the 2-gauge field by a function $h_{\mu\nu}(p,x)$ depending on a path and a point. In this case the 2-gauge field is a nonlocal object of the type of a string. The description of the string and of the field by means of the functions ψ and $h_{\mu\nu}$ contains some arbitrariness. Going over to other functions ψ', $h'_{\mu\nu}$ which describe the same physical situation is a generalization of the concept of gauge transformation. Such a transformation leaves invariant the representation $\chi(K)$ of the group of 2-loops (representation which gives an adequate description of the 2-gauge field) as well as the representation $U_\chi(S)$ of the group of 2-paths, which describes the action of this field on the string.

Prescribing in the original space a 1-form A and a 2-form H allows one to define a 2-gauge field $\chi_{H,A}$ of a special type. In the particular case when $H=DA$ this 2-gauge field effectively reduces to the gauge field with the vector potential A. In the case when the 2-form H does not depend on A the 2-gauge field $\chi_{H,A}$ may possibly correspond to the generalized gauge field having a tensor potential $H_{\mu\nu}$ which was postulated by Nambu[5] as the field mediating the interaction of strings. However, in our opinion, the interaction of the strings is mediated by a 2-gauge field which is nonlocal and cannot be described in terms of a finite number of local fields.

[1)]These results were reported at the III International Seminar "Group-Theoretical Methods in Physics" (Yurmala, USSR, May 1985).

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Translated by M. E. Mayer