Fermion and boson mass shifts in an electric field: relation to quantum electrodynamics at short distances

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The correct asymptotic behavior in the limit of a strong electric field ε is derived for the mass shift Δm obtained in Ref. 1 for an electron in a constant uniform electromagnetic field. The real part of the shift decreases monotonically with increasing electric field, and its dependence on $e\varepsilon \gg m^2$ turns out to be analogous to the dependence (with the sign reversed) of the radiative mass δm of an electron in a vacuum on the square of the cutoff momentum, Λ^2, i.e., $\mathrm{Re}\, \Delta m(e\varepsilon) \simeq -\delta m(\Lambda^2)$, provided $e\varepsilon \sim \Lambda^2 \gg m^2$. The eigenvalue of the mass operator and the mass shift of a boson in a constant uniform electromagnetic field are determined. The real part of the boson mass shift is also found to decrease monotonically with increasing ε, and depends on the radiative mass of the boson as indicated above for $e\varepsilon \gg m^2$. This means that, when they are uniformly accelerated, charged fermions and bosons cease to interact with virtual quanta of their own field, the square of the momentum of which is less than $e\varepsilon$. The fermion and boson mass shifts in a strong magnetic field increase with increasing field, and are unrelated to the radiative mass. The properties of the mass shift revealed by the analysis given in this paper are a further confirmation of the connection between strong-field quantum electrodynamics and quantum electrodynamics at short distances.

1. INTRODUCTION

The eigenvalue of the mass operator and the mass shift of an electron in a constant uniform electromagnetic field were derived in Ref. 1. In the special case of a pure electric field ε and zero transverse electron momentum ($p_\perp = 0$), the mass shift Δm is given by Eq. (78) in Ref. 1, i.e.,

$$\Delta m = m\frac{\alpha}{2\pi}\int\int_0^\infty \frac{dx\,du}{x(1+u)}\left\{\frac{\sinh y}{\sinh x}\left(2\coth x\coth y - \frac{\sinh y}{\sinh x}\right)\right.$$
$$\left.\times\exp\left[-i\frac{x-y}{\beta}\right] - \frac{1+2u}{(1+u)^2}\exp\left[-i\frac{xu}{\beta(1+u)}\right]\right\} e^{-i\lambda x/u},$$
$$y = \mathrm{Arccoth}(\coth x + u/x), \qquad (1)$$

where $\beta = |e\varepsilon| m^{-2}$ and $\lambda = \mu^2/|e\varepsilon|$ are dimensionless parameters, μ is the photon mass introduced to remove the infrared divergence, and λ is the smallest parameter which, whenever possible, is assumed equal to zero.

In a weak field $\beta \ll 1$ and the shift (1) has the following asymptotic behavior[1]:

$$\Delta m = m\frac{\alpha}{2\pi}\left\{-\pi\beta + \beta^2\left(\frac{4}{3}\ln\frac{\gamma}{2\beta} + \frac{4}{9}\right)\right.$$
$$\left. + \ldots - i\left[\beta\left(2\ln\frac{2\beta m}{\gamma\mu} - 1\right) - \frac{2\pi}{3}\beta^2 + \ldots\right]\right\}, \qquad (2)$$

where $\gamma = 1.781\ldots$ and the dots represent high-order terms in the parameter β. Thus, the leading terms in the expression for the mass in a weak field are classical and independent of $\hbar$ since in classical theory, the mass of the photon must be replaced with the minimum wave number $k_{\perp\min} = \mu c/\hbar$. The classical part of the shift and its manifestations in quantum-mechanical processes have been examined in detail in Refs. 1–4.

The asymptotic properties of the shift (1) in a strong field $\beta \gg 1$, obtained in Ref. 1, and in a corrected form in Ref. 4, are both erroneous. The difficulty of extracting from (1) its correct asymptotic behavior for $\beta \gg 1$ is due to the presence of the two large parameters β and λ^{-1} in (1). This difficulty is overcome in the present paper by transforming (1) to a form in which the infrared part, which depends on taking $\lambda \to 0$, is explicitly isolated in a simple term, so that the remaining complicated part depends only on the single parameter β. The correct asymptotic behavior of the shift for $\beta \gg 1$ is thus obtained in the form

$$\Delta m = m\frac{\alpha}{2\pi}\left\{-\frac{3}{2}\ln\frac{\beta}{\gamma} - 2.63655\ldots\right.$$
$$\left. + \ldots - i\left[\beta\ln\frac{m^2}{\mu^2} - \frac{3\pi}{4} + \ldots\right]\right\} \qquad (3)$$

and exhibits some remarkable properties. First, $\mathrm{Re}\,\Delta m$ is a negative, monotonically decreasing function of β. Second, the dependence of $\mathrm{Re}\,\Delta m$ on the strong electric field (or, more precisely, on $e\varepsilon$) is similar to the functional dependence (with the sign reversed) of the radiative mass δm of an electron in a vacuum on the square of the cutoff momentum (see Refs. 5–8):

$$\delta m = m\frac{\alpha}{2\pi}\left(\frac{3}{2}\ln\frac{\Lambda^2}{m^2} + \frac{3}{4} + \ldots\right), \quad \Lambda^2 \gg m^2. \qquad (4)$$

This means that, as the electron is accelerated in a strong electric field $\varepsilon \gg m^2/e$, it is deprived of the part of its radiative mass that is due to the interaction with the virtual quanta with $|k^2| \lesssim e\varepsilon$.

We shall show that this analogy between the dependence of the mass shift on the strong electric field and the

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dependence of the radiative mass on the square of the cutoff momentum is also encountered in scalar electrodynamics.

