Nonlinear interactions in closed and open systems

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Nonlinear interactions in open and closed systems are considered. It is shown that in open systems nonlinear interaction of nonresonant waves with resonant waves can lead to amplification (damping) of the nonresonant waves, while in closed systems this instability is absent and an adiabatic invariant—the number of quanta of the nonresonant waves—is conserved.

1. INTRODUCTION

Nonlinear interactions of waves in various media are now being actively studied. The purpose of this paper is to draw attention to specific features of nonlinear interactions in open and closed systems. We shall be especially interested in strongly nonequilibrium systems subjected to the action of external sources that can transfer energy to the system. If besides this there is an energy sink, then in a system far from equilibrium a stationary spectrum of excitations, waves, and particles can be established. It is known that open systems with external pumping and an energy sink are capable of self-organization.[1,2] Here we shall direct attention to the possibility that the law of conservation of the adiabatic invariants may be violated in such systems. As a consequence, periodic waves of very high frequencies (higher than all the characteristic frequencies of the system) can be amplified (damped) possible in such systems. This instability is absent in closed systems. One much-studied system that can be open is a turbulent plasma with pumping and dissipation of energy. Using this example we shall illustrate here the phenomenon of amplification (damping) of high-frequency waves. In many laboratory experiments on magnetic confinement of a plasma the system is essentially open. An example of such a system is heated electrons, which can give up energy both to other particles of the plasma and to radiation. Radiation losses also play a large role in an astrophysical plasma (the solar atmosphere, pulsars, and other objects), which also provides examples of open systems. The physical mechanism of the exchange of energy between particles and nonresonant waves in the presence of resonant waves is discussed in detail in Ref. 3. [Below, by resonant waves $(\omega_1, \mathbf{k}_1)$ we shall mean vibrations satisfying the Čerenkov-resonance condition $\omega_1 = \mathbf{k}_1 \cdot \mathbf{v}$ with certain particles of the distribution; by nonresonant waves $(\omega, \mathbf{k})$ we shall mean oscillations that satisfy neither the Čerenkov condition nor the scattering condition: $\omega \neq \mathbf{f} \cdot \mathbf{v}$, $\omega - \omega_1 \neq (\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{v}$.] This mechanism, as will be demonstrated, ensures conservation of the number of quanta in closed systems, and amplification (damping) of waves in open systems.

2. INTERACTION OF PARTICLES AND RANDOM FIELDS IN A PLASMA

For illustration we shall consider a system of particles and random fields in a turbulent plasma. In the first approximation the interaction of the particles with the turbulent fields is described, as is well known, by the quasilinear equation[4] (see also Ref. 5)

$$\frac{\partial \Phi_\mathbf{p}}{\partial t} = \pi e^2 \int dk_1 \frac{|E_{k_1}|^2}{k_1^2} \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \delta(\omega_1 - \mathbf{k}_1 \mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_\mathbf{p}}{\partial \mathbf{p}}\right) \quad (1)$$

(the spectrum $|E_{k_1}|^2$ is assumed to be essentially subthermal). Here $\Phi_\mathbf{p} = \langle f_\mathbf{p} \rangle$ is the regular part of the distribution function, and for simplicity is assumed to depend only on the time: $\Phi_\mathbf{p} = \Phi_\mathbf{p}(t)$. The turbulent oscillations are assumed to be longitudinal, and $k_1 = (\omega_1, \mathbf{k}_1)$, $dk_1 = d\omega_1 d\mathbf{k}_1$; $|E_{k_1}|^2$ is the correlation function of the Fourier components of the turbulent field:

$$\langle E_{k_1, i} E_{k_1', j} \rangle = |E_{k_1}|^2 (k_{1i} k_{1j}/k_1^2) \delta(\mathbf{k}_1 + \mathbf{k}_1') \delta(\omega_1 + \omega_1'), \quad (2)$$

where the angular brackets denote averaging over a statistical ensemble. We note that Eq. (1) describes, in principle, the interaction not only with weakly turbulent fields but also with strongly turbulent ones (when the dependence of $|E_{k_1}|^2$ on the frequency ω_1 does not reduce to the δ-function dependence $\delta(\omega_1 - \omega_{\mathbf{k}_1})$; on the use of (1) in a strongly turbulent plasma, see Ref. 6.

