

Hydrodynamic equations for the quantum liquid-quantum crystal phase interface ($T = 0$)

M. Yu. Kagan

Institute of Physical Problems

(Submitted 21 May 1985)

Zh. Eksp. Teor. Fiz. **90**, 498–511 (February 1986)

Linear $T = 0$ surface-hydrodynamics equations consistent with the surface-energy conservation law are derived. A general expression is established for the frequency and angular dependences of the sound-transmission coefficient on the basis of these equations. The problem of the stability of the tangential flow of a superfluid liquid slipping along the surface of a quantum crystal is investigated. The threshold velocity at which instability sets in is determined and found to be much smaller than the velocity of sound.

The large energy scale of the zero-point particle vibrations at a quantum liquid-quantum crystal phase interface, as well as the closeness of the solid- and liquid-phase densities, allows us to assume that thermodynamic equilibrium can be established in the system over fairly short periods of time, and, consequently, that the melting-crystallization processes are essentially nondissipative. The correctness of this assumption is confirmed by the existence at the phase interface of weakly damped melting-crystallization waves, which were theoretically predicted by Andreev and Parshin,¹ and experimentally observed by Keshishev, Parshin, and Babkin.²

The rapid establishment of thermodynamic equilibrium at a quantum liquid-quantum crystal interface also manifests itself in the effect, discovered by Castaing *et al.*,^{3,4} of anomalously weak low-frequency-sound transmission across the interface. Indeed, if the period of the incident wave is longer than the time required for the establishment of thermodynamic equilibrium, then, besides the normally required continuity of the pressure across the interface, we must also have equality of the chemical potentials of the phases at the interface. In this case the variable part of the pressure is equal to zero, and the incident acoustic wave should be totally reflected from the boundary, as happens at a boundary with vacuum.

But allowance for the purely surface effects in the boundary conditions gives rise to a small but finite sound-transmission coefficient D . Correspondingly, we obtain a finite, though anomalously large, Kapitza thermal resistance R_K . As will be shown below, of greatest importance for the sound-transmission problem is the allowance for three main types of surface effect in the boundary conditions. The first type is described by terms connected with the surface-tension coefficient α and with the surface anisotropy. The second type is described by a term connected with the additional surface kinetic energy introduced by Castaing and Puech,^{5a} which is quadratic in the difference between the solid- and liquid-phase velocities perpendicular to the boundary. This term is derived in Ref. 5b on the basis of model arguments. The effect is due to the fact that, when matter runs over from the solid into the liquid phase, its short-range order must be substantially reconstructed at the

phase interface. Finally, the third type is described by terms introduced by Marchenko and Parshin⁶ and connected with the surface stresses (surface elasticity). Allowance for the surface stresses leads to a situation in which the crystal is elastically deformed even in the state of total thermodynamic equilibrium.

In Sec. 1 of the present paper we derive linear, $T = 0$ surface-hydrodynamics equations (generalized boundary conditions) consistent with the surface energy conservation law and taking simultaneously into account all the important surface effects. Using these equations, we establish a general expression for the frequency and angular dependences of the sound-transmission coefficient. As analysis of the general formula for the transmission coefficient, carried out with the use of the problem's two small parameters

$$(\rho_{\text{sol}} - \rho_{\text{liq}})/\rho_{\text{sol}} \approx (\rho_{\text{sol}} - \rho_{\text{liq}})/\rho_{\text{liq}} \approx \Delta\rho/\rho \ll 1, \quad kd \ll 1$$

(ρ_{sol} and ρ_{liq} are the densities of the solid and liquid phases, k is the wave vector, and d is the interatomic separation), shows, in the two opposite limiting cases of near-normal incidence and incidence at glancing angles, the nonvanishing of the amplitude of the transmitted wave is due largely to the effects of the reconstruction of the short-range order at the phase interface and the effect of the surface tension. At the same time, when the acoustic waves are incident at angles close to the critical angles for total internal reflection, the dominant contribution to the transmission coefficient is due to the surface stresses (the surface elasticity).

In Sec. 2 of the present paper we derive the linear—in the deviations from equilibrium—equations of surface hydrodynamics under conditions when there occurs a stationary surface flow of superfluid liquid in the direction parallel to the phase interface. Using the system obtained, we analyze the spectrum of the small surface oscillations. It turns out that the stationary flow of superfluid liquid is unstable against weak perturbations even at velocities $v \ll c$.

1. SURFACE HYDRODYNAMICS IN THE ABSENCE OF STATIONARY SURFACE FLOWS

As shown in Ref. 7, the slightly deformed state of a crystal surface in the absence of the melting—crystallization processes is completely determined when its tangential—

strain tensor u'_{ik} is specified. To describe the deformation of the surface in the case when the melting-crystallization processes occur in the system, we must introduce another independent variable: the additional;—with respect to the elastic displacement—recrystallization-induced displacement $\zeta(x, y, t)$ of the surface points. In this case the total displacement will be given by the sum of the elastic, u_z , and recrystallization-induced, ζ , displacements: $Z = u_z + \zeta$, and, accordingly, the velocity of the phase interface will be equal to the sum of the velocity \dot{u}_z of the solid and the recrystallization rate $\dot{\zeta}$, viz. $V_b = \dot{u}_z + \dot{\zeta}$ (the normal to the undeformed surface coincides, as usual, with the z axis).

The total surface energy consists of three terms:

$$\mathcal{E}_s = \mathcal{E}_{\text{sel}} + \mathcal{E}_{\text{sinert}} + \mathcal{E}_{\text{skin}},$$

where $\mathcal{E}_{\text{sel}}(u'_{ik}, \nabla_x \zeta, \nabla_y \zeta)$ is the elastic surface energy, which depends on the gradients of the elastic and recrystallization-induced displacements; $\mathcal{E}_{\text{sinert}}(\dot{\zeta})$ is the energy due to the necessity of reconstruction of the short-range order when matter flows over from phase to phase, and quadratic in the recrystallization rate; and $\mathcal{E}_{\text{skin}}$ is the surface kinetic energy, which depends on the surface momentum density $\mathbf{P}^s = (P^s_\alpha, P^s_z)$ ($\alpha = 1, 2$) and the surface mass ν , as well as the velocities of the liquid \mathbf{v}_{liq} , crystal $\mathbf{v}_{\text{sol}} = \dot{\mathbf{u}}$ and phase interface V_b . Let us go over into a reference frame K_0 in which, first, the phase interface is at rest ($V_b = 0$) and, second, the velocity of the liquid has no tangential component ($\mathbf{v}_{\text{liq}\alpha} = 0$). Then the Galilean transformation formulas

$$\mathbf{P}^s_\alpha = \mathbf{P}^s_{0\alpha} + \nu \mathbf{v}_{\text{liq}\alpha}, \quad P^s_z = P^s_{0z} + \nu V_b, \quad (1)$$

$$\mathcal{E}_s = \mathcal{E}_{s0} + P^s_{0\alpha} v_{\text{liq}\alpha} + P^s_{0z} V_b + \frac{1}{2} \nu V_b^2 + \frac{1}{2} \nu v_{\text{liq}\alpha}^2 \quad (2)$$

relate the surface energy density \mathcal{E}_s and the surface momentum density \mathbf{P}^s in the laboratory reference system K to the corresponding quantities in the K_0 system. Therefore, taking account of the fact that the kinetic part of the surface energy in the reference frame K_0 can depend only on the Galilean-invariant combinations of the velocities $\dot{\mathbf{u}}$, \mathbf{v}_{liq} , and V_b , we have

$$d\mathcal{E}_{s0} = \lambda_{ik} du'_{ik} + F_\alpha d\nabla_\alpha \zeta + d\mathcal{E}_{\text{sinert}} + (v_{\text{sol}\alpha} - v_{\text{liq}\alpha}) dP^s_{0\alpha} + (v_{\text{sol}z} - V_b) dP^s_{0z}. \quad (3)$$

