Instability of parametric spin wave systems in antiferromagnets

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The stability of the steady state of parametrically excited spin waves is investigated with allowance for their nonlinear relaxation and for magnon-magnon interactions of higher orders. It is shown that the parametric system can develop aperiodic and oscillatory instabilities. Existence criteria are found for these instabilities, and several examples are considered. The phenomenon of multistability in the parametric system, i.e., the observation of different (not necessarily steady) microwave absorption levels at the same level of pumping, is discussed.

INTRODUCTION

The study of parametric excitation of magnons by a microwave field yields important information on the properties of spins systems in ferromagnets and antiferromagnets.[1,2] In addition, a system of parametrically excited spin waves (PSW) is in itself very convenient object for studying and modeling the processes which occur in a many-particle medium. Substantial progress in understanding the physics of PSW has been made since Zakharov, L'vov, and Starobinets developed their theoretical model of spin-wave turbulence (the so-called S theory).[3] According to this theory,[3] a pump field $h\cos\omega_p t$ above the threshold ($h > h_c$) for the parametric resonance ($\omega_p = \omega_{\mathbf{k}} + \omega_{-\mathbf{k}}$, where $\omega_{\mathbf{k}}$ is the PSW frequency) gives rise to a phase correlation for pairs of waves having wave vectors of equal magnitude and opposite direction and thus establishes a strict connection between the number $n(\mathbf{k})$ of excited magnons and the phase $\psi(\mathbf{k})$ of the pair. A number of the conclusions of the S theory are in agreement with the experimental data and permit a satisfactory interpretation of many observed effects. For ferromagnets, in which the anisotropy of spin-wave processes is substantial because of the dipole-dipole interaction, the "parametric system" is made up of separate groups of PSW having different wave vectors. For this reason, it is possible to have different steady or oscillatory regimes in the system.[1,3–5] For antiferromagnets, in which one can in practice neglect[6] the anisotropy of the amplitudes of the magnon interactions, the theory[3] implies only a spherically isotropic (in $\mathbf{k}$) stable steady state of the PSW. The properties of this state were studied experimentally in Refs. 7–9, and the results were in satisfactory agreement with the S theory at power levels P_c slightly above the threshold ($P/P_c - 1 \sim 1$). However, in the experiments of Refs. 9–15 it was found that, beginning at a certain pump power, the steady state loses stability, and regular or irregular oscillations arise in the microwave power absorbed by the sample. Importantly, the instabilities in the parametric system were observed for different types of PSW (electronic spin waves in Refs. 10–14 and nuclear spin waves in Refs. 9 and 15) and in different antiferromagnetic crystals: $MnCO_3$ (Refs. 9, 13), $CsMnF_3$ (Refs. 10, 15), $CsMnCl_3$ (Ref. 11), $FeBO_3$ (Ref. 12), and $CuCl_2 \cdot 2H_2O$ (Ref. 14). One naturally wonders whether the instabilities of the steady state of the PSW system in antiferromagnets can be described in terms of the basic physical principles of the theory of spin-wave turbulence.[3] This question is the subject of the present paper.

Instabilities in a PSW system can result from the following obvious causes (either individually or in concert): 1) As the pump power increases, the magnon dissipation parameters change and the higher-order nonlinear magnon interactions become important; 2) the formation of the parametric instability can involve other degrees of freedom of the crystal (other magnons, phonons, etc); 3) there is a whole class of possible instabilities which lead to spatial inhomogeneities in a parametric system. Several aspects of the theory of PSW instabilities (mainly the spatially inhomogeneous instabilities) have been discussed in the literature. For example, Borovik[16] considered the possibility that soliton solutions leading to oscillations of the absorbed power can form in a PSW system.[1)] Fal'kovich[18] proposed a mechanism whereby a kinetic instability of the parametric system leads to the excitation of natural elastic oscillations of the sample, like those observed experimentally.[12,13]

