The spectrum of a hydrogen-like atom in high-frequency electromagnetic radiation: analytic solution

V. P. Gavrilenko

All-Union Scientific Research Center for the Study of Surfaces and Vacuum

A hydrogen-like atom in a high-frequency electromagnetic field is investigated analytically. The effective operator V for the interaction between the atom and the hf field is found. It is shown that, for circularly and linearly polarized hf fields, the operator V is diagonal in the basis of spherical wave functions corresponding to a fixed principal quantum number n. The diagonal form of the operator V is used to determine the complete set of quasienergy states of the hydrogen-like atom in the hf field for the two types of wave polarization (circular and linear). This means that the results obtained by Ritus[2] for the hydrogen-like atom in an hf field are valid for arbitrary n and not only for $n \leqslant 2$. All the quasienergy states of the hydrogen-like atom corresponding to $n \leqslant 3$ have been found for elliptically polarized hf fields, and twelve such states (out of the total number equal to n^2) have been found for any $n \geqslant 4$.

§1. INTRODUCTION

When the effect of a monochromatic electric field of frequency ω on a hydrogen-like atom in a state with principal quantum number n is investigated, two cases may be considered, namely, (a) the low-frequency case $\omega \ll |\omega_{nn'}|$, where $|\omega_{nn'}|$ is the separation (on the frequency scale) between the level n and nearest levels n' and (b) the high-frequency case $\omega \gg |\omega_{nn'}|$. It is well known[1,2] that the solutions of the Schroedinger equation for the atom in a periodic (in time) field can be conveniently taken to be the wave functions of the quasienergy states. In the low-frequency case (a), the wave functions of the quasienergy states of a hydrogen-like atom are well-known for different polarizations of the monochromatic field. They have been examined for linear polarization in Ref. 3, for circular polarization in Refs. 4 and 5, and for elliptic polarization in Ref. 6. The high-frequency case (b) has been investigated to a much lesser extent.

The foundations of the theoretical study of the interaction between an hf electromagnetic field ($\omega \gg |\omega_{nn'}|$) and a hydrogen-like atom were laid in Ref. 2, where the wave functions of quasienergy states of atomic levels with $n \leqslant 2$ were found for linearly and circularly polarized hf fields. The perturbation operator U is diagonal in spherical wave functions, as noted in Ref. 2. However, the high-frequency condition $\omega \gg |\omega_{nn'}|$ for $n = 1, 2$, is difficult to satisfy using modern lasers. The determination of the wave functions of quasienergy states of hydrogen-like atoms for $n > 2$ in an hf field is therefore optical not only from the theoretical but also practical points of view. The present paper is largely devoted to this. As far as we know, since the publication of Ref. 2, the analytic determination of the wave functions for $n > 2$ has been considered in the literature only for linear polarization. The view has been expressed[7,8] that, for $n > 2$ in the hf case $\omega \gg |\omega_{nn'}|$, and for a given state n, the operator U mixes states with orbital quantum numbers l and $l \pm 2$, so that the determination of quasienergy states with $n > 2$ becomes much more complicated. It will be shown below that this mixing is unimportant at high frequencies ω.

One of the main results reported below is very simple: for both linearly and circularly polarized hf fields, the operator U is diagonal, for any given n, in the basis of wave functions φ_{nlm} written in terms of spherical coordinates. Hence, for any n, the quasienergies measured from the unperturbed level n are

$$\nu_{nlm} = \langle nlm | U | nlm \rangle,$$

where the wave functions of quasienergy states have the form $\Phi_{nlm} = \exp(-i\nu_{nlm}t)\varphi_{nlm}$. The results obtained in Ref. 2 for the hydrogen-like atom in an hf field are thus seen to be valid for arbitrary n, and not only for $n \leqslant 2$.

Another important result reported in this paper is the wave function of quasienergy states of a hydrogen atom in an elliptically polarized hf field. Moreover, we investigate the spontaneous-emission spectrum of a hydrogen-like atom, emitted as a result of $n \rightarrow n'$ transitions in the presence of the hf field.