Here

$$d\mathcal{E}_{\text{sinert}} = \dot{\zeta} d(\rho_1 \dot{\zeta}), \quad (4)$$

where $\rho_1 \sim \rho d$ is the coefficient of proportionality between $\mathcal{E}_{\text{sinert}}$ and $\dot{\zeta}^2/2$, and has the dimensions of surface density; u'_{ik} is the tangential strain tensor, which is connected with the strain tensor u_{ik} of bulk elasticity theory by the relation (see Ref. 8)

$$u_{ik}{}^t = \Lambda_{ip} \Lambda_{ki} u_{pi}, \quad \Lambda_{ip} = \delta_{ip} - n_i n_p / [1 + (\nabla_\alpha Z)^2], \quad (5)$$

where $\mathbf{n} = (1, -\nabla_\alpha Z)$ is the normal to the deformed surface. The right-hand side of the identity (3) is a total differential; therefore, the cross derivatives for the elastic part of the surface energy should be equal to each other:

$$\partial^2 \mathcal{E}_{s0} / \partial \zeta_\alpha \partial u_{ik}{}^t = \partial \lambda_{ik} / \partial \zeta_\alpha = \partial F_\alpha / \partial u_{ik}{}^t. \quad (6)$$

In the relation (3) $P^s_{0\alpha} = \rho_2 (v_{\text{sol}\alpha} - v_{\text{liq}\alpha})$, where $\rho_2 \sim \rho d$ is the effective surface density. The expression (3) is the most general expression for the surface-energy differential at $T = 0$. With the substitutions

$$u_{ik} \rightarrow \frac{1}{2} (\nabla_k X_i + \nabla_i X_k), \quad \dot{u}_k \rightarrow v_{\text{sol}k}; \quad \dot{X}_i + v_{\text{sol}k} \nabla_k X_i = 0$$

(the X_i are variables considered in Ref. 9) for the elastic part of the surface energy and

$$P^s_\alpha \rightarrow (P^s_i)^t = \Lambda_{ik} P^s_k, \quad P^s_z \rightarrow P_{ni}{}^s = (P^s_k n_k) n_i / [1 + (\nabla_\alpha Z)^2]$$

for the kinetic part made in it, we can use it to derive surface hydrodynamics equations that are nonlinear in the velocities and the deformations. We limit ourselves below to the derivation of the boundary conditions that are linear in the deviations of the quantities from their equilibrium values.

We shall derive the equations of surface hydrodynamics of the basis of the laws of conservation of the surface quantities in the differential form. Then, as usual, we define the boundary as the place where the surface particle-number density ν is equal to zero. Let us emphasize that we can define the boundary without resort to the requirement that the surface mass there be equal to zero. In this case it is not difficult to show that each of the observable physical quantities of the type of the sound-transmission coefficient D contains a displacement-invariant combination of the surface particle-number density ν and the surface inertial mass ρ_1 . At low frequencies this combination has the simple form:

$$M_s = \rho_1 + \nu \rho_{\text{sol}} / \rho_{\text{liq}} = \text{invar}.$$

According to Ref. 3, the laws of conservation of surface mass and surface momentum have, when the condition $\nu = 0$ is taken into account, the form

$$\nabla_\alpha P_\alpha = (j_k^{\text{sol}} - j_k^{\text{liq}}) n_k - (\rho_{\text{sol}} - \rho_{\text{liq}}) V_b, \quad (7)$$

$$\dot{P}_\alpha + \nabla_\beta \pi_{\alpha\beta} = (\Pi_{\alpha k}^{\text{sol}} - \Pi_{\alpha k}^{\text{liq}}) n_k - (j_\alpha^{\text{sol}} - j_\alpha^{\text{liq}}) V_b, \quad (8)$$

$$\dot{P}_z + \nabla_\beta \pi_{z\beta} = (\Pi_{zk}^{\text{sol}} - \Pi_{zk}^{\text{liq}}) n_k - (j_z^{\text{sol}} - j_z^{\text{liq}}) V_b. \quad (9)$$

In the absence of the melting-crystallization processes, the system of equations (7)–(9) is a complete system, and the law of conservation of surface energy is a consequence of it. In the presence of these processes there is an independent variable ζ , and the surface energy conservation law, as expressed in the form

$$\dot{\mathcal{E}} + \nabla_\alpha \Theta_\alpha = (Q_{\text{sol}k} - Q_{\text{liq}k}) n_k - (E_{\text{sol}} - E_{\text{liq}}) V_b, \quad (10)$$

constitutes an independent equation. In Eqs. (7)–(10)

$$\Pi_{ik}^{\text{sol}} = \rho_{\text{sol}} v_{\text{sol}i} v_{\text{sol}k} + \mathcal{P}_{\text{sol}} \delta_{ik} - \sigma_{ik}^{\text{sh}}$$

is the volumetric momentum flux in the solid phase;

$$\sigma_{ik}^{\text{sh}} = \sigma_{ik} - \sigma_{ll} \delta_{ik} / 3,$$

where

$$\sigma_{ik} = 2\rho_{\text{sol}} c_t^2 u_{ik} + \rho_{\text{sol}} (c_l^2 - 2c_t^2) u_{ll} \delta_{ik},$$

c_t and c_l being the transverse- and longitudinal-sound velocities, is the shear part of the stress tensor;

$$\Pi_{ik}^{\text{liq}} = \rho_{\text{liq}} v_{\text{liq}i} v_{\text{liq}k} + \mathcal{P}_{\text{liq}} \delta_{ik}$$

is the momentum flux in the liquid; \mathcal{P}_{sol} and \mathcal{P}_{liq} are the

pressures in the solid and liquid phases; $j_i^{\text{sol}} = \rho_{\text{sol}} v_{\text{sol}i}$ and $j_i^{\text{liq}} = \rho_{\text{sol}} v_{\text{liq}i}$ are the volumetric matter fluxes;

$$Q_{\text{sol}k} = (\mu_{\text{sol}} + v_{\text{sol}}^2/2)j_{\text{sol}k} - \sigma_{ik}^{\text{sh}}v_{\text{sol}i},$$

$$Q_{\text{liq}k} = (\mu_{\text{liq}} + v_{\text{liq}}^2/2)j_{\text{liq}k}$$

are the volumetric solid- and liquid-phase-energy fluxes;

$$E_{\text{sol}} = (\mu_{\text{sol}} \rho_{\text{sol}} - \mathcal{P}_{\text{sol}}) + \sigma_{ik}^{\text{sh}}u_{ik}/2 + \rho_{\text{sol}}v_{\text{sol}}^2/2,$$

$$E_{\text{liq}} = (\mu_{\text{liq}} \rho_{\text{liq}} - \mathcal{P}_{\text{liq}}) + \rho_{\text{liq}}v_{\text{liq}}^2/2$$

are the volume energy densities; μ_{sol} and μ_{liq} are the chemical potentials, which are connected with the pressure by the standard thermodynamic identities

$$d\mathcal{P}_{\text{sol}} = \rho_{\text{sol}}d\mu_{\text{sol}}, \quad d\mathcal{P}_{\text{liq}} = \rho_{\text{liq}}d\mu_{\text{liq}};$$

P_α , P_z , and \mathcal{E} are the components of the surface momenta P_α^s and P_z^s and the surface energy \mathcal{E}_s in the (x, y) plane:

$$P_\alpha = P_\alpha^s [1 + (\nabla_\beta Z)^2]^{1/2}, \quad P_z = P_z^s [1 + (\nabla_\beta Z)^2]^{1/2},$$

$$\mathcal{E} = \mathcal{E}_s [1 + (\nabla_\beta Z)^2]^{1/2}; \quad (11)$$

and $\pi_{\alpha\beta}$, $\pi_{\alpha z}$, and Θ_α are surface momentum and energy fluxes to be determined.