The structure of this paper is as follows. The electron mass shift (1) is transformed in Section 2 into a simpler and more convenient form in which the entire infrared part, i.e., the part that depends on $\lambda\to 0$, is concentrated in a separate simple term. The essential aspects of the derivation of asymptotic expressions for weak and strong fields are discussed next. The mass operator for a scalar charged particle in a constant uniform electromagnetic field is discussed in Section 2, and is diagonalized with the aid of its eigenfunctions. The resulting mass shift of a charged boson is simplified in Section 4 in the special case of an electric field and boson states with zero transverse momentum. Asymptotic expansions are obtained for the shift in strong and weak fields. As in the case of the electron, the boson mass shift in an electric field is negative and decreases monotonically with increasing $\beta = e\varepsilon m^{-2}$. For $\beta \ll 1$, it is equal to the fermion mass shift, and for $\beta \gg 1$, it is found to be very different, but its dependence on the field is similar to the dependence of the boson radiative mass on the square of the cutoff momentum. Finally, Section 5 discusses the fermion and boson mass shift in a magnetic field, and the results are compared with the mass shift in an electric field. The section concludes with a discussion of the fundamental aspects of strong-field quantum electrodynamics that are due to the behavior of the particle mass shift.

2. TRANSFORMATION OF THE EXPRESSION FOR THE ELECTRON MASS SHIFT IN AN ELECTRIC FIELD AND THE RELATION TO THE RADIATIVE MASS

It is noted in Ref. 1 that the coefficient functions in the first term of (1) can be written explicitly in terms of the integration variables:

$$\frac{\sinh y}{\sinh x}\left(2\coth x \coth y - \frac{\sinh y}{\sinh x}\right) = \frac{1 + 2\sinh^2 x + 2ux^{-1}\sinh x \coth x}{1 + 2ux^{-1}\sinh x \coth x + u^2x^{-2}\sinh^2 x}. \quad (5)$$

Careful examination of the integrand in (1) in terms of the variables x, u shows that only the term $2\sinh^2 x$ in the first term in (1) exhibits the infrared singularity. Isolating this term, and transforming from the variables x, u to x, $z = x - y$ in which

$$u = x\sinh z/\sinh x \sinh(x - z), \quad (6)$$

we obtain the following transformed expression:

$$\Delta m = m(\alpha/2\pi)(I_1 + I_2),$$

$$I_1 = 2\int_0^\infty dz\, e^{-iz/\beta}\int_z^\infty dx \exp[-i\lambda\sinh^2 x(\coth z - \coth x)],$$

$$I_2 = \int_0^\infty dz\, e^{-iz/\beta}\int_z^\infty dx\left[\frac{u}{1+u}\left(\frac{\coth z + \coth x}{x} - 2\right) - \frac{1}{x^2} - \frac{z}{x^3}\right]. \quad (7)$$

Since, as $\lambda\to 0$, large values of x become effective in the integral I_1, it can be shown that, to within terms that vanish for $\lambda\to 0$,

$$I_1 = i\beta\left[\ln\frac{i\gamma\lambda}{2} + \psi\left(1 + \frac{i}{2\beta}\right) - \psi(1)\right] + \ldots, \quad (8)$$

where $\psi(x)$ is the logarithmic derivative of the Γ function.

The representation given by (7) has the form of the Laplace transformation in β, and is more convenient than (1) if we wish to determine the asymptotic behavior for $\beta \ll 1$ and $\beta \gg 1$. To do this, we need only know the asymptotic behavior of the function $\Phi(z)$ in the integral

$$I_2(\beta) = \int_0^\infty dz\, e^{-iz/\beta}\left[\Phi(z) - \frac{3}{2z}\right],$$

$$\Phi(z) = \int_z^\infty dx\frac{u}{1+u}\left(\frac{\coth z + \coth x}{x} - 2\right). \quad (9)$$

For $\beta \ll 1$, small $z \sim \beta$ are important, for which

$$\Phi(z) = \frac{3}{2z} - 1 + \frac{4}{3}z\ln 2z - \frac{7}{9}z + \ldots, \quad z \ll 1. \quad (10)$$

The expansion given by (2) is obtained after integration.

When $\beta \gg 1$, the behavior of $\Phi(z)$ for large z is important. It can be shown that

$$\Phi(z) = -\ln[2z(1 - e^{-2z})] + \Phi_1(z),$$

$$\Phi_1(z) = 4\int_z^\infty \frac{dx\, e^{-2x}[1 + (1 - e^{-2z})(1 - e^{-2x})^{-1}]}{e^{2x-2z} + 2x(1 - e^{-2z}) - 1 - e^{-2z} + e^{-2x}}, \quad (11)$$

where $\Phi_1(z)$ decreases exponentially for $z \gg 1$. Using this relation for $\Phi(z)$, and Eq. (8) for I_1, we obtain one further representation for the electron mass shift:

$$\Delta m = m\frac{\alpha}{2\pi}\left[\frac{3}{2}\mathrm{Ei}\left(-\frac{i}{\beta}\right) - i\beta\ln\frac{m^2}{\mu^2} + K(\beta)\right],$$

$$K(\beta) = \int_0^1 dz\, e^{-iz/\beta}\left[\Phi_1(z) - \frac{3}{2z}\right] + \int_1^\infty dz\, e^{-iz/\beta}\Phi_1(z). \quad (12)$$

This is convenient in determining the asymptotic behavior for $\beta \gg 1$ because the integral $K(\beta)$ can then be expanded into a series in powers of $(-i/\beta)$. The leading terms of the resulting asymptotic expression for Δm are given in (3), where

$$-2.63655\ldots = K(\infty)$$

$$= \int_0^\infty dx\left\{\frac{4e^{-2x}}{e^{2x} - 2x - 1}\left[\frac{e^{2x} - x - 1}{2x - 1 + e^{-2x}}\ln\frac{2x}{1 - e^{-2x}} - \frac{x}{1 - e^{-2x}}\right] - \frac{3}{2x}\theta(1 - x)\right\} - \frac{5}{4}, \quad (13)$$

and $\theta(x)$ is the Heaviside step function.