In Section 2, we determine the effective operator for the interaction between the hydrogen-like atom and the hf electromagnetic field. We then use this operator in section 3 to find the quasienergy states of the hydrogen-like atom for different polarizations of the monochromatic hf field (circular, linear, and elliptic). The range of validity of the solution is examined in Section 4, and a comparison is given between the analytic results obtained here and known numerical calculations of dynamic polarizabilities of the hydrogen atom.[9] The results are discussed in Section 5, and some of their generalizations are analyzed.

§2. EFFECTIVE OPERATOR FOR THE INTERACTION BETWEEN A HYDROGEN-LIKE ATOM AND AN hf ELECTROMAGNETIC FIELD

The Schrodinger equation for the hydrogen-like atom in a quasimonochromatic electromagnetic field (or a super-

position of such fields) with vector potential

$$\mathbf{A}(t)=(A_x(t),\ A_y(t),\ A_z(t)) \qquad (1)$$

is (we shall be using the atomic system of units, in which $\hbar=m=e=1$)

$$i\,\partial\Psi/\partial t=[H_0+c^{-1}\mathbf{p}\cdot\mathbf{A}(t)]\Psi,\qquad H_0=\frac{\mathbf{p}^2}{2}-\frac{Z}{r}+\frac{A^2(t)}{2c^2},$$
$$(2)$$

where Z is the nuclear charge.

We shall seek the solution of (2) in the form

$$\Psi=\exp[-i\alpha(t)]\Phi,\qquad \alpha(t)\equiv c^{-1}\mathbf{p}\cdot\mathbf{B}(t), \qquad (3)$$

where it is assumed that the vector

$$\mathbf{B}(t)=(B_x(t),\ B_y(t),\ B_z(t))$$

satisfies the equation $d\,\mathbf{B}(t)/dt=\mathbf{A}(t)$ and has a zero mean in time. Substituting (3) in (2), we obtain

$$i\partial\Phi/\partial t=H_1\Phi,$$
$$H_1\equiv\exp[i\alpha(t)]H_0\exp[-i\alpha(t)]=H_0+i[\alpha,\ H_0]$$
$$+(i^2/2)[\alpha,\ [\alpha,\ H_0]]+\ldots=\bar{H}_1+\tilde{H}_1, \qquad (4)$$

where $\bar{H}_1$ is the component of the Hamiltonian H_1 averaged over the hf oscillations of the field $\mathbf{A}(t)$. In the hf case that we are examining, $\omega\gg|\omega_{nn'}|$ and the main contribution to the solution (4) is due to $\bar{H}_1$. To within terms quadratic in the field, the operator $\bar{H}_1$ can be written in the form [1)]

$$\bar{H}_1=\frac{\mathbf{p}^2}{2}-\frac{Z}{r}+\frac{\overline{A^2}}{2c^2}+V,$$

$$V=\frac{Z}{2c^2}\left[\frac{(\varepsilon+1)\overline{B^2}}{r^{3+\varepsilon}}-\frac{(3+4\varepsilon)\overline{(\mathbf{B}\cdot\mathbf{r})^2}}{r^{5+\varepsilon}}\right]\quad(\varepsilon\to0). \qquad (5)$$

A bar over a mathematical expression in (5) signifies averaging over the high-frequency oscillations of the field $\mathbf{A}(t)$. The term $\overline{\mathbf{A}^2}/2c^2$ is the average oscillatory energy of a free electron in the wave.

§3. QUASIENERGY STATES OF A HYDROGEN-LIKE ATOM IN A MONOCHROMATIC hf ELECTROMAGNETIC FIELD

The interaction between a hydrogen-like atom and a monochromatic hf electromagnetic field will be examined below using (4) and (5) for different polarizations of this field.

1. Elliptically polarized field. Suppose that

$$\mathbf{A}(t)=A_0(1+\zeta^2)^{-1/2}(\cos\omega t,\ \zeta\sin\omega t,\ 0),\qquad A_0=-cE_0/\omega,$$
$$(6)$$

where ζ is the degree of elliptic polarization. Substituting (6) in (5), and using spherical coordinates, we find that, as $\varepsilon\to0$,

$$V=V(\zeta^2)=-\frac{ZE_0^2}{8\omega^4 r^{3+\varepsilon}}\left[(1-3\cos^2\theta)-2\varepsilon\cos 2\theta\right.$$

$$\left.+\frac{3(1-\zeta^2)}{1+\zeta^2}\sin^2\theta\cos 2\varphi\right]. \qquad (7)$$