In the problem we consider the oscillations of the quantities under a hydrostatic stress due to an equilibrium pressure $\mathcal{P}_0 = 26$ atm. Therefore, the expression for the volume energy of the crystal contains terms linear, besides those quadratic, in the deformations. (At equilibrium it is the thermodynamic potential, and not the energy, that has a minimum.) In view of this, it is convenient to separate out in this expression the purely quadratic—in the deformations—shear part. Notice that, to the accuracy required for the derivation of the linear boundary conditions,

$$\mathcal{P}_{\text{sol}} = \mathcal{P}_0 - \sigma_{11}/3, \quad \mu_{\text{sol}} = \mu_0 - \sigma_{11}/3\rho_{\text{sol}},$$

where \mathcal{P}_0 and μ_0 are the equilibrium values of the pressure and the chemical potential. Therefore, the volumetric momentum flux in the solid can be represented in the form

$$\Pi_{ik}^{\text{sol}} = \rho_{\text{sol}}v_{\text{sol}i}v_{\text{sol}k} + \mathcal{P}_0\delta_{ik} - \sigma_{ik}.$$

It must be emphasized that the equations (8) and (9) expressing the law of conservation of surface momentum are consistent with the law of conservation of surface angular momentum (see Ref. 10) provided

$$\pi_{\alpha\beta} = \pi_{\beta\alpha}, \quad \pi_{\alpha z} = \pi_{\alpha\beta}\nabla_\beta Z + P_\alpha V_b. \quad (12)$$

Furthermore, since momentum and mass flux are one and the same thing in nonrelativistic hydrodynamics, Eq. (7) expresses the equality of the bulk-momentum components perpendicular to the surface in the rest frame K_0 of the phase interface. Owing to the continuity of the bulk momentum density j_{0z} , there is no need to introduce a surface momentum density P_{0z} into the hydrodynamics. Notice that there is, generally speaking, no reason why $P_{0\alpha}$ should also vanish. But in the linear equations of surface hydrodynamics the contribution of the terms containing $P_{0\alpha}$ always turns out to be small.

Substituting into the formula (11) for the surface-energy component in the (x, y) plane the expressions (2)–(5) with $v = 0$ and $P_{0z} = 0$, and using the relation (6), we obtain in the approximation quadratic in the variable parts of the quantities the expression

$$\mathcal{E} = (\alpha_0\delta_{\mu\nu} + \alpha_{\text{an}\mu\nu})\xi_\mu\xi_\nu/2 + \lambda_{1\alpha\beta\gamma}\xi_\gamma u_{\alpha\beta} + \rho_1\dot{\xi}^2/2,$$

$$\alpha_{\text{an}\mu\nu} = \partial F_\mu/\partial \xi_\nu, \quad \lambda_{1\alpha\beta\gamma} = \partial F_\gamma/\partial u_{\alpha\beta} = \partial \lambda_{\alpha\beta}/\partial \xi_\gamma, \quad (13)$$

where α_0 is the surface energy density in the absence of deformation (the coefficient of surface tension), $\alpha_{\text{an}\mu\nu}$ is the surface-anisotropy coefficient, $\lambda_{1\alpha\beta\gamma}$ is the coefficient of the off-diagonal term in the expansion of the elastic part of the surface energy: $\lambda_{1\alpha\beta\gamma} = \lambda_{1\beta\alpha\gamma}$. In the expression (13) we have discarded all the terms of order $u^2\xi^2$, terms which, as can be shown with the use of the parameters $\Delta\rho/\rho \ll 1$ and $kd \ll 1$ of the problem, are small at not too high frequencies ω , when $\omega(d/c)(\rho/\Delta\rho) < 1$. Of the small terms of order u/ξ we have retained only one, namely, $\lambda_{1\alpha\beta\gamma}u_{\alpha\beta}\xi_\gamma$, which, as will be shown below, is important when the angle of incidence of the acoustic wave is close to the critical angle for total internal reflection.

Let us emphasize that the volume equations for the liquid are usually written in the Euler variables. Therefore, to obtain the correct boundary conditions at the boundary between two contiguous semi-infinite phases, the volume equations for the solid should also be written in the Euler variables. Thus, the Euler volume fluxes and energy density of the solid that figure in Eqs. (7)–(10) are not the same as the Lagrange fluxes and energy density used in the linear theory of elasticity. Owing to the presence of linear terms in the expression for the bulk energy of the crystal, these differences (due to the presence of the factor $\delta\rho_{\text{sol}}/\rho_{0\text{sol}}$, where $\rho_{0\text{sol}}$ and $\delta\rho_{\text{sol}}$ are the constant and variable parts of the density of the solid) occur even when the linear expression for the volume momentum flux is used. But they can simply be reduced to a renormalization of the elastic moduli when we go over from the Lagrange to the Euler variables.¹¹⁾

Differentiating (13) with respect to the time, and using the forms (7)–(9) of the laws of conservation of surface mass and momentum under conditions when there are no stationary surface flows and $v = P_{0z} = P_{0\alpha} = 0$, we obtain in the approximation quadratic in the variable parts of the quantities, after simple transformations, the equation

$$\mathcal{E} + \nabla_\beta \Theta_\beta - (Q_{\text{sol}k} - Q_{\text{liq}k})n_k + (E_{\text{sol}} - E_{\text{liq}})V_b$$

$$= -v_{\text{sol}z}\nabla_\alpha\pi_{\alpha z} - v_{\text{sol}\alpha}\nabla_\beta(\pi_{\alpha\beta} + \lambda_{1\beta\alpha\gamma}\xi_\gamma)$$

$$- \dot{\xi} [\rho_{\text{sol}}(\mu_{\text{liq}} - \mu_{\text{sol}}) + \mathcal{P}_{\text{sol}} - \mathcal{P}_{\text{liq}}$$

$$+ \nabla_\beta(\tilde{\alpha}_{\mu\beta}\xi_\mu + \lambda_{1\alpha\gamma\beta}u_{\alpha\gamma}) - \rho_1\dot{\xi}^2], \quad (14)$$

where

$$\Theta_\beta = -\tilde{\alpha}_{\mu\beta}\xi_\mu\dot{\xi} - \lambda_{1\alpha\gamma\beta}\dot{\xi}u_{\alpha\gamma} - \lambda_{1\beta\alpha\gamma}v_{\text{sol}\alpha}\xi_\gamma,$$

$$\tilde{\alpha}_{\mu\beta} = \alpha_0\delta_{\mu\beta} + \alpha_{\text{an}\mu\beta}.$$

In order for Eq. (14) to have the form (10) of the surface-energy conservation law, the following conditions must be fulfilled:

$$\pi_{\alpha\beta} = -\lambda_{1\beta\alpha\gamma}\xi_\gamma, \quad \pi_{\alpha z} = 0,$$

$$\dot{\xi}\{\rho_{\text{sol}}(\mu_{\text{liq}} - \mu_{\text{sol}}) + \mathcal{P}_{\text{sol}} - \mathcal{P}_{\text{liq}}$$

$$+ \nabla_\beta(\tilde{\alpha}_{\mu\beta}\xi_\mu + \lambda_{1\alpha\gamma\beta}u_{\alpha\gamma}) - \rho_1\dot{\xi}^2\} = 0.$$

Let us emphasize that the expression

$$\pi_{\alpha\beta}Z_\beta + P_\alpha V_b = -\lambda_{1\beta\alpha\gamma}\xi_\gamma Z_\beta + P_\alpha V_b$$

is of second order in smallness. Therefore, the condition $\pi_{\alpha z}$

$= 0$ is consistent with the law of conservation of surface angular momentum, (12). If the melting—crystallization processes occur in the system, then $\dot{\xi} \neq 0$. Therefore,

$$\rho_{\text{sol}} (\mu_{\text{liq}} - \mu_{\text{sol}}) + \mathcal{P}_{\text{sol}} - \mathcal{P}_{\text{liq}} + \nabla_{\beta} (\tilde{\alpha}_{\mu\beta} \xi_{\mu} + \lambda_{1\alpha\gamma\beta} u_{\alpha\gamma}) - \rho_1 \ddot{\xi} = 0.$$