In this paper we consider only spatially homogeneous instabilities of the parametric system. The description of PSW follows the basic physical principles[2)] of the S theory[3]: a) the phase correlation of the parametric pairs is determined by an interaction Hamiltonian that is diagonal in the wave pairs; b) the interaction with the heat bath is taken into account by the introduction of a dissipative term in the equation of motion. Before turning to the main exposition, let us briefly review the known results. The initial equations of the theory[3] can be written in the form (see also Ref. 8)

$$\frac{1}{2}\frac{d}{dt}\theta + hV_0\sin\theta = \frac{\omega_p}{2} - \omega_k - 2T_0N - S_0N, \tag{1a}$$

$$\frac{1}{2}\frac{d}{dt}N = N(hV_0\cos\theta - \gamma_0), \tag{1b}$$

where $\theta(k) \equiv \pi/2 - \psi(k)$; $N(k) \equiv n(k)/\mathscr{N}$ is the number of PSW per magnetic cell ($\mathscr{N}$ is the number of magnetic cells in the sample); $V_0(k)$ is the coupling coefficient of the PSWs with the pump (with allowance for the "indirectness" of the pumping[20]); $\gamma_0(k)$ is the magnon relaxation rate, and $S_0(k)$ and $T_0(k)$ are the four-magnon interaction amplitudes responsible for the phase shift and the renormalization of the PSW frequency, respectively. For convenience we as-

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sume here and below that the dimensional quantities are given in frequency units. According to Ref. 3, the magnons excited in the steady state are those whose wave vectors lie on the resonant surface

$$\omega_p/2-\omega_k-2T_0N_0=0, \tag{2}$$

where the steady-state values N_0 and θ_0 are determined from (1a) and (1b) with the derivatives set to zero. The stability analysis of system (1) is done by the standard methods (see, e.g., Ref. 21):

$$\theta=\theta_0+\delta\theta, \quad N=N_0+\delta N; \quad \delta\theta,\ \delta N \propto \exp(\lambda t).$$

The following characteristic numbers result:

$$\lambda_\pm=-\gamma_0\pm i\{4S_0(2T_0+S_0)N_0^2-\gamma_0^2\}^{1/2}. \tag{3}$$

Usually Imλ is called the frequency of collective oscillations of the PSW, and Reλ characterizes the damping of these oscillations. The development of instability requires that at least one of the characteristic numbers have a positive real part.[21] It follows from (3) that this condition is realizable only for

$$S_0(2T_0+S_0)<0. \tag{4}$$

Inequality (4) is known as the criterion for the onset of aperiodic (Im$\lambda = 0$) instability of the steady state of the PSW.[3] This inequality can be satisfied for certain modes in ferromagnets,[3] in which case a numerical modeling (including a second group of excited magnons) shows that the parametric system develops self-oscillations analogous to the self-oscillations of the magnetization that are observed experimentally.[1,4] For antiferromagnets, however, inequality (4) is not satisfied ($S_0 = T_0$), i.e., as we have said, in this case Eqs. (1a) and (1b) imply only a stable steady state of the PSW above the excitation threshold. In what follows we consider physical mechanisms which lead to modifications of (1a) and consider several physical mechanisms which lead to modifications of (1a) and (1b) and the consequences of these modifications.

GENERALIZED CRITERION OF INSTABILITY OF THE STEADY STATE OF PSW

As we know, the relaxation of waves in solids is the process whereby energy stored in one mode "spreads out" over an infinite number of degrees of freedom of the crystal. At small deviations from equilibrium the relaxation toward the ground state occurs exponentially with a rate determined by the thermodynamic equilibrium properties of the object. Parametric instability increases the population of the excited mode above its thermal level by several orders of magnitude. Significant changes can thus occur in the PSW dissipation processes, since the properties of one or more of the subsystems taking part in the relaxation of these waves can change appreciably. There are two distinct cases of nonlinear relaxation of PSW. In the first case the dissipation increases. This is called positive differential nonlinear damping (the medium inhibits wave excitation). Alongside the phase mechanism, positive nonlinear damping is an independent mechanism limiting the amplitude of the excited modes. The second case, in which the PSW dissipation falls off, is called negative differential nonlinear damping (the medium promotes wave excitation). Negative nonlinear relaxation of PSW is intimately related to one model for "hard" PSW excitation, which is characterized by the presence of different amplitudes for the onset (h_{c1}) and extinction (h_{c2}) of the parametric instability ($h_{c1} > h_{c2}$).