The quasilinear equation (1) can be obtained easily from the general kinetic collisionless equation

$$\frac{\partial f_\mathbf{p}}{\partial t} + \mathbf{v} \frac{\partial f_\mathbf{p}}{\partial \mathbf{r}} + e\mathbf{E} \frac{\partial f_\mathbf{p}}{\partial \mathbf{p}} = Q_\mathbf{p}. \quad (3)$$

For $Q_\mathbf{p} = 0$, representing $f_\mathbf{p}$ in the form of the sum of the regular part $\Phi_\mathbf{p}$ and fluctuating part $\delta f_\mathbf{p}$ and neglecting the nonlinear terms in the equation for $\delta f_\mathbf{p}$, we obtain Eq. (1). In the right-hand side of (3) we have introduced the term $Q_\mathbf{p}$ to describe phenomenologically the more general case of open systems in which there are external sources or sinks of energy, momentum, or particles. The properties of $Q_\mathbf{p}$ can vary. We consider the case of regular sources $\langle Q_\mathbf{p} \rangle = Q_\mathbf{p}$, when the quantity $Q_\mathbf{p}$ will appear only in the equation for the regular part $\Phi_\mathbf{p}$ of the distribution function (but not in the equation for $\delta f_\mathbf{p}$). As a result, an additional term $Q_\mathbf{p}$ arises in the right-hand side of Eq. (1). It plays a fundamental role, inasmuch as it can lead to the establishment of a stationary distribution function (see below).

A different situation arises when the source $Q_\mathbf{p}$ is not regular, e.g., when $Q_\mathbf{p}$ describes losses of various kinds. In this case $Q_\mathbf{p}$ can often be written in the form

$$Q_\mathbf{p} = -(\partial/\partial \mathbf{p}) \mathbf{F}(\mathbf{p}) f_\mathbf{p}. \quad (4)$$

Then, apart from the fact that in the right-hand side of (1)

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the term $\langle Q_{\mathbf{p}}\rangle = -(\partial/\partial\mathbf{p})\cdot\mathbf{F}(\mathbf{p})\Phi_{\mathbf{p}}$ will appear, in the equation for the fluctuating part $\delta f_{\mathbf{p}}$ of the distribution function there also appears a source $\delta Q_{\mathbf{p}} = -(\partial/\partial\mathbf{p})\cdot\mathbf{F}(\mathbf{p})\delta f_{\mathbf{p}}$. If we assume that the characteristic time of the losses is much longer than the period of the turbulent oscillations (which, as a rule, is the case), then the term $\delta Q_{\mathbf{p}}$ in the equation for $\delta f_{\mathbf{p}}$ can be treated using perturbation theory. As a result, for the regular part of the distribution function we obtain the equation

$$\frac{\partial\Phi_{\mathbf{p}}}{\partial t}=\pi e^2\int dk_1\frac{|E_{k_1}|^2}{k_1^2}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1\mathbf{v})\left(\mathbf{k}_1\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right)-\frac{\partial}{\partial\mathbf{p}}F(\mathbf{p})\Phi_{\mathbf{p}}+$$
$$+e^2\int dk_1\frac{|E_{k_1}|^2}{k_1^2}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega_1-\mathbf{k}_1\mathbf{v}}\frac{\partial}{\partial\mathbf{p}}\mathbf{F}(\mathbf{p})\times\frac{1}{\omega_1-\mathbf{k}_1\mathbf{v}}\left(\mathbf{k}_1\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right). \quad (5)$$

In the general case the last term is small in comparison with the second, but in those cases when the losses [the second term in (5)] exactly cancel the quasilinear heating [the first term in (5)], resulting in the establishment of a stationary distribution function, the last term in (5) becomes of the same order as the other nonlinear interactions.