Finally, in the absence of stationary surface flows, the linear boundary conditions have the form

$$\begin{aligned} \rho_{\text{liq}} (v_{\text{liq}z} - V_b) &= \rho_{\text{sol}} (v_{\text{sol}z} - V_b), \\ \sigma_{z\alpha} &= \nabla_{\beta} (\lambda_{1\beta\alpha\gamma} \xi_{\gamma}), \quad \sigma_{zz} + \mathcal{P}_{\text{liq}} - \mathcal{P}_0 = 0, \\ \rho_{\text{sol}} (\mu_{\text{liq}} - \mu_{\text{sol}}) + \mathcal{P}_{\text{sol}} - \mathcal{P}_{\text{liq}} + \nabla_{\beta} (\tilde{\alpha}_{\mu\beta} \xi_{\mu} + \lambda_{1\alpha\gamma\beta} u_{\alpha\gamma}) - \rho_1 \ddot{\xi} &= 0. \end{aligned} \quad (15)$$

The first equation in the system (15) is a matter-balance equation; the second and third equations constitute a condition for mechanical equilibrium at the boundary; and the last equation constitutes a generalized condition for phase equilibrium. This last equation can also be regarded as the equation of motion of the boundary relative to the solid. In the zeroth approximation the system (15) is consistent with the conditions for total phase equilibrium:

$$\mathcal{P}_{0\text{liq}} = \mathcal{P}_{0\text{sol}} = \mathcal{P}_0, \quad \mu_{0\text{liq}} = \mu_{0\text{sol}} = \mu_0.$$

In the approximation linear in the deviation from equilibrium, we have, when account is taken of the thermodynamic identities $\delta\mathcal{P}_{\text{sol}} = \rho_{\text{sol}} \delta\mu_{\text{sol}}$ and $\delta\mathcal{P}_{\text{liq}} = \rho_{\text{liq}} \delta\mu_{\text{liq}}$ for the variable parts of the chemical potentials and pressures, the equations

$$\begin{aligned} \Delta \rho \delta\mu_{\text{liq}} + \nabla_{\beta} (\tilde{\alpha}_{\mu\beta} \delta\xi_{\mu} + \lambda_{1\alpha\gamma\beta} \delta u_{\alpha\gamma}) - \rho_1 \delta\ddot{\xi} &= 0, \\ \delta\sigma_{z\alpha} &= \nabla_{\beta} (\lambda_{1\beta\alpha\gamma} \delta\xi_{\gamma}), \quad \delta\sigma_{zz} + \rho \delta\mu_{\text{liq}} = 0, \\ \rho \delta v_{\text{sol}z} - \rho \delta v_{\text{liq}z} - \Delta \rho \delta \xi &= 0. \end{aligned} \quad (16)$$

Let us emphasize that, in the absence of stationary surface flows, the velocities, displacements, and stress-tensor components coincide with their variable parts, and the symbol δ in front of them can be dropped. In (16) $\Delta\rho = \rho_{\text{sol}} - \rho_{\text{liq}}$, $\rho \approx \rho_{\text{sol}} \approx \rho_{\text{liq}}$ ($\Delta\rho/\rho \ll 1$). In the balance equation, as in the expression (13), we have discarded the small terms of order u/ξ . It follows from the system of equations (16) that, for sound incident at the phase interface, the amplitude of the wave transmitted into the other medium is nonzero only because of the capillary effects. Indeed, setting $\tilde{\alpha}_{\mu\beta} = \lambda_{1\alpha\gamma\beta} = \rho_1 = 0$ in (16), we obtain $\delta\mu_{\text{liq}} = 0$ and $\delta\sigma_{zz} = \delta\sigma_{z\alpha} = 0$. Therefore, the liquid and solid phases become “decoupled” and, for acoustic waves incident at the boundary from any one of the media, this boundary is equivalent to a boundary with vacuum. As a result, no sound is transmitted into the other medium.

2. TRANSMISSION OF SOUND ACROSS A QUANTUM LIQUID-QUANTUM CRYSTAL PHASE INTERFACE

A. In the sound transmission problem it is convenient to represent the velocity and displacement fields in the isotropic solid in the form of sums of longitudinal and transverse parts:

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}_l + \dot{\mathbf{u}}_t,$$

where

$$\text{rot } \mathbf{u}_l = 0, \quad \mathbf{u}_l = \nabla \Phi, \quad \text{div } \mathbf{u}_t = 0, \quad \mathbf{u}_t = \text{rot } \Psi,$$

and Φ and Ψ are the longitudinal- and transverse-acoustic-wave potentials. Assuming in the particular case of the two-dimensional problem that all the quantities depend only on x and z , where z is the normal to the undeformed boundary, we can choose the potential Ψ in the form $\Psi = \Psi \mathbf{1}_y$, where $\mathbf{1}_y$ is the unit vector along the y axis. Then the volume vector equation $\rho \ddot{\mathbf{u}}_i = \nabla_k \sigma_{ik}$ for the displacement field in the solid reduces to two scalar wave equations for the potentials Ψ and Φ :

$$\ddot{\Phi} = c_l^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \Phi, \quad \ddot{\Psi} = c_t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \Psi. \quad (17)$$

The displacements in the solid are given by the formulas

$$u_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z}, \quad u_y = 0, \quad u_z = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x}. \quad (18)$$

The velocity field in the superfluid liquid is described by the potential φ (where $\mathbf{v}_{\text{liq}} = \nabla \varphi$) satisfying the wave equation

$$\ddot{\varphi} = c_{\text{liq}}^2 (\partial^2 / \partial x^2 + \partial^2 / \partial z^2) \varphi, \quad (19)$$

where c_{liq} is the velocity of sound in the liquid. The variable part of the pressure in the acoustic wave for the liquid is given by the standard formula

$$\delta \mathcal{P}_{\text{liq}} = \rho_{\text{liq}} \delta \mu_{\text{liq}} = -\rho_{\text{liq}} \dot{\varphi} \approx -\rho \dot{\varphi}. \quad (20)$$

Let us consider a plane acoustic wave incident at the phase interface from the liquid. Taking account of Eqs. (17) and (19), which give for plane waves the relation between the frequency and wave vector, we have for the incident, reflected, and the two transmitted waves the expressions

$$\begin{aligned} \varphi &= \varphi_{10} \exp \left[-i\omega t + i \frac{\omega}{c_{\text{liq}}} \sin \theta_{1\text{liq}} x + i \frac{\omega}{c_{\text{liq}}} \cos \theta_{1\text{liq}} z \right] \\ &+ \varphi_{20} \exp \left[-i\omega t + i \frac{\omega}{c_{\text{liq}}} \sin \theta_{2\text{liq}} x - i \frac{\omega}{c_{\text{liq}}} \cos \theta_{2\text{liq}} z \right], \\ \Phi &= \Phi_0 \exp \left[-i\omega t + i \frac{\omega}{c_l} \sin \theta_l x + i \frac{\omega}{c_l} \cos \theta_l z \right], \\ \Psi &= \Psi_0 \exp \left[-i\omega t + i \frac{\omega}{c_t} \sin \theta_t x + i \frac{\omega}{c_t} \cos \theta_t z \right], \end{aligned} \quad (21)$$

where φ_{10} and φ_{20} are the amplitudes of the incident and reflected waves in the liquid, Φ_0 and Ψ_0 are the amplitudes of the longitudinal and transverse waves generated in the solid, and $\theta_{1\text{liq}}$, $\theta_{2\text{liq}}$, θ_l , and θ_t are the angles between the wave vectors of the corresponding waves and the normal to the surface. The problem is homogeneous in the coordinate x ; therefore,

$$\frac{\sin \theta_l}{c_l} = \frac{\sin \theta_t}{c_t} = \frac{\sin \theta_{1\text{liq}}}{c_{\text{liq}}} = \frac{\sin \theta_{2\text{liq}}}{c_{\text{liq}}}. \quad (22)$$

In particular, we have *a priori* the obvious equalities

$$\sin \theta_{1\text{liq}} = \sin \theta_{2\text{liq}} = \sin \theta_{\text{liq}}.$$