Continuing the discussion of the mass shift (3), we note that the linear fall in the shift for small β in the range $\beta \gtrsim 1$, which is due to the classical term $-\frac{1}{3}m\alpha\beta$, is replaced by the much slower logarithmic decrease due to quantum-mechanical effects. For exponentially strong fields, it therefore follows that Re Δm approaches $-m$, and the mass of the electron vanishes. This is not a dramatic effect because the particle is unstable (i.e., it has a large imaginary mass Δm), not to mention the fact that all the radiative corrections involving α must be taken into account in this range.

3. BOSON MASS OPERATOR IN THE ELECTROMAGNETIC FIELD AND ITS DIAGONALIZATION

Let us now consider the elastic scattering amplitude for a spin-zero charged particles in an external electromagnetic field A_μ. In the second order in the radiation field, the Rohrlich rules,[9] generalized to the case of an external field, ensure that the amplitude consists of three terms, namely,

$$T_{1fi}=ie^2\int d^4x\,d^4x'\,D_0(x-x')\psi_f^*(x)\,(\Pi_\mu^*+\Pi_\mu)\,G_0(x,x')\times(\Pi_\mu'^*+\Pi_\mu')\psi_i(x'), \tag{14.1}$$

$$T_{2fi}=-4e^2\int d^4x\,d^4x'\,D_0(x-x')\psi_f^*(x)\psi_i(x')\delta(x-x'), \tag{14.2}$$

$$T_{3fi}=ie^2\int d^4x\,d^4x'\,D_0(x-x')\psi_f^*(x)\,(\Pi_\mu^*+\Pi_\mu)\psi_i(x)\times(x'|\Pi_\mu G_0+G_0\Pi_\mu|x'), \tag{14.3}$$

which correspond to diagrams a, b, and c in Fig. 1. In this expresison, $G_0(x,x')$ is the propagation function of a scalar particle in the external field and $D_0(x-x')$ is the photon propagation function. Neither includes the radiative corrections. The operators $\Pi_\mu=-i\partial_\mu-eA_\mu$ and $\Pi_\mu^*=i\partial_\mu-eA_\mu$ act, respectively, on the nearest functions to the right and left. We shall also use the matrix notation (see Ref. 1)

$$G_0(x,x')=(x|G_0|x'),\quad \Pi_\mu G_0(x,x')=(x|\Pi_\mu G_0|x'),\quad G_0(x,x')\Pi_\mu'^*=(x|G_0\Pi_\mu|x'). \tag{15}$$

In our case of a constant uniform field, $T_3=0$ because the mean current

$$\langle j_\mu(x)\rangle=e(x|\Pi_\mu G+G\Pi_\mu|x),$$

induced by this field in a vacuum is zero.

For the amplitude $T=T_1+T_2$, we obtain an expression in the form of the matrix element

$$T_{fi}=-\int d^4x\,d^4x'\,\psi_f^*(x)M(x,x')\psi_i(x') \tag{16}$$

of the mass operator in the coordinate representation and, if we integrate this by parts, we can put it in the form of a nonoperator function:

$$M(x,x')=ie^2[4D_0(x-x')(x|\Pi_\mu G_0\Pi_\mu|x')-2i\partial_\mu D_0(x-x')(x|\Pi_\mu G_0+G_0\Pi_\mu|-i\delta(x-x')G_0(x,x')+4i\delta(x-x')D_0(x-x')]. \tag{17}$$

For the constant uniform electromagnetic field $F_{\alpha\beta}$, it is convenient to use the proper-time representation of the Green's function[10,11]

$$G_0(x,x')=-\frac{ie^{i\varphi}}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[-im^2s-L(s)+\frac{i}{4}z\beta z\right],\qquad z_\alpha=x_\alpha-x_\alpha', \tag{18}$$

in which β and L are matrix and scalar functions of the field tensor $F_{\alpha\beta}$ and the proper time s:

$$\beta_{\alpha\beta}=(eF\,\mathrm{cth}\,eFs)_{\alpha\beta},\qquad L=\frac{1}{2}\,\mathrm{tr}\ln\left(\frac{\mathrm{sh}\,eFs}{eFs}\right), \tag{19}$$

and φ is the phase of the Green's function that has not been diagonalized in x, x' and is given by the following integral along the line joining the points x, x':

$$\varphi(x,x')=e\int_{x'}^{x}dy_\alpha A_\alpha(y). \tag{20}$$

It is shown in Ref. 12 that

$$(x|\Pi_\mu G_0\Pi_\mu|x')=-i\delta(x-x')-m^2G_0(x,x')+{}^1/_2e^2zFFzG_0(x,x'), \tag{21}$$

and the matrix elements $(x|\Pi_\alpha G_0|x')$ and $(x|G_0\Pi_\alpha|x')$ differ from $G_0(x,x')$ by the additional factors $\frac{1}{2}(\beta+eF)_{\alpha\beta}z_\beta$ and $\frac{1}{2}(\beta-eF)_{\alpha\beta}z_\beta$ under the integral in (18), respectively. Finally,

$$D_0(z)=-\frac{i}{(4\pi)^2}\int_0^\infty\frac{dt}{t^2}\exp\left(\frac{iz^2}{4t}\right). \tag{22}$$

These formulas define completely the function $M(x,x')$.