It is shown in Ref. 11 that

$$\int_0^\infty r^{-s}R_{nl}(r)R_{nl'}(r)r^2\,dr=0,\qquad s=2,3,\ldots,l-l'+1, \qquad (8)$$

where $R_{nl}(r)$ are the radial wave functions of the hydrogen-like atom. In view of this, the matrix elements of the operator $V(\zeta^2)$ given by (7) satisfy the equation

$$\langle nlm|V(\zeta^2)|nl'm'\rangle=0\qquad(l'=l\pm2). \qquad (9)$$

Let us begin by considering circular polarization ($\zeta^2=1$). The term containing $\cos 2\varphi$ in (7) is then absent, so that, because of (9), the spherical wave functions φ_{nlm} are the correct eigenfunctions of the zero-order Hamiltonian $\bar{H}_1$ in (5) [with the perturbing operator $V(1)$ given by (7)]. The following energy eigenvalues correspond to these eigenfunctions:

$$F_{nlm}=E_n^{(0)}+\langle nlm|V(1)|nlm\rangle$$
$$=\begin{cases}E_n^{(0)}-\dfrac{Z^4E_0^2}{2\omega^4}\dfrac{3m^2-l(l+1)}{n^3(2l-1)l(l+1)(2l+1)(2l+3)},& l\ne0\\[2ex]E_n^{(0)}+\dfrac{Z^4E_0^2}{3n^3\omega^4},& l=0\end{cases}$$
$$(10)$$

$$(E_n^{(0)}=-Z^2/2n^2+E_0^2/4\omega^2).$$

These energy eigenvalues are the same as those obtained in Ref. 2, and are valid for arbitrary n and not only for $n\leqslant2$.

Let us now examine the case of arbitrary degree of elliptic polarization ζ. When $\zeta^2\ne1$, the term containing $\cos 2\varphi$ in (7) will mix the states φ_{nlm}, $\varphi_{nlm'}$ ($l\geqslant1$, $m'=m-2$). This mixing will not, however, occur for the states φ_{n00}, φ_{n10}. Hence, the energy values for these states, F_{n00} and F_{n10}, do not depend on ζ and are given by (10). To find the energies corresponding to the other states φ_{nlm}, we must solve the secular equation. For the states φ_{nl1}, $\varphi_{nl,-1}$ ($l=1,2$) the secular equation is a quadratic whose solution gives the two energy values F_{ns} ($s=1,2$):

$$F_{ns}=E_n^{(0)}+\frac{Z^4E_0^2}{2\omega^4}\frac{1}{n^3(2l-1)(2l+1)(2l+3)}\left[\frac{l^2+l-3}{l+1}\right.$$

$$\left.+(-1)^{s+1}\frac{3}{2}\frac{1-\zeta^2}{1+\zeta^2}\right]\qquad(l=1,2;\ s=1,2) \qquad (11)$$

The corresponding eigenfunctions are

$$\varphi_{ns}=2^{-1/2}[(-1)^{s+1}\varphi_{nl1}+\varphi_{nl,-1}]\quad(l=1,2;\ s=1,2). \qquad (12)$$

For the states φ_{nl0}, φ_{nl2}, $\varphi_{nl,-2}$ ($l=2,3$), the secular equation is a cubic. The solution of this equation gives the following eigenenergies F_{np} ($p=1,2,3$):

$$F_{n1}=E_n^{(0)}+V_{22}, \qquad (13)$$

$$F_{np}=E_n^{(0)}+2^{-1}\{V_{11}+V_{22}+(-1)^p[(V_{11}-V_{22})^2+8V_{12}^2]^{1/2}\}$$

$$(p=2,3),$$

where

$$V_{11}\equiv\langle nl0|V(\zeta^2)|nl0\rangle=\frac{Z^4E_0^2}{2\omega^4}\frac{1}{n^3(2l-1)(2l+1)(2l+3)},$$

$$V_{22}\equiv\langle nl2|V(\zeta^2)|nl2\rangle$$

$$= -\frac{Z^4 E_0^2}{2\omega^4} \cdot \frac{12 - l(l+1)}{n^3(2l-1)l(2l+1)(l+1)(2l+3)},$$