Here we have arrived at the problems of when nonequilibrium systems are stationary and the role of so-called adiabatic interactions.

3. CONSERVATION OF AN ADIABATIC INVARIANT (THE NUMBER OF QUANTA) IN A TIME-DEPENDENT CLOSED SYSTEM

Numerous investigations of quasilinear processes have shown that Eq. (1) does not in fact have time-independent solutions. This has become clear in a particularly striking way in the study of ion-acoustic turbulence excited by an electric field.[7] It is also clear from general considerations, inasmuch as Eq. (1) contains not only diffusion over the angles in momentum space but also diffusion in energy, as a consequence of which the particles continuously acquire energy (if $|E_{k_1}|^2$ is stationary). This acceleration (heating), like Fermi acceleration, is due to the fact that (1) contains only a diffusion term and no friction term (as, e.g., in the Landau collision integral). Therefore, a balance leading to a stationary distribution is impossible. In open nonequilibrium systems, when $Q_{\mathbf{p}}\neq 0$, such a balance and a stationary distribution are, in principle, possible.

For $Q_{\mathbf{p}}=0$ a system obeying (1) will vary weakly in time. In this case the particles will not radiate nonresonant waves, and as a result of this the number of quanta of nonresonant waves will be adiabatically conserved. The question of the adiabatic interaction was discussed in detail some time ago (see the review in Ref. 8). However, this question, as will be seen from the following, requires a more precise treatment. This is due to the uncertainty in the definition of the slowly time-varying dielectric permittivity, which depends simultaneously on the frequency. We shall show how this uncertainty arises.

In time-varying spatially uniform systems the relationship between the spatial Fourier components $\mathbf{D}_{\mathbf{k}}(t)$ and $\mathbf{E}_{\mathbf{k}}(t)$ of the electric induction and electric-field intensity, respectively (for the present we consider only longitudinal fields), has the form

$$\mathbf{D}_{\mathbf{k}}(t)=\int_{-\infty}^{t}\tilde{\varepsilon}_{\mathbf{k}}(t,t')\mathbf{E}_{\mathbf{k}}(t')\,(dt'/2\pi), \quad (6)$$

which expresses in general form, with allowance for causality, the linear dependence between $\mathbf{D}_{\mathbf{k}}(t)$ and $\mathbf{E}_{\mathbf{k}}(t)$.

In stationary systems $\varepsilon_{\mathbf{k}}$ depends only on the difference $\tau=t-t'$ (the factor 2π in the denominator of (6) has been introduced in order that the relation $D_{\omega,\mathbf{k}}=\varepsilon_{\omega,\mathbf{k}}E_{\omega,\mathbf{k}}$ hold for the Fourier components. For time-dependent systems it is therefore customary to write $\tilde{\varepsilon}_{\mathbf{k}}(t,t')$ as a function of the arguments $\tau=t-t'$ and $(t+t')/2$ (see Ref. 8):

$$\tilde{\varepsilon}_{\mathbf{k}}(t,t')=\varepsilon_{\mathbf{k}}(t-t',(t+t')/2). \quad (7)$$

When the parameters of the system change slowly in comparison with the characteristic period of the oscillations, $\varepsilon_{\mathbf{k}}$ is a rapid function of $\tau=t-t'$ and a slow function of $(t+t')/2=t-\tau/2$. Therefore, approximately, we have

$$\tilde{\varepsilon}_{\mathbf{k}}(t,t')\approx\varepsilon_{\mathbf{k}}(\tau,t)-(\tau/2)\,\partial\varepsilon_{\mathbf{k}}(\tau,t)/\partial t. \quad (8)$$