The recrystallization-induced displacement in the plane acoustic wave has the form

$$\xi = \xi_0 \exp [-i\omega t + i(\omega/c_l) \sin \theta_l x]. \quad (23)$$

It is well known that the velocities of sound in the liquid and crystal satisfy the inequalities (see, for example, Ref. 11)

$$c_t < c_{\text{liq}} < c_l. \quad (24)$$

Therefore, in the case of sound incident from the liquid there exists a critical angle for total internal reflection for longitudinal sound in the solid: the "Brewster angle." It is given by the relation $\sin\theta_{\text{liq}} = c_{\text{liq}}/c_l$, with $\sin\theta_l = 1$ and $\cos\theta_l = 0$. When the sound is incident at an angle greater than the critical angle, only a transverse acoustic wave is generated in the solid. Notice that the phenomenon of total internal reflection manifests itself in the sharply nonmonotonic character of the angular dependence of the sound transmission coefficient.^{12,13}

B. Let us, to begin with, analyze the case when $\sin\theta_{\text{liq}} < c_{\text{liq}}/c_l$. Substituting the formulas (21) and (23) into the system of equations (16) at the point $z = 0$, with allowance made for (18), (20), and (22), and taking account of the fact that for the two-dimensional problem $u_{zy} = u_{xy} = \sigma_{zy} = 0$ and, consequently, only the components $\tilde{\alpha}_{xx}$ and λ_{1xxx} of the tensors $\tilde{\alpha}_{\mu\beta}$ and $\lambda_{1\alpha\gamma\beta}$ are important, we obtain for the sound transmission coefficient $D = 1 - R$ (where $R = |\varphi_{20}|^2/|\varphi_{10}|^2$ is the reflection coefficient) at not too high frequencies ω , specifically, at frequencies ω , such that $\omega(d/c)(\rho/\Delta\rho) < 1$, the expression

$$D = 1 - R$$

$$= \frac{4\omega^2(\rho/\Delta\rho)^4(\cos\theta_{\text{liq}}/c_{\text{liq}})G[d_2^2\cos\theta_l/c_l + (\Delta\rho/\rho)^2d_1^2\cos\theta_l/c_l]}{G^2 + (\rho/\Delta\rho)^4(\omega^2d_2^2/c_l^2)[\cos\theta_l + c_l(\cos\theta_{\text{liq}}/c_{\text{liq}})G]^2}$$

$$G = \cos^2 2\theta_l + \frac{c_l^2}{c_t^2} \sin 2\theta_l \sin 2\theta_l, \quad (25)$$

$$d_1 = \frac{\lambda_{1xxx}}{\rho c_l^2} \sin^2 \theta_l, \quad d_2 = \frac{1}{\rho} \left(\rho_1 - \frac{\tilde{\alpha}_{xx}}{c_l^2} \sin^2 \theta_l \right),$$

where $d_1, d_2 \sim d$. In the formula (25) we have retained only the leading terms in the problem's small parameters $\Delta\rho/\rho \ll 1$ and $kd \ll 1$. The numerator in the transmission coefficient is proportional to a sum of two terms. The first term, which depends on the parameter d_2 , is connected with the coefficient of surface tension and the anisotropy coefficient $\tilde{\alpha}_{xx}$, as well as the effects of the reconstruction of the short-range order: the inertial surface mass ρ_1 . The second term, which depends on the parameter d_1 , is connected with the surface-elastic constant λ_{1xxx} . The lengths d_1 and d_2 are of the same order of magnitude. Therefore, it follows from (25) that, at small values of $\cos\theta_l$ and, consequently, for angles of incidence close to the Brewster angle, i.e., for

$$\arcsin(c_{\text{liq}}/c_l) - |\eta_1|(\Delta\rho/\rho)^2 < \theta_{\text{liq}} < \arcsin(c_{\text{liq}}/c_l)$$

(η_1 is a numerical factor composed of the velocities of sound), the transmission coefficient is determined largely by the surface elasticity. At the same time, in the entire remaining angle range

$$0 < \theta_{\text{liq}} < \arcsin(c_{\text{liq}}/c_l) - |\eta_1|(\Delta\rho/\rho)^2,$$

it is determined by the surface tension, the surface anisotropy,

and the inertial mass. Let us emphasize that, at an angle of incidence exactly equal to the Brewster angle, the transmission coefficient is nonzero only because of the allowance made for the surface elasticity. (In this case it is, in magnitude, $(\Delta\rho/\rho)^2$ times smaller than the transmission coefficient for angles not close to the Brewster angle.) This circumstance justifies the retention, in the formula (13) for the surface energy, of the small term $\lambda_{1\alpha\beta\gamma}\zeta_\gamma u_{\alpha\beta}$. The frequency dependence of D is governed by the value of the characteristic dimensionless parameter $\varepsilon = (\omega d/c)(\rho/\Delta\rho)^2$ of the problem (here and below we assume that all the sound velocities are of the same order of magnitude). For values of $\varepsilon < 1$ the frequency-dependent term in the denominator of (25) is small compared to the frequency independent term G^2 . Therefore, in the region far from the Brewster angle $D \sim \varepsilon^2 \ll 1$. In the vicinity of the Brewster angle $D \sim \varepsilon^2(\Delta\rho/\rho)^2$, i.e., it is a quantity of even greater degree of smallness. We are considering not too high frequencies, i.e., frequencies ω , such that $\omega(d/c)(\rho/\Delta\rho) < 1$. This corresponds to the variation of ε in the range $0 < \varepsilon < \rho/\Delta\rho$. Thus, there exists a frequency region where $\varepsilon > 1$, but $\omega(d/c)(\rho/\Delta\rho) < 1$. For $\varepsilon \gtrsim 1$, $D \sim \varepsilon^2/(1 + \varepsilon^2) \sim 1$ in the region of angles far from the Brewster angle, i.e., at large values of the parameter ε the quantum liquid-quantum crystal phase interface is "transparent."

Let us emphasize that the small-surface-oscillation spectrum can also be obtained from the formula (25) for the transmission coefficient. For the purpose, we must equate the denominator of the formula (25) to zero (see Ref. 14), and restore in it the dependence on the frequency and wave vector with the aid of the relations

$$\begin{aligned} \cos\theta_l &= i|k_z^l|c_l/\omega, & \sin\theta_l &= c_l k_x/\omega, \\ \cos\theta_l &= i|k_z^l|c_l/\omega, & \sin\theta_l &= c_l k_x/\omega, \\ \cos\theta_{\text{liq}} &= i|k_z^{\text{liq}}|c_{\text{liq}}/\omega, & \sin\theta_{\text{liq}} &= c_{\text{liq}} k_x/\omega. \end{aligned}$$

Here we have taken into account the fact that the surface oscillations attenuate exponentially with distance from the surface into the interior of each of the phases. As a result we obtain (see also Ref. 15)

$$\begin{aligned} -\bar{G} \left\{ \omega^2 \left[1 + \frac{\rho\rho_1|k_x|}{(\Delta\rho)^2} \right] - \frac{\tilde{\alpha}_{xx}\rho k_x^2|k_x|}{(\Delta\rho)^2} \right\} \\ = \frac{|k_z^l|}{|k_x|} \frac{\rho\rho_1|k_x|}{(\Delta\rho)^2} \left(\omega^2 - \frac{\tilde{\alpha}_{xx}}{\rho_1} k_x^2 \right), \end{aligned} \quad (26)$$

where

$$\bar{G} = [(\omega^2 - 2c_l^2 k_x^2)^2 - 4c_l^4 k_x^2 |k_z^l| |k_z^l|] / \omega^4.$$