(14)

$$V_{12} = \langle nl0 | V(\zeta^2) | nl2 \rangle$$

$$= \frac{3Z^4 E_0^2}{4\omega^4} \frac{1-\zeta^2}{1+\zeta^2} \frac{1}{n^3(2l-1)(2l+1)(2l+3)}$$

$$\left[\frac{(l-1)(l+2)}{l(l+1)} \right]^{1/2}.$$

The corresponding wave functions are

$$\varphi_{n1} = 2^{-1/2}(\varphi_{nl2} - \varphi_{nl,-2}),$$

(15)

$$\varphi_{np} = a(p)\varphi_{nl0} + b(p)\varphi_{nl2} + c(p)\varphi_{nl,-2} \quad (p=2,3),$$

where

$$a(p) = 2^{-1/2}\left\{ 1 + \frac{(-1)^{p+1}(V_{22}-V_{11})}{[(V_{22}-V_{11})^2 + 8V_{12}^2]^{1/2}} \right\}^{1/2},$$

$$b(p) = c(p) = \frac{(-1)^p}{2}\left\{ 1 + \frac{(-1)^p(V_{22}-V_{11})}{[(V_{22}-V_{11})^2 + 8V_{12}^2]^{1/2}} \right\}^{1/2}.$$

(16)

Equations (11)–(16) define completely the quasienergy states of the hydrogen-like atom in elliptically polarized hf fields (and, consequently, in the linearly polarized hf field $\zeta = 0$, as well) for levels $n = 1, 2, 3$. We now proceed to the determination of the quasienergy states of a hydrogen-like atom in a linearly polarized hf field for any n.

2. Linearly polarized field. Let

$$\mathbf{A}(t) = A_0(0, 0, \cos \omega t), \quad A_0 = -cE_0/\omega.$$

(17)

Substituting this in (5), we find that

$$V = 2E_0^2 v/\omega^4,$$

$$v = \frac{Z}{8r^{3+\varepsilon}}[(1-3\cos^2\theta) + \varepsilon(1-4\cos^2\theta)].$$

(18)

By virtue of (8), the wave functions φ_{nlm} in a linearly polarized field are the correct eigenfunctions (as in the case of circular polarization) of the zero-order Hamiltonian $\bar{H}_1$ in (5) [with perturbing operator V given by (18)]. The corresponding energy eigenvalues are

$$F_{nlm} = E_n^{(0)} + \langle nlm | V | nlm \rangle$$

$$= \begin{cases} E_n^{(0)} + \frac{Z^4 E_0^2}{\omega^4} \frac{3m^2 - l(l+1)}{n^3(2l-1)l(2l+1)(l+1)(2l+3)}, & l \neq 0 \\ E_n^{(0)} + \frac{Z^4 E_0^2}{3n^3\omega^4}, & l=0 \end{cases}$$

$$(E_n^{(0)} = -Z^2/2n^2 + E_0^2/4\omega^2).$$

(19)

These energies [like those in (10)] are equal to the energies obtained in Ref. 2, and are valid for any n and not merely $n \leqslant 2$.

The spontaneous-emission spectrum of a hydrogen-like atom in an hf monochromatic electromagnetic field, due to $n \rightarrow n'$ transitions, can be written in the form (if the states corresponding to the level n have the equilibrium populations at the initial time):

$$I^{(e)}(\Delta\omega) = \sum_{i \in n, f \in n'} I_{if}^{(e)}(\Delta\omega),$$

(20)

$$I_{if}^{(e)}(\Delta\omega) = |\langle \varphi_i | \mathbf{re} | \varphi_f \rangle|^2 \delta(\Delta\omega - F_i + F_f),$$

where **e** is a unit vector representing the polarization of the photons. The level energies in the hf field are determined by (10), (11), (13), and (19), depending on the polarization of $\mathbf{A}(t)$, and the wave functions φ_i, φ_f correspond to the energies F_i, F_f.

Thus, by recording the spontaneous-emission spectrum for different positions of the transmitting planes of a polaroid, we can determine from (10) the polarization of the vector $\mathbf{A}(t)$, the ellipticity ζ, and (for known frequency ω), the amplitude of the electric field E_0.