Defining $\varepsilon_{\mathbf{k}}(\omega,t)$ by

$$\varepsilon_{\mathbf{k}}(\omega,t)=\int_0^{\infty}(d\tau/2\pi)\,e^{i\omega\tau}\varepsilon_{\mathbf{k}}(\tau,t), \quad (9)$$

we obtain from (8)

$$\int_0^{\infty}\frac{d\tau}{2\pi}e^{i\omega\tau}\tilde{\varepsilon}_{\mathbf{k}}(t,t')\approx\varepsilon_{\mathbf{k}}(\omega,t)+\frac{i}{2}\frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\varepsilon_{\mathbf{k}}(\omega,t). \quad (10)$$

The imaginary part of $\varepsilon_{\mathbf{k}}$, described by the second term of (10), corresponds to the result of Pitaevskiĭ.[9]

The nonresonant oscillations will be described by the equation (here and everywhere below, for illustration, only longitudinal waves appear)

$$\varepsilon_{\mathbf{k}}(\omega_{\mathbf{k}}(t),t)=0,\quad \omega=\omega_{\mathbf{k}}(t), \quad (11)$$

which gives the frequency ω of the oscillations as a function of $\mathbf{k}$ and t. Differentiating (11) with respect to the time, we obtain

$$\frac{d\omega_{\mathbf{k}}(t)}{dt}=-\left\{\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial t}\left(\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega}\right)^{-1}\right\}_{\omega=\omega_{\mathbf{k}}(t)}. \quad (12)$$

We write the field $\mathbf{E}_{\mathbf{k}}(t)$ in a weakly time-dependent medium in the form

$$\mathbf{E}_{\mathbf{k}}(t)=\mathbf{E}_{\mathbf{k}}^{(0)}(t)\exp\left\{-i\int^{t}\omega_{\mathbf{k}}(t')\,dt'\right\}, \quad (13)$$

separating the slowly time-varying amplitude $\mathbf{E}_{\mathbf{k}}^{(0)}(t)$ and the eikonal. The induction $\mathbf{D}_{\mathbf{k}}(t)$ should also be represented as in (13), by separating out the amplitude $\mathbf{D}_{\mathbf{k}}^{(0)}(t)$.

Substituting (8) and (13) into (6), we obtain

$$\mathbf{D}_{\mathbf{k}}^{(0)}(t)=\int_0^{\infty}\frac{d\tau}{2\pi}\left(\varepsilon_{\mathbf{k}}(\tau,t)-\frac{\tau}{2}\frac{\partial\varepsilon_{\mathbf{k}}(\tau,t)}{\partial t}\right)\mathbf{E}_{\mathbf{k}}^{(0)}(t-\tau)\times\exp\left\{-i\int_{t}^{t-\tau}\omega_{\mathbf{k}}(t')\,dt'\right\}. \quad (14)$$

In the same way that in Eq. (8) $\varepsilon_{\mathbf{k}}(\tau,t-\tau/2)$ was expanded in powers of τ, in the second, "slow" argument, we must expand $\mathbf{E}_{\mathbf{k}}^{(0)}(t-\tau)$, and also $\exp(-i\int\omega_{\mathbf{k}}(t')dt')$, in powers of τ:

$$\exp\left\{-i\int_{t}^{t-\tau}\omega_{\mathbf{k}}(t')\,dt'\right\}\approx\exp\{i\omega_{\mathbf{k}}(t)\tau\}\left(1-\frac{i\tau^2}{2}\frac{d\omega_{\mathbf{k}}(t)}{dt}\right). \quad (15)$$