Let us draw attention to the fact that the surface-elastic constant λ_{1xxx} does not figure anywhere in Eq. (26). For small values of the dimensionless parameter $\tilde{\varepsilon}$, which coincides in order of magnitude with the parameter ε introduced earlier ($\tilde{\varepsilon} = \rho\rho_1|k_x|/(\Delta\rho)^2 < 1$), the right-hand of (26) is small compared to the left-hand side, and the spectral equation breaks up into two independent equations

$$\bar{G} = 0, \quad \omega^2(1 + \tilde{\varepsilon}) - \tilde{\alpha}_{xx}\rho k_x^2|k_x|/(\Delta\rho)^2 = 0, \quad (27)$$

determining respectively the linear spectrum of the Rayleigh waves on the free surface of the solid: $\omega = p_1 c_l |k_x|$, where p_1

is a numerical factor (of the order of unity) composed of the velocities c_t and c_l , and the $\omega \propto k^{3/2}$ spectrum

$$\omega^2 = \tilde{\alpha}_{xx} \rho k_x^2 |k_x| / (\Delta\rho)^2 (1 + \tilde{\epsilon}),$$

of the melting-crystallization waves, slightly renormalized as a result of the presence of the inertial surface mass. Notice that, for $\tilde{\epsilon} \ll 1$, the phase velocity of the melting-crystallization waves is much smaller than that of the Rayleigh waves. The two branches of the spectrum begin to interact with each other when $1 \ll \tilde{\epsilon} \ll (\rho/\Delta\rho)$. But in the leading approximation in $\Delta\rho/\rho < 1$ and $kd < 1$, in which Eq. (26) was obtained, this interaction leads only to the renormalization of the Rayleigh waves as a result of the presence of the capillary effects. The spectrum of the Rayleigh waves will have the form $\omega = p_2 c_t |k_x|$, where $p_2 < p_1$ is a numerical factor. The spectrum of the melting-crystallization waves can be determined as before from the second equation in (27). It is linear in the region $\tilde{\epsilon} \gg 1$: $\omega^2 = \tilde{\alpha}_{xx} k_x^2 / \rho_1$. The relative disposition of the two spectral branches is determined in the region $\tilde{\epsilon} \gg 1$ by the relation between α_{xx}/ρ_1 and $p_2^2 c_t^2$. The spectral branches do not intersect when $\tilde{\alpha}_{xx}/\rho_1 < p_2^2 c_t^2$.

Let us return to the formula (25), and consider the particular case of normal incidence of sound from the liquid, i.e., the case when $\theta_{\text{liq}} = \theta_l = \theta_t = 0$. For normal incidence

$$D = 1 - R = \frac{4e^2 c_{\text{liq}} c_t}{1 + e^2 (c_{\text{liq}} + c_t)^2 / c_t^2}, \quad \epsilon = \frac{\rho \rho_1}{(\Delta\rho)^2} \frac{\omega}{c_{\text{liq}}},$$

which coincides with the result obtained by Castaing and Puech.⁵ For the values of the parameter $\epsilon < 1$ we have $D = 4e^2 c_{\text{liq}} / c_t \ll 1$; for $1 < \epsilon < \rho/\Delta\rho$ we have $D = 4c_{\text{liq}} c_t / (c_{\text{liq}} + c_t)^2 \sim 1$, in accord with the discussion carried out above. Let us draw attention to the fact that the formula for D in the case when $1 < \epsilon < \rho/\Delta\rho$ coincides with the formula for D in the case of normal incidence of acoustic waves at a classical boundary between two immiscible liquids of the same density (see Ref. 16, §65). Indeed, estimates carried out on the basis of the system of equations (16) show that $(\Delta\rho/\rho) \zeta \ll v_{\text{liqz}} \approx v_{\text{solz}} \ll \zeta$ for $1 < \epsilon < \rho/\Delta\rho$. Consequently, on the one hand, $u \ll \zeta$, and the assumption under which the formula (13) was derived is confirmed. (Let us emphasize that, for $\epsilon < 1$, the displacement u is also small compared to ζ .) On the other hand, $v_{\text{liqz}} \approx v_{\text{solz}} \gg (\Delta\rho/\rho) \zeta$, and the matter balance equation in (16) reduces to the condition $v_{\text{liqz}} = v_{\text{solz}}$. In this case the boundary conditions (16), with allowance for the fact that at normal incidence of the wave from the liquid transverse sound is not generated in the solid and that $\delta\sigma_{zz} = -\delta\mathcal{P}_{\text{sol}}$, will coincide with the boundary conditions at the interface between two classical immiscible liquids: $\delta\mathcal{P}_{\text{liq}} = \delta\mathcal{P}_{\text{sol}}$, $v_{\text{liqz}} = v_{\text{solz}}$. This analysis shows that the characteristic parameter ϵ of the problem can be written in the form $\epsilon = \omega\tau$, where $\tau = (d/c) (\rho/\Delta\rho)^2$ is the hydrodynamic surface-reconstruction time. At $\omega\tau \gg 1$ the quantum boundary becomes a classical one, and D increases significantly. Below we limit ourselves to the consideration of low frequencies: $\omega\tau < 1$. The formula (25) contains the total transmission coefficient for incidence of sound from the liquid at angles smaller than the Brewster angle. Let us

now consider separately the longitudinal- and transverse-sound transmission coefficients, which make up D in the low-frequency region ($\epsilon < 1$):

$$D_l = \frac{|\dot{\mathbf{u}}_l|^2 c_l \cos\theta_l}{|\mathbf{v}_{\text{liq in}}|^2 c_{\text{liq}} \cos\theta_{\text{liq}}} = \frac{4\cos\theta_l \cos\theta_{\text{liq}} \rho^4 \omega^2}{c_l c_{\text{liq}} (\Delta\rho)^4 G^2} \times [d_2 \cos 2\theta_l - (\Delta\rho/\rho) \sin 2\theta_l d_1]^2$$

$$D_t = \frac{|\dot{\mathbf{u}}_t|^2 c_t \cos\theta_t}{|\mathbf{v}_{\text{liq in}}|^2 c_{\text{liq}} \cos\theta_{\text{liq}}} = \frac{4\cos\theta_t \cos\theta_{\text{liq}} \rho^4 \omega^2}{c_t c_{\text{liq}} (\Delta\rho)^4 G^2} \times [d_2 (c_t^2/c_l^2) \sin 2\theta_t + (\Delta\rho/\rho) d_1 \cos 2\theta_t]^2, \quad (28)$$

where $\mathbf{v}_{\text{liq in}}$ is the velocity field in the acoustic wave incident from the liquid. Comparison of the formula (25) for low frequencies and the formulas (28) leads to the relation $D_l + D_t = D = 1 - R$, which is in accord with the law of conservation of energy. Naturally, $D_l = 0$ when the angle of incidence is equal to the Brewster angle, and $D_t = 0$ at normal incidence.

Notice that, with the aid of the formulas (25) and (28), we can experimentally determine the values of all the surface constants important for the present problem. To begin with, from the transmission coefficient for longitudinal sound in the case of normal incidence we can find the inertial mass ρ_1 , and from the transmission coefficient for transverse sound in the case when the angle of incidence is equal to the Brewster angle we can determine the surface-elastic constant λ_{1xxx} . Then from the transmission coefficient for the case of oblique incidence we can find the value of the constant $\tilde{\alpha}_{xx}$.

C. Let us now consider the case when the sound is incident from the liquid at an angle greater than the Brewster angle (i.e., the case when $\sin\theta_{\text{liq}} > c_{\text{liq}}/c_l$). In this case the longitudinal acoustic wave attenuates with distance from the boundary into the interior of the crystal. The quantity $\cos\theta_l$ is purely imaginary: $\cos\theta_l = i|\cos\theta_l|$. With allowance for this circumstance, D has the form

$$D = 1 - R = \frac{4 \cos\theta_{\text{liq}} \cos\theta_l \rho^4 \omega^2}{c_{\text{liq}} c_t (\Delta\rho)^4} \times \frac{[(\Delta\rho/\rho)^2 d_1^2 \cos^2 2\theta_l + 4d_2^2 (c_l/c_t)^4 \sin^2 \theta_l |\cos\theta_l|^2]}{[\cos^4 2\theta_l + 4(c_l/c_t)^4 \sin^2 2\theta_l \sin^2 \theta_l |\cos\theta_l|^2]}. \quad (29)$$

Here by the quantity $\sin\theta_l > 1$, we mean the quantity $(c_l/c_{\text{liq}}) \sin\theta_{\text{liq}}$. The transmission coefficient given by the formula (29) coincides, in accord with the law of conservation of energy, with the transverse-sound transmission coefficient given by the formula

$$D_t = \frac{|\dot{\mathbf{u}}_t|^2 c_t \cos\theta_t}{|\mathbf{v}_{\text{liq in}}|^2 c_{\text{liq}} \cos\theta_{\text{liq}}}.$$