§4. RANGE OF VALIDITY. COMPARISON WITH NUMERICAL CALCULATIONS

The formulas obtained above for the sublevel shifts due to the third term in the Hamiltonian H_1 in (4) are valid to within the quadratic correction in the field, $\Delta\varepsilon_{nlm}$, which, in second-order perturbation theory, is due to the oscillating term $i[\alpha, H_0]$ in H_1. Let us estimate this correction, assuming, to be specific, a linearly polarized hf field. In the isolated-level approximation, it is then readily shown (see, for example, Ref. 12)[2)]

$$\Delta\varepsilon_{nlm} = \frac{1}{2}\frac{E_0^2}{\omega^4}\sum_{n'}\frac{\omega_{nn'}|[p_z, H_0]_{nn'}|^2}{\omega_{nn'}^2 - \omega^2}$$

$$= \frac{1}{2}\frac{E_0^2}{\omega^4}\sum_{n'}\frac{\omega_{nn'}^5|z_{nn'}|^2}{\omega_{nn'}^2 - \omega^2}.$$

(21)

We shall assume a high-frequency field, i.e.,

$$\omega \gg \max\left[\frac{Z^2}{2n^2}, \frac{Z^2}{2}\left(\frac{1}{L^2} - \frac{1}{n^2}\right) \right], \quad L = \max(l, 1),$$

(22)

which means that the frequency ω must exceed the distance from the levels with $n' > n$ and those with $n' < n$, which are coupled to the level n by the dipole matrix element $z_{nn'}$.[3)] When (22) is satisfied, Eq. (21) becomes

$$\Delta\varepsilon_{nlm} = -\frac{1}{2}\frac{E_0^2}{\omega^4}\sum_{n'}\left(\frac{\omega_{nn'}}{\omega}\right)^2 \omega_{nn'}^3 |z_{nn'}|^2.$$

(23)

At the same time, the above shift of the sublevel (nlm) due to the third term in the Hamiltonian H_1 can be written in the form[4)]

$$\Delta F_{nlm} = \frac{1}{4}\frac{E_0^2}{\omega^4}\left[\frac{\partial^2}{\partial z^2}\left(-\frac{Z}{r}\right) \right]_{nn} = -\frac{E_0^2}{2\omega^4}\sum_{n'}\omega_{nn'}^3 |z_{nn'}|^2.$$

(24)

Comparison of (23) with (24) shows that, when (22) is satisfied, we also have $|\Delta\varepsilon_{nlm}| \ll |\Delta F_{nlm}|$ provided only that the quantity ΔF_{nlm} is not anomalously low as a result of "random" mutual cancellation of large terms with opposite signs in (24). If, nevertheless, this cancellation does occur for some particular sublevel (nlm), the shift of this sublevel

TABLE I. Dynamic polarizabilities of hydrogen levels for linearly polarized radiation from the ruby laser ($\omega = 14\,400$ cm^{-1}): I—present results; II—numerical calculations from Ref. 9.

nlm	I Re β, a. e.	II Re β, a. e.	II Im β, a. e.	nlm	I Re β, a. e.	II Re β, a. e.	II Im β, a. e.
540	−229.8	−230.0	0.632	64±3	−232.8	−232.9	0.349
54±1	−230.2	−230.4	0.607	64±4	−234.3	−234.4	0.194
54±2	−231.3	−231.5	0.530	650	−231.5	−231.6	0.042
54±3	−233.2	−233.4	0.404	65±1	−231.6	−231.7	0.041
54±4	−235.8	−236.0	0.225	65±2	−231.8	−231.9	0.037
640	−230.9	−230.9	0.546	65±3	−232.2	−232.3	0.031
64±1	−231.1	−231.1	0.524	65±4	−232.8	−232.8	0.023
64±2	−231.7	−231.8	0.459	65±5	−233.5	−233.5	0.013

will no longer be described by the analytic formula given above but, by virtue of (22), it will be numerically small (in comparison with the shift of the other sublevels). Comparison with the numerical calculations reported in Ref. 9 (see below) will also show that, when the high-frequency condition (22) is satisfied, the third term in the Hamiltonian H_1 plays the leading part in the shift of the sublevel (nlm).