Nonlinear damping of PSW is easily included in the calculational scheme of the S theory.[3] In the case when the medium has time to readjust to the instantaneous values of N, it is sufficient to replace γ_0 in (1b) by the nonlinear function $\tilde{\gamma}(N)$:

$$\gamma_0\to\tilde{\gamma}(N)=\gamma_0+\gamma_1N+\gamma_2N^2+\dots\ . \tag{5}$$

Estimates of γ_1 for ferromagnets[3,22] and antiferromagnets (after a processing of the experimental data[8,11,23,-25]) show that it lies in the range $|\gamma_1/S_0| \sim 0.1$–1.

Another way of going beyond system of equations (1) (i.e., beyond the standard S theory) is to take into account the next higher orders in the interactions of the parametric pairs. It is easily shown that for magnets with isotropic magnon-magnon interactions such a procedure leads to equations for θ and N that are analogous to (1a) and (1b) but with the following substitutions:

$$S_0\to\tilde{S}(N)=S_0+S_1N+S_2N^2+\dots, \tag{6a}$$

$$T_0\to\tilde{T}(N)=T_0+T_1N+T_2N^2+\dots, \tag{6b}$$

$$V_0\to\tilde{V}(N)=V_0+V_1N+V_2N^2+\dots\ . \tag{6c}$$

Here the coefficients S_1 and T_1 are determined by six-magnon scattering processes; S_2 and T_2 by eight-magnon processes, etc. It should be noted that besides the change in the coefficients S_0 and T_0 responsible for the phase shift and the renormalization of the PSW frequency, there is also a change in the coupling coefficient V_0 of the parametric system with the alternating field,[26] i.e., there is a nonlinear response to the pump. The factors V_1, V_2, etc., are determined by the effective interactions of the pump field with four magnons, six magnons, etc. A rough algebraic estimate of the expansion coefficients in (6a)–(6c) for $T_0 \gg \omega_k$ gives $R_n \propto R_0(T_0/\omega_k)^n$, where $R = S, T, V$.

The steady states of a parametric system with allowance for generalizations (5) and (6) are determined by the following equations:

$$h\tilde{V}(N_0)\cos\theta_0=\tilde{\gamma}(N_0), \tag{7a}$$

$$N_0^2[\tilde{S}(N_0)]^2+[\tilde{\gamma}(N_0)]^2=[h\tilde{V}(N_0)]^2, \tag{7b}$$

while the condition for the resonant surface is analogous to (2). It should be noted that now Eq. (7b) can have a number of solutions with $N_0(h) > 0$, leading to multistability in the parametric system, a frequent occurrence in highly nonequilibrium systems.[27]

The stability of the steady states of the PSW is studied (as above) by linearizing the modified equations (1a) and (1b). The following characteristic numbers result:

$$\lambda_{\pm}=-\tilde{\gamma}-\frac{\delta\tilde{\gamma}}{\delta N}N_0+\frac{\tilde{\gamma}}{\tilde{V}}\frac{\delta\tilde{V}}{\delta N}N_0\pm i\Big\{4\Big[\tilde{S}(2\tilde{T}+\tilde{S})N_0^2+\tilde{S}N_0^3\frac{\delta}{\delta N}(2\tilde{T}+\tilde{S})-\frac{N_0}{2}\frac{\delta}{\delta N}(h^2\tilde{V}^2-\tilde{\gamma}^2)\Big]-\Big(\tilde{\gamma}+\frac{\delta\tilde{\gamma}}{\delta N}N_0-\frac{\tilde{\gamma}}{\tilde{V}}\frac{\delta\tilde{V}}{\delta N}N_0\Big)^2\Big\}^{1/2}. \quad (8)$$

It is easy to see that the inequality $\mathrm{Re}\lambda>0$ has two solutions in this case (depending on the sign of the expression in curly brackets). The first corresponding to a modification of the aperiodic instability criterion (4) and can be written

$$\tilde{S}(2\tilde{T}+\tilde{S})+\tilde{S}N_0\frac{\delta}{\delta N}(2\tilde{T}+\tilde{S})-\frac{1}{2N_0}\frac{\delta}{\delta N}(h^2\tilde{V}^2-\tilde{\gamma}^2)<0,$$
$$\mathrm{Im}\,\lambda_{\pm}=0. \quad (9)$$

The second solution is fundamentally new criterion for the onset of PSW instability:

$$\tilde{\gamma}+\frac{\delta\tilde{\gamma}}{\delta N}N_0-\frac{\tilde{\gamma}}{\tilde{V}}\frac{\delta\tilde{V}}{\delta N}N_0<0, \quad (10)$$
$$\Omega=|\mathrm{Im}\,\lambda_{\pm}|\neq 0.$$

This solution corresponds to the development of an oscillatory instability of a parametric system with frequency Ω.