From the formula (29) it follows that in the angle range

$$\arcsin(c_{\text{liq}}/c_l) < \theta_{\text{liq}} < \arcsin(c_{\text{liq}}/c_l) + |\eta_2| \Delta\rho/\rho,$$

where η_2 is a constant composed of the sound velocities, D is nonzero largely because of the effects connected with the surface elasticity. In the entire remaining region of angles

$$\arcsin(c_{\text{liq}}/c_l) + |\eta_2| \Delta\rho/\rho < \theta_{\text{liq}} < \pi/2$$

(in particular, when the angles of incidence are close to $\pi/2$, i.e., are glancing angles) D is related with the surface constants ρ_1 and $\tilde{\alpha}_{xx}$. When the angle of incidence, θ_{liq} , is equal to the Brewster angle, the formula (29) and the second formula in (28) coincide, and lead to the expression

$$D = \frac{4 \cos \theta_i \cos \theta_{\text{liq}} \lambda_{1xxx} \omega^2}{c_i c_t \cos^2 2\theta_i (\Delta\rho)^2 c_t^4} \sim \left(\frac{\Delta\rho}{\rho}\right)^2 \varepsilon^2.$$

As the angle of incidence $\theta_{\text{liq}} \rightarrow \pi/2$, the transmission coefficient D tends to zero like $\cos\theta_{\text{liq}}$.

D. Let us also briefly discuss the case of incidence at the phase interface of a transverse plane acoustic wave from the solid.

For the incident, transmitted, and two reflected waves we have

$$\begin{aligned} \Psi &= \Psi_{10} \exp[-i\omega t + i(\omega/c_t) \sin \theta_i x + i(\omega/c_t) \cos \theta_i z] \\ &+ \Psi_{20} \exp[-i\omega t + i(\omega/c_t) \sin \theta_i x - i(\omega/c_t) \cos \theta_i z], \\ \Phi &= \Phi_0 \exp[-i\omega t + i(\omega/c_t) \sin \theta_i x - i(\omega/c_t) \cos \theta_i z], \\ \varphi &= \varphi_0 \exp[-i\omega t + i(\omega/c_{\text{liq}}) \sin \theta_{\text{liq}} x + i(\omega/c_{\text{liq}}) \cos \theta_{\text{liq}} z], \end{aligned} \quad (30)$$

where

$$\frac{\sin \theta_i}{c_t} = \frac{\sin \theta_t}{c_t} = \frac{\sin \theta_{\text{liq}}}{c_{\text{liq}}}$$

Ψ_{10} is the amplitude of the incident transverse acoustic wave, Ψ_{20} and Φ_0 are the amplitudes of the reflected longitudinal and transverse acoustic waves, and φ_0 is the amplitude of the acoustic wave generated in the liquid. The recrystallization-induced displacement ξ is given as before by the formula (23). In the case when transverse sound is incident at an angle $0 < \theta_i < \arcsin(c_t/c_l)$ all the waves have real wave-vector components along the normal to the interface, i.e., along the z axis.

For angles of incidence $\arcsin(c_t/c_l) < \theta_i < \arcsin(c_t/c_{\text{liq}})$ the reflected longitudinal acoustic wave attenuates exponentially with distance into the interior of the crystal. For angles of incidence $\arcsin(c_t/c_{\text{liq}}) < \theta_i < \pi/2$ both the reflected longitudinal acoustic wave in the crystal and the acoustic wave generated in the liquid are exponentially damped.

Substituting (23) and (30) into the system of equations (16), we find that the coefficient of transmission by the liquid of transverse sound incident at the interface from the solid

$$D = |\mathbf{v}_{\text{liq}}|^2 c_{\text{liq}} \cos \theta_{\text{liq}} / |\dot{\mathbf{u}}_{\text{in}}|^2 c_t \cos \theta_i,$$

where \mathbf{u}_{in} is the elastic-displacement field in the incident transverse wave, coincides with the transmission coefficient for transverse sound incident at the interface from the liquid,

$$D_t = |\dot{\mathbf{u}}_t|^2 c_t \cos \theta_t / |\mathbf{v}_{\text{liq in}}|^2 c_{\text{liq}} \cos \theta_{\text{liq}},$$

and is given by the second formula in (28) when $\sin \theta_t < c_t/c_l$, and by the formula (29) when $c_t/c_l < \sin \theta_t < c_t/c_{\text{liq}}$. This result is a consequence of the principle of reversibility.

Notice that the transmission coefficient for normal incidence (i.e., in the case when $\theta_i = \theta_t = \theta_{\text{liq}} = 0$) is equal to zero. For small angles of incidence θ_i , the coefficient D is

nonzero largely because of the effects connected with the inertial surface mass ρ_1 . In the angle range

$$\arcsin(c_t/c_l) - |\eta_s| \Delta\rho/\rho < \theta_i < \arcsin(c_t/c_l) + |\eta_s| \Delta\rho/\rho$$

D is nonzero because of the effects due to the surface-elastic constant λ_{1xxx} .

Let us also note that, at low frequencies, the coefficient of conversion of transverse sound into longitudinal sound in the case when $\sin \theta_t < c_t/c_l$ is, up to the small frequency-dependent terms $\sim \omega^2$, equal to

$$R_t = \frac{|\dot{\mathbf{u}}_t|^2 c_t \cos \theta_t}{|\dot{\mathbf{u}}_{\text{in}}|^2 c_t \cos \theta_t} = \frac{4c_t \cos \theta_t}{c_t \cos \theta_t} \frac{\cos^2 2\theta_i \sin^2 2\theta_i}{G^2}.$$

This expression coincides with the transformation ratio for the case of transverse sound incident at the boundary between a classical crystal and vacuum.

3. THE QUANTUM LIQUID-QUANTUM CRYSTAL PHASE INTERFACE IN THE PRESENCE OF A TANGENTIAL SUPERFLUID-LIQUID FLOW

Let us consider the problem of the development in time of surface oscillations excited in the background of uniform "slipping" of superfluid liquid along the surface of the initially undeformed stationary crystal. The formulation of such a problem is itself possible only for the superfluid liquid, since, because of viscosity, the normal liquid has zero tangential velocity at the boundary with the solid.

The derivation of the equations of surface hydrodynamics in the presence of a stationary surface superfluid-liquid flow parallel to the boundary is carried out on the basis of the same formulas, (1)–(13), that are used in the derivation of the equations in the absence of surface flows. Let us limit ourselves to the case of sliding velocities that are small compared to the sound velocities. Then it is easy to show that we again can ignore in the problem the dependence of the surface energy (3) on the surface momentum $P_{0\alpha}$. Retaining only the leading terms in the small parameters $\Delta\rho/\rho \ll 1$, $kd \ll 1$, and $v_0/c \ll 1$, where v_0 is the slipping velocity of the liquid, we have for not too high frequencies, i.e., frequencies ω such that $\omega(d/c)(\rho/\Delta\rho) < 1$, the following linear boundary conditions:

$$\begin{aligned} \rho_{\text{liq}} (\mathbf{v}_{\text{liq}} \mathbf{n} - V_b) &= \rho_{\text{sol}} (\mathbf{v}_{\text{sol}} \mathbf{n} - V_b), \\ \nabla_{\beta} &= (\lambda_{1\beta\alpha\gamma} \xi_{\gamma}) = \sigma_{z\alpha} + \rho_{\text{sol}} v_{\text{liq}\alpha} (\mathbf{v}_{\text{sol}} \mathbf{n} - V_b), \\ \sigma_{zz} + \mathcal{P}_{\text{liq}} - \mathcal{P}_0 &= 0, \\ \rho_{\text{sol}} (\mu_{\text{sol}} + v_{\text{liq}}^2/2 - v_{\text{sol}\alpha} v_{\text{liq}\alpha} - \mu_{\text{sol}}) + \mathcal{P}_{\text{sol}} - \mathcal{P}_{\text{liq}} \\ &+ \nabla_{\beta} (\tilde{\alpha}_{\mu\beta} \xi_{\mu} + \lambda_{1\alpha\gamma\beta} u_{\alpha\gamma}) - \rho_1 \dot{\xi} = 0. \end{aligned} \quad (31)$$