For the diagonalization of the mass operator, it is natural to use its eigenfunctions $E_{\omega p}(x)$, obtained from the γ-matrix eigenfunctions of the mass operator in spinor electrodynamics[1] by replacing the spin numbers σ with zero and the γ-matrix $w_{\sigma\gamma}$ with unity, so that, instead of (7) and (8) of Ref. 1, we now have

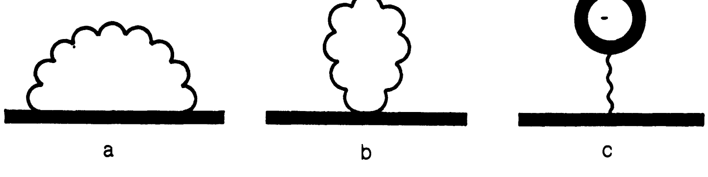

FIG. 1.

$$E_{\omega p}(x)=\frac{e^{i\pi\lambda/4}\Gamma(-\lambda)\exp[i(p_2x_2+p_3x_3)]}{(2\pi|\varepsilon/\eta|)^{1/4}(n!)^{1/2}}D_n(\rho)D_\lambda(\omega\tau), \quad (23)$$

$$n=k-\frac{1}{2}, \quad \lambda=-i\frac{2|e\eta|k-p^2}{2|e\varepsilon|}-\frac{1}{2}, \quad (24)$$

but the arugments of the parabolic cylinder functions are the same as before:

$$\rho=(2|e\eta|)^{1/2}\left(x_1-\frac{p_2}{e\eta}\right), \quad \tau=e^{i\pi/4}(2|e\varepsilon|)^{1/2}\left(x_0+\frac{p_3}{e\varepsilon}\right). \quad (25)$$

We recall that these functions correspond to the potential $A_\mu=(0,\eta x_1,-\varepsilon x_0,0)$ in the coordinate frame in which the vectors **H** and **E** lie along axis 3, and are denoted here by η and ε. The functions (23) are the eigenfunctions of the operators

$$\Pi^2, \quad -i\partial_2, \quad -i\partial_3, \quad \Pi_1^2+\Pi_2^2 \quad (26)$$

with eigenvalues

$$p^2, \quad p_2, \quad p_3, \quad p_\perp^2=2|e\eta|k, \quad (27)$$

where k now assumes the half-integral values $k=1/2, 3/2, 5/2, \ldots$. Thus, for a scalar charged particle in a magnetic field, the square of the transverse momentum is always positive because the charge has a zero-point energy in a magnetic field whereas, for a spinor particle, this energy can be balanced by the interaction energy between the intrinsic magnetic moment and the field. The functions (23) satisfy the orthogonality condition

$$\int E^*_{\omega' p'}(x)E_{\omega p}(x)d^4x$$
$$=(2\pi)^4\delta(p'^2-p^2)\delta(p_2'-p_2)\delta(p_3'-p_3)\delta_{k'k}\delta_{\omega'\omega} \quad (28)$$

and the completeness conditions

$$\sum_{\omega,k}\int E_{\omega p}(x)E^*_{\omega p}(y)\frac{dp^2\,dp_2\,dp_3}{(2\pi)^4}=\delta(x-y) \quad (29)$$

and, generally, play the same part in the electrodynamics with an external field as the plane waves e^{ipx} do in the electrodynamics of particles in a vacuum.

To evaluate the mass operator in the $E_{\omega p}$ representation, we must know the following two integrals:

$$J_0, J_2=\int d^4x\,d^4x'\,E^*_{\omega p}(x)\{1, zBz\}E_{\omega' p'}(x')$$
$$\times\exp\left\{i\varphi+\frac{1}{4}izAz\right\}, \quad (30)$$

where A, B are symmetric matrix functions of the matrix F. In our case, they have the form

$$A=\beta+t^{-1} \quad B=e^2FF+(2t)^{-1}\beta.$$

The method of evaluating this type of integral is given in the appendix of Ref. 1. It is sufficient to find J_0 because J_2 is obtained from it by differentiating J_0 with respect to the matrix A. The result is

$$J_0, J_2=(2\pi)^6\delta(p-p')\delta_{\omega\omega'}\frac{4ie^{-i\bar p w\bar p}}{[\det(A-eF)]^{1/2}}$$
$$\times\{1, 4\bar p(A+eF)^{-1}B(A-eF)^{-1}\bar p+2i\,\mathrm{tr}\,B(A-eF)^{-1}\}. \quad (31)$$

where w is the symmetric matrix function of the field F:

$$w=\frac{1}{eF}\,\mathrm{Arccoth}\,\frac{A}{eF}=\frac{1}{2eF}\ln\frac{A+eF}{A-eF}, \quad (32)$$

the four-vector $\bar p_\mu$ has the components

$$\bar p_1=0, \ \bar p_2=-\mathrm{sgn}\,(e\eta)\,(2|e\eta|k)^{1/2},$$
$$\bar p_0-\mathrm{sgn}\,(e\varepsilon)\bar p_3=\omega(2|e\varepsilon|)^{1/2}, \ \bar p^2=p^2, \quad (33)$$

and $(2\pi)^4\delta(p-p')\delta_{\omega\omega'}$ represents the right-hand side of (28). The matrix w has two doubly-degenerate eigenvalues that are functions of s and t, namely,

$$w_1=\frac{1}{e\eta}\,\mathrm{Arccoth}\left(\mathrm{ctg}\,e\eta s+\frac{1}{e\eta t}\right),$$
$$w_2=\frac{1}{e\varepsilon}\,\mathrm{Arccoth}\left(\mathrm{cth}\,e\varepsilon s+\frac{1}{e\varepsilon t}\right). \quad (34)$$