EXAMPLES

Let us consider several examples illustrating the use of criteria (9) and (10). Our discussion will be limited to the influence of nonlinear PSW dissipation, which is especially interesting because in this particular case all the results of the analysis will be valid for both antiferromagnets and ferromagnets. We write the nonlinear PSW damping in the simplest from

$$\tilde{\gamma}=\gamma_0+\gamma_1 N_0. \quad (11)$$

For this case the steady-state values N_0 are given by the expression[22]

$$N_0^{(\pm)}(h)=(\gamma_0/|S_0|)\{-\varkappa\pm[\xi^2(1+\varkappa^2)-1]^{1/2}\}(1+\varkappa^2)^{-1}, \quad (12)$$

where

$$\varkappa=\gamma_1/|S_0|,\quad \xi=h/h_{c1}.$$

Criteria (9) and (10) assume the forms

$$\gamma_0\gamma_1/N_0+\gamma_1^2+S_0(2T_0+S_0)<0, \quad (13)$$
$$\gamma_0+2\gamma_1 N_0<0. \quad (14)$$

For positive nonlinear PSW damping there is only one branch: $N_0=N_0^{(+)}(h)>0$. It is easy to see that in antiferromagnets the steady state of the PSW on this branch is stable. For ferromagnets with $S_0(2T_0+S_0)<0$ there is an aperiodic instability [see (13)] if the nonlinear damping coefficient is not too large: $\gamma_1/|S_0|<|2T_0/S_0+1|^{1/2}$. It should be noted that this PSW instability does not arise immediately at the excitation threshold [as would be implied by the standard criterion (4)] but only after a certain supercriticality ξ_a is reached and the first term in (13) has become sufficiently small. Allowance for the positive nonlinear PSW damping

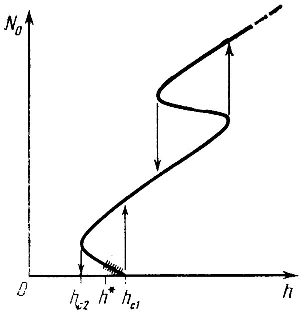

FIG. 1. Steady states of a parametric system for a "hard" type of magnon excitation. With increasing supercriticality, progressively higher-order nonlinearities become important, leading to multistabilities in the parametric system.

thus permits explanation of the experimental data[1,4] revealing a degeneracy of the self-oscillations of the magnetization in the ferromagnet YIG at supercriticalities of 0.1–1 dB (i.e., $\xi_a\simeq 1.01$–1.10).[3)] The nonlinear damping coefficient here amounts to $\gamma_1/|S_0|\sim 0.1$–1, in good agreement with estimates based on the susceptibility (see Ref. 22).

For a negative nonlinear PSW damping, $\gamma_1=-|\gamma_1|$, there are two branches of steady-state values $N_0(h)>0$: one branch with a positive ($\delta N_0/\delta h>0$) slope, $N_0=N_0^{(+)}$, $h\geqslant h_{c2}$ and one branch with a negative ($\delta N_0/\delta h<0$) slope, $N_0=N_0^{(-)}$, $h_{c2}\leqslant h\leqslant h_{c1}$ (see Fig. 1), where

$$h_{c1}=\gamma_0/V_0,\quad h_{c2}=h_{c1}(1+\varkappa^2)^{-1/2}. \quad (15)$$

It follows from inequality (13) that the steady states of the PSWs are aperiodically unstable if

$$N_0<N_0(h^*)=\gamma_0|\gamma_1|[S_0(2T_0+S_0)+\gamma_1^2]^{-1}. \quad (16)$$

For $S_0=T_0$ these states lie on the negative-slope branch (marked with hatches in Fig. 1), since

$$N_0(h_{c2})/N_0(h^*)=(3+\varkappa^2)/(1+\varkappa^2)>1. \quad (17)$$

Let us now consider the criterion of oscillatory instability (14), which implies that for

$$N_0>N_0(\tilde{h})=\gamma_0/2|\gamma_1| \quad (18)$$

the parametric system executes self-oscillations whose fundamental frequency can be estimated from the frequency of the small-amplitude collective oscillations of the PSWs [see Eq. (8)]:

$$\Omega=\gamma_0[(2N_0/\gamma_0)^2S_0(2T_0+S_0)-1]^{1/2}. \quad (19)$$

The amplitude of the self-oscillations of the PSWs is really not small, and because of the nonlinearity of the equations of motion their shape can be arbitrarily complex. We have done a computer modeling of the shape and Fourier spectrum of the self-oscillations of the absorbed power ($P\propto\tilde{\gamma}N$) for the case $\tilde{h}/h_{c1}=5$ (see Fig. 2). It is noteworthy that the spectrum contains oscillations at the subharmonic $\Omega/2$. As the pump power is increased further, the $\Omega/4$, $\Omega/8$, etc. subharmonics appear. This circumstance indicates that dynamical chaos develops easily when one more degree of freedom is

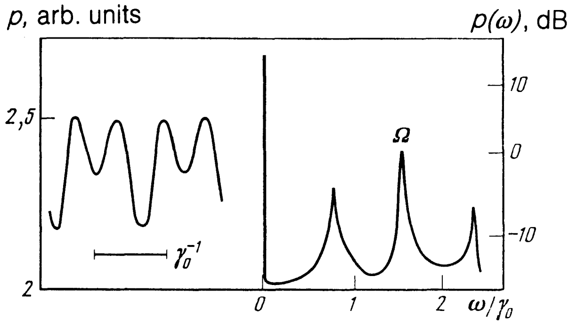

FIG. 2. The form of the self-oscillations of the absorbed power (left) and their Fourier spectrum (right), calculated for $h/h_{c1} = 5.3$ and $|\gamma_1/S_0| = 0.1$. The PSW are excited on the resonant surface given by Eq. (2). The "hardness" of the excitation is $(h_{c1}/h_{c2})^2 = 1.01$ (i.e., 0.43 dB).

present in the parametric system (e.g., when the dynamics of the subsystem that causes the nonlinear PSW relaxation is taken into account). Transition to chaos through a sequence of period doublings of the self-oscillations was observed experimentally in Refs. 14, 15, and 17.

Let us now make some estimates of the threshold for the oscillatory instability. After several transformations, Eqs. (12),(15), and (18) yield

$$\frac{\tilde{h}}{h_{c2}}=\frac{1}{2}\frac{1+\varkappa^2}{|\varkappa|},\qquad 1+\varkappa^2=\left(\frac{h_{c1}}{h_{c2}}\right)^2. \tag{20}$$

For $0<|\varkappa|\leqslant 1$ this formula is valid for the $N_0^{(+)}$ branch of the steady-state PSW values. We see that when the PSW excitation has a "hardness" $(h_{c1}/h_{c2})^2 \geqslant 2$ (i.e., $\geqslant 3$ dB), this entire branch is subject to oscillatory instability. For $|\varkappa|>1$ relation (20) is satisfied for the $N_0^{(-)}$ branch, and for $|\varkappa|\gg 1$ we have $\tilde{h}\approx h_{c1}/2$. Experimentally, only the $N_0^{(+)}$ branch has been studied; this branch was encountered in a field $h \geqslant h_{c1}$, and in a field h_{c2} a breakaway occurred from this branch to a state with $N_0 = 0$. The threshold for excitation of oscillations of the absorbed microwave power for electronic magnons was observed at $(\tilde{h}/h_{c2})^2 = 10$ in $CsMnF_3$ (Ref. 10) and at $(\tilde{h}/h_{c2})^2 = 70$ in $CsMnCl_3$ (Ref. 11). From (20) we get the following estimates for the nonlinear damping coefficients; $|\gamma_1/S_0| \approx 0.16$ and $|\gamma_1/S_0| \approx 0.06$, in order-of-magnitude agreement with the experimental situation. In an experiment on the excitation of nuclear magnons,[15] oscillations of the absorbed power were observed at all values of h up to the extinction field h_{c2} for the parametric process. Since the "hardness" of the excitation here was $(h_{c1}/h_{c2})^2 \approx 2.3$, this situation is in good agreement with the criterion given above for the oscillatory PSW instability.