The system of equations (31) differs from (15) by the presence of terms quadratic in the velocities in the generalized phase-equilibrium equation and in one of the mechanical-equilibrium equations. Let us, moreover, note that the liquid-velocity component along the normal to the deformed surface, $\mathbf{v}_{\text{liq}} \mathbf{n}$, does not, even in the first approximation in the deviation from equilibrium, coincide with $v_{\text{liq}z} \cdot \mathbf{v}_{\text{liq}} \mathbf{n} = v_{\text{liq}z} - v_{0\alpha} \xi_{\alpha}$. In the zeroth approximation the system of equations (31) yields the relations

$$\mathcal{P}_{\text{sol}} = \mathcal{P}_{\text{liq}} = \mathcal{P}_0, \quad \mu_{\text{sol}} = \mu_{\text{liq}} + v_0^2/2 = \mu_0.$$

Notice that the chemical potentials of the solid and liquid do not coincide in the presence in the problem of a stationary surface flow even in the absence of small oscillations in the system. In the first approximation in the deviations of the quantities from their equilibrium values we obtain, after allowance has been made for the thermodynamic identities

$$\delta \mathcal{P}_{\text{sol}} = \rho_{\text{sol}} \delta \mu_{\text{sol}}, \quad \delta \mathcal{P}_{\text{liq}} = \rho_{\text{liq}} \delta \mu_{\text{liq}},$$

the equations

$$\begin{aligned} \Delta \rho \delta \mu_{\text{liq}} + \nabla_{\mathbf{B}} (\tilde{\alpha}_{\mu\beta} \delta \xi_{\mu} + \lambda_{1\alpha\gamma\beta} \delta u_{\alpha\gamma}) - \rho_1 \delta \ddot{\xi} \\ + \rho v_{0\alpha} (\delta v_{\text{liq}\alpha} - \delta v_{\text{sol}\alpha}) = 0, \\ \delta \sigma_{z\alpha} + \rho v_{0\alpha} \delta \dot{\xi} = \nabla_{\beta} (\lambda_{1\beta\alpha\gamma} \delta \dot{\xi}_{\gamma}), \\ \delta \sigma_{zz} + \rho \delta \mu_{\text{liq}} = 0, \\ \rho \delta v_{\text{sol}z} - \rho \delta v_{\text{liq}z} + \rho v_{0\alpha} \delta \dot{\xi}_{\alpha} - \Delta \rho \delta \dot{\xi} = 0. \end{aligned} \quad (32)$$

Let us analyze the spectrum of the surface oscillations generated in the background of uniform slipping of the liquid for small wave vectors: $\varepsilon = (\rho/\Delta\rho)^2 kd \ll 1$. In this case it can be shown that $|\dot{u}| < |v_{\text{liq}}|$. Therefore, if we are interested only in the low-lying branch of the spectrum, i.e., in the melting-crystallization waves, then we can discard the two mechanical-equilibrium equations in the system of equations (32), and set, as is done in Ref. 1, $\mathbf{u} = \dot{\mathbf{u}} = 0$ in the phase-equilibrium and balance equations. Then (32) reduces to

$$\begin{aligned} \Delta \rho \delta \mu_{\text{liq}} + \tilde{\alpha}_{\mu\beta} \nabla_{\beta} \delta \xi_{\mu} - \rho_1 \delta \ddot{\xi} + \rho v_{0\alpha} \delta v_{\text{liq}\alpha} = 0, \\ -\rho \delta v_{\text{liq}z} + \rho v_{0\alpha} \delta \dot{\xi}_{\alpha} - \Delta \rho \delta \dot{\xi} = 0. \end{aligned} \quad (33)$$

The system (33) is valid at the temperature $T = 0$. In a real experiment $T \neq 0$. It is important for us to determine both the real and the imaginary parts of the melting-crystallization wave spectrum. Therefore, it is expedient to include in (33) the dissipation due to the friction of the phase interface on the normal excitations of the liquid and crystal. Analysis similar to the one carried out in Ref. 17 shows that this can be done by adding to the phase-equilibrium equation a term linear in the recrystallization rate, specifically, the term $-\rho \delta \dot{\xi}/K$, where K is the crystal growth coefficient. Assuming $1/K$ to be a small quantity, and solving in this case the system of equations (33) with the aid of perturbation theory together with the volume continuity equation $\text{div} \delta \mathbf{v}_{\text{liq}} = 0$ and the equation $\delta \dot{\mathbf{v}}_{\text{liq}} + \nabla \{ \delta \mu_{\text{liq}} + v_{0\alpha} \delta v_{\text{liq}\alpha} \} = 0$ of superfluid motion, we find the spectrum

$$\begin{aligned} \omega \approx \left[\omega_0 - \frac{\rho}{\Delta\rho} v_0 k_x \right] - \frac{i}{2} \frac{\rho^2}{(\Delta\rho)^2} \frac{|k_x|}{K} \frac{[\omega_0 - (\rho/\Delta\rho) v_0 k_x]}{\omega_0}, \\ \omega_0^2 = \frac{\rho \tilde{\alpha}_{xx} k_x^2 |k_x| / (\Delta\rho)^2}{[1 + \rho_1 |k_x| / (\Delta\rho)^2]}, \end{aligned} \quad (34)$$

which coincides in the case when $K = \infty$ with Nozières's result.¹⁸ In deriving (34), we considered the two-dimensional problem, i.e., the $k = (k_x, k_z)$ case, and assumed the velocity to be parallel to the x axis. In (34) ω_0 is the melting-crystallization wave spectrum in the absence of stationary flows. When $v_0 k_x > 0$ and $(\rho/\Delta\rho) |v_0| |k_x| > \omega_0$, we have Im

$\omega > 0$, and, consequently, instability sets in. Physically, its origin is connected with the fact that the solid is stationary, and therefore the correction to the real part of the frequency, equal to $-(\rho/\Delta\rho) v_0 k_x$, is not just a Galilean shift. This instability is very similar to the instability of a superfluid flow against excitation production at velocities higher than the critical velocity.

Let us consider the region $\varepsilon < 1$. Then $\omega_0^2 \approx \tilde{\alpha}_{xx} \rho k_x^2 |k_x| / (\Delta\rho)^2$ and $(\rho/\Delta\rho) |v_0| |k_x| > \omega_0$ for $|v_0| > (\tilde{\alpha}_{xx} |k_x| / \rho)^{1/2} \rightarrow 0$ if $k_x \rightarrow 0$, i.e., the instability sets in virtually in a nonthreshold manner (the limitation on v_0 is due only to the fact that $|k_x| > 2\pi/L$, where L is the dimension of the crystal). But when we take gravity into account, we have

$$\omega_0^2 = [\tilde{\alpha}_{xx} \rho k_x^2 |k_x| / (\Delta\rho)^2 + \rho g |k_x| / \Delta\rho],$$

and there appears the instability threshold

$$\begin{aligned} (\rho/\Delta\rho) v_{\text{ocr}} = \min(\omega_0 / |k_x|), \\ v_{\text{ocr}} = (4g \tilde{\alpha}_{xx} \Delta\rho / \rho^2)^{1/4} \sim 4 \text{ cm/sec.} \end{aligned}$$

Nevertheless, the critical velocity at which the instability sets in is low compared to the sound velocity, and the assumptions under which we derived the system of equations (31) hold.

I express my profound gratitude to A. F. Andreev for the formulation of the problem and for the fruitful discussions that motivated the present investigations. I am also grateful to Yu. K. Dzhikaev, Yu. A. Kosevich, V. I. Marchenko, A. É. Meïerovich, and A. Ya. Parshin for numerous and very useful discussions, and to A. V. Babkin for acquainting me with the experimental data.

¹¹Here and below we neglect the static surface stresses, which have the order of smallness d/L , where L is the dimension of the crystal. They give rise to small—in comparison with the large equilibrium pressure—elastic-modulus renormalizations when we go from the Lagrange to the Euler variables.

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Translated by A. K. Agyei