By using the above formulas, we can give the mass operator the following diagonal form:

$$M(p,p')=(2\pi)^4\delta(p-p')\delta_{\omega\omega'}M_R(\bar p,F), \quad (35)$$

$$M_R(\bar p,F)=\frac{i\alpha}{4\pi}\int_0^\infty\frac{ds}{s^2}e^{-im^2s}\left(\frac{e^2\eta\varepsilon s^2}{\sin e\eta s\,\sinh e\varepsilon s}-1\right)$$
$$-\frac{\alpha}{\pi}\iint_0^\infty\frac{ds\,dt}{t^2}e^{-im^2s}\left\{\frac{\sin e\eta w_1\,\sinh e\varepsilon w_2}{\sin e\eta s\,\sinh e\varepsilon s}\right.$$
$$\times[2\bar p(A+eF)^{-1}B(A-eF)^{-1}\bar p$$
$$+i\,\mathrm{tr}\,B(A-eF)^{-1}-m^2]e^{-i\bar p w\bar p}$$
$$\left.-\frac{\omega^2}{s^2}\left(p^2\frac{\omega^2}{st}+\frac{2i\omega}{st}-m^2\right)e^{-ip^2\omega}\right\}+M_R^0(\bar p),$$

where $\omega^{-1}=s^{-1}+t^{-1}$. The expression for $M_R(\bar p,F)$ has been renormalized, i.e., the unnormalized expression $M(\bar p,F)$ has had subtracted from it the value of $M(\bar p,0)$ in zero field and has had added to it the renormalized value $M_R(\bar p,0)\equiv M_R^0(\bar p)$. The renormalization of $M_R(\bar p,F)$ has removed from it the ultraviolet divergences and, for $F=0$, $M_R^0(\bar p,F)$ becomes equal to $M_R^0(\bar p)$. The invariant functions of the variables s, t

$$\bar p w\bar p=w_1\bar p_\perp^2+w_2\bar p_\parallel^2=\frac{w_1\varepsilon^2+w_2\eta^2}{\eta^2+\varepsilon^2}p^2+\frac{w_1-w_2}{\eta^2+\varepsilon^2}(F\bar p)^2, \quad (36)$$

$$\bar p(A+eF)^{-1}B(A-eF)^{-1}\bar p=\bar p_\perp^2\left(\frac{\mathrm{cotg}\,e\eta s}{2e\eta t}-1\right)\sin^2 e\eta w_1$$
$$+\bar p_\parallel^2\left(\frac{\coth e\varepsilon s}{2e\varepsilon t}+1\right)\sinh^2 e\varepsilon w_2, \quad (37)$$

$$\mathrm{tr}\,B(A-eF)^{-1}=e\eta\left(\frac{\mathrm{cotg}\,e\eta s}{2e\eta t}-1\right)\sin 2e\eta w_1$$
$$+e\varepsilon\left(\frac{\coth e\varepsilon s}{2e\varepsilon t}+1\right)\sinh 2e\varepsilon w_2 \quad (38)$$

are transcendental functions of the field invariants, η, ε and linear functions of the dynamic invariants p^2, $(F\bar{p})^2$ or $\bar{p}_\perp^2$, $\bar{p}_\parallel^2$:

$$p^2=\bar{p}_\perp^2+\bar{p}_\parallel^2, \quad (F\bar{p})^2=\eta^2\bar{p}_\perp^2-\varepsilon^2\bar{p}_\parallel^2. \tag{39}$$

The eigenvalue of the mass operator is thus a function of the four invariants η, ε, $(F\bar{p})^2$, p^2. This is usefully compared with the γ-matrix eigenvalue of the mass operator of a fermion [see Eq. (52) in Ref. 1].

The expression given by (35) differs from the corresponding results in Refs. 13 and 14 in being much more compact due to the use of the eigenvalues w_1, w_2 and the vector $\bar{p}_\mu$.

Because the external field modifies the radiative effects in the Green's function

$$G_R(\bar{p}, F)=-i[p^2+m^2+M_R(\bar{p}, F)]^{-1} \tag{40}$$

the pole in the complex plane of p^2 shifts from $-m^2$ to the point p_F^2 given by $p_F^2+m^2+M_R(\bar{p}_F, F)=0$, so that the shift of the square of the particle mass is

$$\Delta m^2=-p_F^2-m^2=M_R(\bar{p}_F, F). \tag{41}$$

To first order in α (which we have been considering), we can take $p_F^2\simeq -m^2$ on the right of (41). At this point, $M_R^0(\bar{p})=0$ and the remaining terms in (35) become somewhat simpler.

4. BOSON MASS SHIFT IN AN ELECTRIC FIELD

Let us consider the boson mass shift in a purely electric field in the state with zero transverse momentum. Substituting $p^2=-m^2$, $\eta=0$, and $\bar{p}_\perp^2=0$ in (35), we obtain

$$\Delta m=m\frac{\alpha}{2\pi}\left(\frac{1}{4}i\beta I_1-i\beta I_2+I_3\right),$$

$$I_1=\int_0^\infty \frac{dx}{x^2}\left(\frac{x}{\sinh x}-1\right)e^{-ix/\beta},$$

$$I_2=\iint_0^\infty \frac{dx\,du}{x(1+u)}\left\{\frac{\sinh y}{\sinh x}\left[\frac{u}{(1+u)x}+\left(u\frac{\coth x}{2x}+1\right)\sinh 2y\right]\times\exp\left(-i\frac{x-y}{\beta}\right)-\frac{2u}{(1+u)^2x}\times\exp\left(-i\frac{xu}{\beta(1+u)}\right)\right\}e^{-i\lambda x/u}, \tag{42}$$

$$I_3=\iint_0^\infty \frac{dx\,du}{x(1+u)}\left\{\frac{\sinh y}{\sinh x}\left[2\left(u\frac{\coth x}{2x}+1\right)\sinh^2 y+1\right]\times\exp\left(-i\frac{x-y}{\beta}\right)-\frac{1+3u+u^2}{(1+u)^2}\exp\left(-i\frac{xu}{\beta(1+u)}\right)\right\}e^{-i\lambda x/u}.$$

where the variables $x=|e\varepsilon|s$, $y=|e\varepsilon|w_2$, $u=s/t$ and the parameters $\beta=|e\varepsilon|m^{-2}$, $\lambda=\mu^2/|e\varepsilon|$ have the same significance as in (1).