The results obtained here determine the real part of the dynamic polarizability β of the levels of a hydrogen-like atom in an hf electric field. At the same time, according to condition (22), there is a finite probability of single-photon ionization of the atom and, consequently, the imaginary part of β is nonzero. If the states corresponding to the level n have equilibrium initial populations, the level separations must exceed the ionization widths if the individual sublevels (nlm) are to be distinguishable in the hf electric field. As an example, let us compare the shift of the ($n00$) level in the hf electric field relative to other sublevels (nlm) with its ionization width (the order of magnitude of the shift is $\Delta F_{n00} = Z^4 E_0^2/3n^3\omega^4$). Since the high-frequency condition (22) for the ($n00$) sublevel is

$$\omega \gg Z^2/2, \qquad (25)$$

the photoionization cross section can be described by the Born approximation. Using formula (70.6) from Ref. 13, we

find that the photoionization probability of the sublevel ($n00$) is

$$W_{n00} = 2^{7/2} Z^5 E_0^2/3n^3\omega^{9/2}.$$

It is readily seen that $\Delta F_{n00} \gtrsim W_{n00}$ when (25) is satisfied. Hence, it follows (recalling that $W_{nlm} \ll W_{n00}$) that the sublevel ($n00$) will be appreciably shifted relative to sublevels with $\rho \geqslant 1$. If, on the other hand, the positions of the individual sublevels with equilibrium populations is difficult to determine experimentally, because of photoionization broadening, we still have the possibility that they will be detected by selective population.

The numerical polarizabilities β of the hydrogen atom were calculated in Ref. 9 for $1 \leqslant n \leqslant 6$ in linearly and circularly polarized electromagnetic fields for: (1) a neodymium laser with $\omega = \omega_N = 9440$ cm^{-1} and (2) a ruby laser with $\omega = \omega_R = 14\,400$ cm^{-1}. For $\omega = \omega_N$ and $\omega = \omega_R$, the condition given by (22) is reasonably well satisfied for $n = 5$ in the case of $l = 4$ states; when $n = 6$ it is well satisfied for $l = 4$ and $l = 5$ states. For these states, the real parts of the dynamic polarizabilities, Re β, calculated from (10) and (19), are close to the corresponding numerical values in Ref. 9. This is clear, for example, from Tables I and II which list the calculated Re β, based on (10) and (19), together with numerical calculations of β from Ref. 9 (ruby laser). It is

TABLE II. Dynamic polarizabilities of hydrogen levels for circularly polarized radiation from the ruby laser ($\omega = 14\,400$ cm^{-1}: I—present results; II—numerical calculations from Ref. 9.

nlm	I Re β, a. e.	II Re β, a. e.	II Im β, a. e.	nlm	I Re β, a. e.	II Re β, a. e.	II Im β, a. e.
540	−233.6	−233.8	0.378	64^{+3}_{-3}	−232.1	−232.0 −232.2	0.778 0.076
54^{+1}_{-1}	−233.4	−233.5 −233.6	0.527 0.252	64^{+4}_{-4}	−231.3	−231.2 −231.5	0.972 0.036
54^{+2}_{-2}	−232.8	−232.0 −233.0	0.701 0.156	650	−232.7	−232.7	0.024
54^{+3}_{-3}	−231.9	−232.0 −232.1	0.901 0.086	65^{+1}_{-1}	−232.7	−232.7 −232.7	0.032 0.017
54^{+4}_{-4}	−230.6	−230.7 −230.8	1.126 0.040	65^{+2}_{-2}	−232.5	−232.6 −232.6	0.041 0.012
640	−233.0	−233.1	0.326	65^{+3}_{-3}	−232.3	−232.4 −232.4	0.052 0.007
64^{+1}_{-1}	−232.9	−233.0 −233.0	0.455 0.218	65^{+4}_{-4}	−232.1	−232.1 −232.1	0.063 0.004
64^{+2}_{-2}	−232.6	−232.6 −232.7	0.605 0.135	65^{+5}_{-5}	−231.7	−231.8 −231.7	0.076 0.002

also clear form Tables I and II that the characteristic separation between sublevels with $n = 6$, $l = 5$ for $\omega = 14\,400$ cm^{-1} is usually greater than the corresponding ionization width.