Then (14), after the introduction of $\varepsilon_{\mathbf{k}}(\omega,t)$ in accordance with (9), takes the form

$$\mathbf{D}_{\mathbf{k}}(t)=\exp\left[-i\int^{t}\omega_{\mathbf{k}}(t')\,dt'\right]\times\left\{\varepsilon_{\mathbf{k}}(\omega,t)\mathbf{E}_{\mathbf{k}}^{(0)}(t)+\frac{i}{2}\frac{\partial^2\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega\,\partial t}\mathbf{E}_{\mathbf{k}}^{(0)}(t)+i\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega}\frac{\partial\mathbf{E}_{\mathbf{k}}^{(0)}(t)}{\partial t}+\frac{i}{2}\frac{\partial^2\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega^2}\frac{d\omega_{\mathbf{k}}(t)}{dt}\mathbf{E}_{\mathbf{k}}^{(0)}(t)\right\}_{\omega=\omega_{\mathbf{k}}(t)} \quad (16)$$

The equation obtained by equating $\mathbf{D}_{\mathbf{k}}(t)$ to zero in (16) is the equation describing the longitudinal oscillations; the first term in the curly brackets in (16) corresponds to (11). Equation (16) together with (11) gives the law of variation of the field amplitude:

$$\gamma_{\mathbf{k}}(t)=\frac{1}{2|\mathbf{E}_{\mathbf{k}}^{(0)}(t)|^2}\cdot\frac{\partial|\mathbf{E}_{\mathbf{k}}^{(0)}(t)|^2}{\partial t}=\frac{1}{2}\left\{\frac{\partial^2\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega^2}\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial t}\times\left(\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega}\right)^{-2}-\frac{\partial^2\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega\,\partial t}\left(\frac{\partial\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega}\right)^{-1}\right\}_{\omega=\omega_{\mathbf{k}}(t)}. \quad (17)$$

This law describes the variation of the nonresonant wave amplitudes [the imaginary part of $\varepsilon_{\mathbf{k}}(\omega,t)$ is absent, since the waves are assumed to be nonresonant]. Equation (17) describes conservation of an adiabatic invariant—the number $N_{\mathbf{k}}(t)$ of quanta, equal to

$$N_{\mathbf{k}}(t)=\pi^2[\partial\varepsilon_{\mathbf{k}}(\omega,t)/\partial\omega]_{\omega=\omega_{\mathbf{k}}(t)}|E_{\mathbf{k}}^{(0)}(t)|^2. \quad (18)$$

The energy density $W(t)$ of the waves is connected with the number $N_{\mathbf{k}}(t)$ of quanta by the relation ($\hbar=1$)

$$W(t)=\int\omega_{\mathbf{k}}(t)N_{\mathbf{k}}(t)\frac{d\mathbf{k}}{(2\pi)^3}. \quad (19)$$

Indeed, differentiating (18) with respect to t,

$$((d/dt)|\mathbf{E}_{\mathbf{k}}^{(0)}(t)|^2=2\gamma_{\mathbf{k}}(t)|\mathbf{E}_{\mathbf{k}}^{(0)}(t)|^2),$$

where $\gamma_{\mathbf{k}}(t)$ is determined in (17), we obtain $(d/dt)N_{\mathbf{k}}(t)=0$.

4. CONSERVATION OF THE NUMBER OF QUANTA, AND THE NONLINEAR INTERACTION

We now show where the error occurs in the arguments of the preceding Section. The choice of the second argument in (7) in the form $(t+t')/2$ is, in fact, arbitrary. It would be possible to replace it by an arbitrary linear combination of t and t', equal to t when $t'=t$. For example, choosing the second argument in (7) in the form t', in place of (8) we obtain

$$\bar{\varepsilon}_{\mathbf{k}}(t,t')=\varepsilon_{\mathbf{k}}(\tau,t-\tau)\approx\varepsilon_{\mathbf{k}}(\tau,t)-\tau\partial\varepsilon_{\mathbf{k}}(\tau,t)/\partial t. \quad (20)$$

The factor-of-two difference between the second term in (20) and (8) means that the adiabatic invariant (18) is not conserved. The authors of Ref. 10 asserted that the magnitude of the imaginary part of $\varepsilon_{\mathbf{k}}$ in a slowly varying system depends on the model used to describe the system. But the quasilinear theory is a specific description, and in particular it can be established exactly whether (8), (20), or any other analogous relation holds. We shall show that in reality it is (20) that holds, and not (8), and shall then show how the paradox which then arises in connection with the conservation of the adiabatic invariant (18) is resolved.