While the integral I_1 reduces to the Γ function

$$I_1=2\ln\Gamma\left(\frac{i}{2\beta}+\frac{1}{2}\right)+\frac{i}{\beta}\ln\frac{2\beta}{i}+\frac{i}{\beta}-\ln 2\pi, \tag{43}$$

the integrals I_2 and I_3, which depend on two parameters, can be simplfied by explicitly isolating their dependence on the infrared parameter $\lambda\to 0$. This can be done by introducing the lower (ultraviolet) limit x_0 for the variable x, and then writing each of these integrals in the form of a difference beween the main integral (depending on the field in the s, t representation) and the vacuum integral (by allowing the field to tend to zero in the main integral:

$$I_n=\lim(I_{n1}-I_{n2}), \quad x_0\to 0. \tag{44}$$

In the main integral I_{n1}, we can replace the integration variable u by the variable $z=x-y$, so that u will be the function of x, z given in (6), and the integral itself will have the form

$$I_{n1}=\int_0^\infty dz\,e^{-iz/\beta}\int_{\max(x_0,z)}^\infty dx\,f_{n1}(x,z)\times\exp[-i\lambda\,\mathrm{sh}^2 x(\mathrm{cth}\,z-\mathrm{cth}\,x)]. \tag{45}$$

The integrand $f_{n1}(x, z)$ needs the infrared cutoff only because of the nonzero limit

$$\lim_{x\to\infty} f_{n1}(x,z)=2e^{-z}, \tag{46}$$

to which it tends exponentially, so that the integral with respect to x of the function

$$\varphi_{n1}(x, z)=f_{n1}(x, z)-2e^{-z} \tag{47}$$

converges at the upper limit. Hence, we can use the following expression for the integrand in (45):

$$2e^{-z}\exp[-i\lambda\sinh^2 x(\coth z-\coth x)]+\varphi_{n1}(x,z)\,. \tag{48}$$

We now transform in the vacuum integral I_{n2} from the variable u to $z=xu(1+u)^{-1}$, and note that its integrand does not require an infrared cutoff:

$$I_{n2}=\int_0^\infty dz\,e^{-iz/\beta}\int_{\max(x_0,z)}^\infty dx\,f_{n2}(x,z),$$
$$f_{22}=\frac{2z}{x^4},\quad f_{32}=\frac{1}{x^2}+\frac{z}{x^3}-\frac{z^2}{x^4}. \tag{49}$$

Using (45), (48), and (49) in (44), and passing to the limit as $x_0\to 0$, we obtain the following expression after integrating the infrared term in accordance with (7) and (8):

$$I_n=\frac{i\beta}{1-i\beta}\left[\ln\frac{i\gamma\lambda}{2}+\psi\left(\frac{3}{2}+\frac{i}{2\beta}\right)-\psi(1)\right]+\int_0^\infty dz\,e^{-iz/\beta}\int_z^\infty dx[\varphi_{n1}(x,z)-f_{n2}]. \tag{50}$$

where

$$\varphi_{21}=2e^{-z}\left(-1+\frac{1+e^{-2y}}{(1+u)(1-e^{-2x})}\right)+\frac{u}{x(1+u)\,\mathrm{sh}\,x\,\mathrm{sh}\,y}\times\left(\frac{1}{2}\,\mathrm{sh}\,2y\,\mathrm{cth}\,x+\frac{1}{1+u}\right), \quad (51)$$

$$\varphi_{31}=-\frac{2ue^{-z}}{1+u}-\frac{2e^{-x}\sinh z}{(1+u)\sinh x}+\frac{\sinh y}{(1+u)\sinh x}\left(\frac{1}{\sinh^2 y}+\frac{u}{x}\coth x\right), \quad (52)$$

$$y=x-z,$$

and the subtraction terms f_{n2} are given by (49).

Thus, Eqs. (43) and (50) express the shift (42) in terms of known transcendental functions and the Laplace integrals

$$J_2(\beta)=\int_0^\infty dz\,e^{-iz/\beta}\left[\Phi_2(z)-\frac{2}{3z^2}\right],$$

$$J_3(\beta)=\int_0^\infty dz\,e^{-iz/\beta}\left[\Phi_3(z)-\frac{7}{6z}\right], \quad (53)$$

$$\Phi_n(z)=\int_z^\infty dx\,\varphi_{n1}(x,z).$$

When $\beta\ll 1$, small $z\sim\beta$ will be important in the integrals (53), for which

$$\Phi_2(z)=\frac{2}{3z^2}-2\ln 2z-\frac{37}{18}+\frac{10}{3}z+\ldots,\quad z\ll 1, \quad (54)$$

$$\Phi_3(z)=\frac{7}{6z}-1+z\left(\frac{10}{3}\ln 2z+\frac{13}{36}\right)+\ldots,\quad z\ll 1. \quad (55)$$

Integrating these expansions and using the expansions of the corresponding functions in terms of β, we obtain

$$\Delta m=m\frac{\alpha}{2\pi}\left\{-\pi\beta+\beta^2\left(\frac{4}{3}\ln\frac{\gamma}{2\beta}+\frac{23}{72}\right)+\ldots -i\left[\beta\left(2\ln\frac{2\beta m}{\gamma\mu}-1\right)-\frac{2\pi}{3}\beta^2+\ldots\right]\right\}. \quad (56)$$

As expected, the dependence on the particle spin appears only in terms of order β^2 or higher [cf. (2)].