Let us now consider the $N_0^{(-)}$ branch of steady states (the stability of PSWs on this branch has not been discussed previously in the literature). As we showed above, for $h^* \leqslant h < h_{c1}$ the states with $N_0^{(-)}$ are aperiodically unstable. For $|\varkappa|<1$ there is a region of stable steady states at fields $h_{c2} < h < h^*$, while for $|\varkappa|>1$ states which are subject to oscillatory instability arise in this region for $h_{c2} < h < \tilde{h}$. For sufficiently large $|\varkappa|$ the regions of oscillatory and aperiodic instabilities overlap ($\tilde{h} \geqslant h^*$).

Thus in the field region $h_{c2} < h < h^*$ there exist three aperiodically stable steady states of the PSW, two of which have $N_0 \neq 0$.[4)] To study the PSW states on the negative-slope branch, one must start with initial conditions in the attraction domain of this branch. Such conditions can be arranged experimentally by a rapid switching of the pump power.

As we have said, allowance for the nonlinear terms $\tilde{S}(N)$, $\tilde{T}(N)$, and $\tilde{V}(N)$ [in addition to $\tilde{\gamma}(N)$] leads to a multiplicity of steady states of the parametric system [a typical $N_0(h)$ curve is shown in Fig. 1]. In addition, criteria (9) and (10) imply that, depending on the relationships among the PSW parameters, there are various ways of realizing the oscillatory and aperiodic instabilities. To check the conclusions of the theory will require special experimental study.

DISCUSSION

An important question is the stable state reached by the parametric system when one of the instabilities discussed above is realized.

For an oscillatory loss of stability it is valid to model the self-oscillations in the framework of system of equations (1) [with nonlinear coefficients (5) and (6)] if the collective oscillation frequency $\Omega = \mathrm{Im}\lambda$ [see Eq. (8)] is appreciably higher than the relaxation rate t_0^{-1} of the PSW packet. In this case the excited magnons remain on the same resonant surface. In the opposite limiting case $\Omega \ll t_0^{-1}$ the singular (in k) distribution of the PSW is able to follow the changes in N, with the result that the parametric packet drifts over different ω_k.

For an aperiodic instability there are several paths of transition to stability. On one of these paths the PSW system jumps from the unstable state to the stable state on one of the branches $N_0(h)$ (see Fig. 1). Another version involves going beyond the framework of Eqs. (1a) and (1b). In this case, which is analogous to the case of the aperiodic instability in ferromagnets, it is necessary to consider other degrees of freedom of the crystal in order to limit the growth of the absorbed power. For a ferromagnet these other degrees of freedom comprised a second group of magnons, allowance for which gave rise to self-oscillations.[3] For an antiferromagnet the "second group" can be parametric pairs (magnon-phonon or phonon-phonon). Their excitation has a threshold and can be brought about in an antiferromagnet by a strong magnetoelastic coupling. A computer modeling of this process shows that the absorbed power can have a periodic or stochastic oscillatory steady state. It would be reasonable to do a detailed study of these self-oscillations for a specific magnetic crystal, in close connection with experiment.

We have considered the possiblities for the onset of only spatially homogeneous PSW instabilities. It would be of interest to do a similar study with allowance for spatial inhomogeneities. In Ref. 28 the spatially inhomogeneous collective oscillations of PSWs were treated as a perturbation of the homogeneous distribution of wave pairs from the S theory.[3] In particular, it was shown there for an isotropic model of the magnon interaction that the most stable modes were the homogeneous collective modes. We believe that substantial inhomogeneities can arise in a parametric system as a result of differences in the character of the PSW relaxation in different parts of the crystal (for example, near the boundary and in the interior of the sample).

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[1)]Ref. 16 gives a possible interpretation of the results of the experiment of Ref. 10. Recently a detailed experimental study[17] has revealed that the regular dips of the absorbed microwave power that was detected in Ref. 10 are accompanied by changes in the density of parametric magnons in the interior of the sample.

[2)]According to Ref. 3, the *S*-theory approximation is valid when processes in which the PSW decay into two secondary waves are unimportant. A numerical modeling of the instability in a system with a clearly expressed decay spectrum was carried out in Ref. 19.

[3)]It was pointed out previously in Ref. 3 that a positive nonlinear PSW damping is one possiblity for increasing the threshold for self-oscillations of the magnetization.

[4)]Recall that different values of N_0 correspond to different wave numbers of the excited PSW [see Eq. (2)].

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