Numerical calculations of Re β for sublevels with high n, l (Ref. 9) are thus in agreement with the average values of the operator V in (5) (wave functions φ_{nlm}), which are readily calculated analytically.

§5. DISCUSSION

We have found the complete set of quasienergy states corresponding to an arbitrary level n of a hydrogen-like atom in an hf electromagnetic field and two types of polarization, namely, circular and linear. In the elliptically polarized hf field, the analogous problem was solved above for levels with $n \leqslant 3$. Moreover, we found twelve quasienergy states (out of the total number equal to n^2) for any level with $n \geqslant 4$ in elliptically polarized hf fields.

The effective interaction operator V in (5) can be used to analyze other more general cases of the interaction between an hf electromagnetic field and a hydrogenlike atom. For example, suppose that a linearly polarized field with vector potential $\mathbf{A}(t)$ can be written as a superposition of hf oscillations with different frequencies ω_j:

$$\mathbf{A}(t) = \left(0, 0, -\sum_j \frac{cE_{0j}}{\omega_j} \cos(\omega_j t + \varphi_j) \right),$$

where it is assumed that oscillations at the difference frequencies $|\omega_j - \omega_{j'}|$ that appear in $B^2(t)$ are also high-frequency oscillations in relation to the hydrogen-like atom. In that case, the results given by (19) remain valid provided we introduce the replacement

$$E_0^2/\omega^4 \rightarrow \sum_j (E_{0j}^2/\omega_j^4).$$

Let us now consider the situation where a linearly polarized hf field is amplitude-modulated with characteristic frequency $\Omega \ll |\omega_{nn'}| \ll \omega$. For example, suppose that

$$\mathbf{B}(t) = (0, \ 0, \ B_0 \cos \Omega t \cos \omega t). \tag{26}$$

Substituting (26) in (5), we then obtain

$$V(t) = \frac{B_0^2}{c^2}(1 + \cos 2\Omega t) v. \tag{27}$$

This expressions shows that a hydrogen-like atom in a particular state φ_{nlm} will function in the hf electromagnetic field as a kind of quadratic detector which removes the hf component and responds to the low-frequency intensity. Because $\Omega \ll |\omega_{nn'}|$, the energy levels of the hydrogen-like atom will follow adiabatically the variation in the perturbation $V(t)$ in (27), and this, in turn, will give rise to a phase modulation of the corresponding wave functions:

$$\Phi_j(t) = \exp\left[-i\left(E_n^{(0)} + \frac{B_0^2}{c^2} v_{jj} \right) t - i \int_0^t \overline{\frac{A_0^2(t')}{2c^2}} \, dt' \right. $$
$$\left. - \frac{iB_0^2 v_{jj}}{2c^2 \Omega} \sin 2\Omega t \right] \varphi_j$$

and, consequently, to an additional splitting of each component of the spontaneous-emission spectrum during an $n \rightarrow n'$ transition. The form of this splitting can be used to deduce the nature of the amplitude modulation of the hf field.

We note in conclusion that our results, obtained in the dipole approximation, are valid for the interaction between a hydrogen-like atom and not only the hf field of an electromagnetic wave, but also the hf electric field of other oscillations, e.g., Langmuir plasma oscillations.

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[1] The inclusion of an infinitesimal quantity ε in the operator V corresponds to a nuclear potential of the form $-Zr^{-(1+\varepsilon)}$, $\varepsilon \rightarrow 0$. This representation enables us to remove the indeterminacy that would arise when some of the matrix elements of the operator V are evaluated for $\varepsilon = 0$. This device is used in the present paper at the suggestion of E. A. Oks, and has already been used in the brief report in Ref. 10 to calculate the quasienergies of the hydrogen atom in a linearly polarized field.

[2] In deriving (21), we have used the result $[p_z, H_0]_{nn'} = -i\omega_{nn'}^2 z_{nn'}$.

[3] Equation (22) takes account of the fact that the matrix element $z_{nn'}$ couples states for which $|l - l'| = 1$.

[4] We have used the well-known sum rule in (24) (see, for example, Ref. 13).

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