When $\beta\gg 1$, we can use the representations

$$J_2(\beta)=\int_0^1 dz\,e^{-iz/\beta}\left[\Phi_2(z)-\frac{2}{3z^2}\right]+\int_1^\infty dz\,e^{-iz/\beta}\,\Phi_2(z)-\frac{2}{3}e^{-i/\beta}-\frac{2i}{3\beta}\mathrm{Ei}\left(-\frac{i}{\beta}\right), \quad (57)$$

$$J_3(\beta)=\int_0^1 dz\,e^{-iz/\beta}\left[\Phi_3(z)-\frac{7}{6z}\right]+\int_1^\infty dz\,e^{-iz/\beta}\,\Phi_3(z)+\frac{7}{6}\mathrm{Ei}\left(-\frac{i}{\beta}\right). \quad (58)$$

Since $\Phi_n(z)$ decreases exponentially as $z\to\infty$, the integals in (57) and (58) can be represented by series in powers of $(-i/\beta)$, with real coefficients $K_{n0}, K_{n1}, \ldots$. We then have

$$J_2(\beta)=K_{20}-\frac{2}{3}-\frac{i}{\beta}\left(K_{21}-\frac{2}{3}-\frac{2}{3}\ln\frac{\beta}{i\gamma}\right)+\ldots, \quad (59)$$

$$J_3(\beta)=-\frac{7}{6}\ln\frac{\beta}{i\gamma}+K_{30}+\ldots,\quad \beta\gg 1. \quad (60)$$

Hence, for $\beta\gg 1$, we obtain

$$\Delta m=m\frac{\alpha}{2\pi}\left\{-\frac{\pi}{2}\beta-\frac{3}{4}\ln\frac{\beta}{\gamma}+\frac{2}{3}+\frac{7+\ln 2-\pi^2}{4}-K_{21}+K_{30}+\ldots -i\left[\beta\left(\ln\frac{2\beta m^2}{\gamma\mu^2}-\frac{8}{3}+\frac{9}{4}\ln 2+K_{20}\right]-\frac{3\pi}{8}+\ldots\right]\right\}. \quad (61)$$

The numerical values of the coefficients K_{ni} that appear in this expression are unimportant for our purposes here. The essential point is that, even in high fields, the real part of the shift is a negative, monotonically decreasing function of β. As in the case of $\beta\ll 1$, the leading term in the asymptotic expression is linear in β, but has a coefficient that is smaller by a factor of two. This means that, as β increases in the region $\beta\gtrsim 1$, the quantum-mechanical effects that come into play slow down the decrease in mass shift, but not to the same degree as for the fermion. The striking fact is that the asymptotic dependence of the mass shift on the field of β is the same as the dependence (with the sign reversed) of the radiative mass of the boson on the square of the cutoff momentum, first found by Neuman and Furry[15]:

$$\delta m=m\frac{\alpha}{2\pi}\left(\frac{3\Lambda^2}{2m^2}+\frac{3}{4}\ln\frac{\Lambda^2}{m^2}+\ldots\right),\quad \Lambda^2\gg m^2. \quad (62)$$

This means that, as it is accelerated in the strong electric field $\varepsilon\gg m^2/e$, the charge ceases to interact with the virtual quanta of its own field, the absolute value of the square of the momentum of which is less that $|e\varepsilon|$, so that its mass is reduced by the amount $\delta m(\Lambda^2)$, where $\Lambda^2\sim|e\varepsilon|$.

5. FERMION AND BOSON MASS SHIFTS IN A MAGNETIC FIELD: DISCUSSION

The electron mass shift in a magnetic field in a state with zero transverse momentum was first found by Demeur[16]:

$$\Delta m=m\frac{\alpha}{2\pi}\int_0^\infty\frac{dx}{x}\int_0^1 dv\,e^{-xv/\beta}\left(\frac{1+ve^{-2x}}{1-v+vx^{-1}e^{-x}\,\mathrm{sh}\,x}-1-v\right), \quad (63)$$

and has the following asymptotic expression for $\beta=|e\eta|m^{-2}\ll 1$:

$$\Delta m=m\frac{\alpha}{2\pi}\left[-\frac{1}{2}\beta+\beta^2\left(\frac{4}{3}\ln\frac{1}{2\beta}-\frac{13}{18}\right)+\ldots\right], \quad (64)$$

whereas, for $\beta\gg 1$, the asymptotic form is

$$\Delta m = m\frac{\alpha}{2\pi}\left[\frac{1}{2}\ln^2\frac{2\beta}{\gamma} - \frac{3}{2}\ln\frac{2\beta}{\gamma} + \ldots\right]. \qquad (65)$$

The last three expressions have frequently been reproduced in the literature (see Refs. 17–20). The shift (63) can also be derived from Eq. (57) of Ref. 1. It takes the form of the Laplace integral if we replace v with $z = xv$ and change the order of integration.

The first term in the expansion in (64), which is linear in β, is a quantum-mechanical term despite the absence of Planck's constant. It is due to the spin of the fermion which, in the state with $p_\perp = 0$, lies along the magnetic field and is equal to the interaction energy between the anomalous part of the magnetic moment and the magnetic field. This term arises at distances of order m^{-1} and, in contrast to the term that is linear in the modulus of the electric field, it can be evaluated from perturbation theory in the external field, as was done by Schwinger in Ref. 11.

It s clear from the above expansions that the mass shift decreases with increasing magnetic field, reaching a negative minimum in the region of $\beta \sim 1$. Thereafter, it increases with increasing β, but is basically different from the monotonic decrease in the mass shift in a strong electric field.

The mass shift of a charged boson in a magnetic field was found by Tsai.[21] For the ground state with the minimum value of $p_\perp^2 = |e\eta|$ [see (27)], he found that the expression that he obtained for Δm could be written in the form

$$\Delta m = m\frac{\alpha}{2\pi}\left(-\frac{1}{4}\beta I_1 + \beta I_2 + I_3\right),$$

$$I_1 = \int_0^\infty \frac{dx}{x^2}\left(\frac{x}{\sinh x} - 1\right)e^{-x/\beta},$$

$$I_2 = \int_0^\infty \frac{dx}{x}\int_0^1 dv\, e^{-xv/\beta - xv}\frac{(1-v)}{D}\left(1 - \frac{e^{-2x}}{D}\right) \qquad (66)$$

$$I_3 = \int_0^\infty \frac{dx}{x}\int_0^1 dv\, e^{-xv/\beta}(1-v)\left(\frac{e^{-xv}}{D} - 1\right),$$

$$D = 1 - v + vx^{-1}e^{-x}\sinh x.$$

The expression given by (66) can also be deduced from our formula (35) if we substitute $p^2 = -m^2$, $\varepsilon = 0$, $\bar{p}_\perp^2 = |e\eta|$. When $\beta \ll 1$, the shift is given by the asymptotoic formula[21]

$$\Delta m = m\frac{\alpha}{2\pi}\left[\beta^2\left(\frac{4}{3}\ln\frac{1}{2\beta} - \frac{7}{72}\right) + \ldots\right], \qquad (67)$$

and, for $\beta \gg 1$, by the formula

$$\Delta m = m\frac{\alpha}{2\pi}\left[C\beta - \frac{3}{4}\ln\frac{\beta}{\gamma} + \ldots\right], \qquad (68)$$

in which

$$C = \frac{1}{4}\ln 2 + \int_0^\infty \frac{dx}{x}\int_0^1 dv(1-v)\frac{e^{-xv}}{D}\left(1 - \frac{e^{-2x}}{D}\right) > 0. \qquad (69)$$

Thus, the fermion and boson mass shifts are positive in a strong magnetic field, and increase with increasing field. Their functional dependence on the field differs from the dependence of the radiative mass on the square of the cutoff momentum, and requires the sort of clear physical interpretation given for the mass shift in the electric field. The general impression is that quantum-mechanical effects lead to an increase in the shift which is described in weak fields by the universal term $\frac{4}{3}\beta^2\ln(1/2\beta)$ which does not depend on the spin of the particle or the form of the field, but these effects cannot overcome the fall in the electric field due to the classical term $\mathrm{Re}\,\Delta m^{\mathrm{cl}} = -\alpha e\varepsilon/2m$, and only reduce this fall in a strong field.

The analogy between the field dependence and the dependence on the square of the momentum was previously found for effective charges determined by the exact strong-field Lagrangian and the exact photon propagator (see Refs. 22 and 12). It is explained by the fact that the dynamic porperties of these quantities are determined by the effective value of the square of the kinetic momentum Π^2, where $\Pi_\alpha = p_\alpha - eA_\alpha$, of virtual charges interacting with the field or quanta. The mean value of Π^2 in strong fields is of the order of $(eA)^2 \sim (eFx)^2 \sim eF \gg m^2$, since the effective formation lengths $x \sim (eF)^{-1/2}$ are small in comparison with the Compton length. However, for high momenta of the quanta, we have $\Pi^2 \sim p^2 \gg m^2$, so that eF is analogous to p^2. This analogy establishes the fundamental connection beween strong field electrodynamics and quantum electrodynamics at short distances.

Our analysis does not capture the difference between the electric and magnetic fields which appears in the asymptotic behavior of the mass shift. This is probably connected with the fact that the physical system (charged particle) has a nonzero four-momentum, and with the attendant dependence on mean values of other invariants apart from Π^2. Nevertheless, the properties of the mass shift in a strong electric field that we have established show that the shift plays a fundamental part in quantum electrodynamics, just as the effective charge does. In its turn, the analogy between the measured shift Δm and the radiative mass δm gives the latter quantity a more realistic physical status, as, indeed, should be the case because of its gauge invariance.

[1] V. I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978) [Sov. Phys. JETP **48**, 788 (1978)]; correction, Zh. Eksp. Teor. Fiz. **76**, 383 (1979) [*sic*].
[2] V. I. Ritus, Zh. Eksp. Teor. Fiz. **80**, 1288 (1981) [Sov. Phys. JETP **53**, 659 (1981).
[3] V. I. Ritus, Zh. Eksp. Teor. Fiz. **82**, 1375 (1982) [Sov. Phys. JETP **55**, 799 (1982)].
[4] V. I. Ritus, Dokl. Akad. Nauk SSSR **275**, 611 (1984) [Sov. Phys. Dokl. **29**, 227 (1984)].
[5] V. Weisskopf, Zh. Phys. **89**, 27 (1934); **90**, 817.
[6] R. P. Feynman, Phys. Rev. **74**, 1430 (1948); **76**, 769 (1949).
[7] J. Schwinger, Phys. Rev. **75**, 651 (1949).
[8] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya elektrodinamika, Nauka, Moscow, 1981 [Translation of edition: Elements of Quantum

Electrodynamics, Oldbourne, London (1962)].
[9]F. Rohrlich, Phys. Rev. **80**, 666 (1950).
[10]V. Fock, Phys. J. Sowjetunion **12**, 404 (1937).
[11]J. Schwinger, Phys. Rev. **82**, 664 (1951).
[12]V. I. Ritus, Zh. Eksp. Teor. Fiz. **73**, 807 (1977) [Sov. Phys. JETP **46**, 423 (1977)].
[13]V. N. Baier, V. M. Katkov, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **67**, 453 (1974) [Sov. Phys. JETP **40**, 225 (1974)].
[14]J. Schwinger, Phys. Rev. D **7**, 1696 (1973).
[15]M. Neuman and W. H. Furry, Phys. Rev. **76**, 1677 (1949).
[16]M. Demeur, Mem. Acad. R. Belgique, Classe Sci. **28**, Fasc. 5, N1643 (1953).
[17]R. G. Newton, Phys. Rev. **96**, 523 (1954).
[18]B. Jancovici, Phys. Rev. **187**, 2275 (1969).
[19]R. G. Newton, Phys. Rev. D **3**, 626 (1971).
[20]Wu-yang Tsai and Asim Yildiz, Phys. Rev. D **8**, 3446 (1973).
[21]Wu-yang Tsai, Phys. Rev. D **8**, 3460 (1973).
[22]V. I. Ritus, Zh. Eksp. Teor. Fiz. **69**, 1517 (1975) [Sov. Phys. JETP **42**, 774 (1975)].

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