We shall find the perturbation $\delta f_{\mathbf{p},\mathbf{k}}(t)$ of the distribution function $f_{\mathbf{p},\mathbf{k}}(t)$ of the particles. For the Fourier components $\delta f_{\mathbf{p},\mathbf{k},\omega}$ we have

$$i(\omega-\mathbf{kv})\delta f_{\mathbf{p},\mathbf{k}\omega}=e\int\mathbf{E}_{\mathbf{k}}(t')(\partial\Phi_{\mathbf{p}}(t')\partial\mathbf{p})e^{i\omega t'}\frac{dt'}{2\pi},$$

whence

$$\delta f_{\mathbf{p},\mathbf{k}}(t)=e\int_0^\infty\frac{e^{-i\omega\tau}\,d\omega\,d\tau}{2\pi i(\omega-\mathbf{kv})}\mathbf{E}_{\mathbf{k}}(t-\tau)\frac{\partial\Phi_{\mathbf{p}}(t-\tau)}{\partial\mathbf{p}}. \quad (21)$$

Since $\mathbf{D}_{\mathbf{k}}(t)=\mathbf{E}_{\mathbf{k}}(t)+4\pi\mathbf{P}_{\mathbf{k}}(t)$, where the polarization is

$$\mathbf{P}_{\mathbf{k}}(t)=ie\frac{\mathbf{k}}{k^2}\int\delta f_{\mathbf{p},\mathbf{k}}(t)\frac{d\mathbf{p}}{(2\pi)^3},$$

we have, on the basis of (21),

$$\mathbf{D}_{\mathbf{k}}(t)=\int_0^\infty\varepsilon_{\mathbf{k}}(\tau,t-\tau)\mathbf{E}_{\mathbf{k}}(t-\tau)\frac{d\tau}{2\pi}, \quad (22)$$

$$\varepsilon_{\mathbf{k}}(\tau,t)=2\pi\delta_+(\tau)+\frac{4\pi e^2}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{e^{-i\omega\tau}\,d\omega}{\omega-\mathbf{kv}+i0}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}(t)}{\partial\mathbf{p}}\right),$$

$$\int_0^\infty\delta_+(\tau)\,d\tau=1.$$

As we see, it is (20), and not (8) or any other relation, that is fulfilled. In accordance with the definition (9) we now have

$$\varepsilon_{\mathbf{k}}(\omega,t)=1+\frac{4\pi e^2}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega-\mathbf{kv}}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}(t)}{\partial\mathbf{p}}\right). \quad (23)$$

We now show how to solve the question of the conservation of the adiabatic invariant (18). For this we write explicitly, using (23), the imaginary part of (20) that arises after Fourier transformation:

$$\operatorname{Im}\varepsilon_{\omega,\mathbf{k}}^{L}=\frac{\partial^2\varepsilon_{\mathbf{k}}(\omega,t)}{\partial\omega\,\partial t}=-\frac{4\pi e^2}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{(\omega-\mathbf{kv})^2}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\frac{\partial\Phi_{\mathbf{p}}}{\partial t}, \quad (24)$$

and for $\partial\Phi_{\mathbf{p}}/\partial t$ we use the quasilinear equation (1). Thus,

$$\operatorname{Im}\varepsilon^{L}_{\omega,\mathbf{k}}=-\frac{4\pi^2e^4}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{dk_1}{k_1^2}\frac{|E_{k_1}|^2}{(\omega-\mathbf{kv})^2}\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1\mathbf{v})\times\left(\mathbf{k}_1\frac{\partial\Phi_\mathbf{p}}{\partial\mathbf{p}}\right). \quad (25)$$

It can be seen from (25) that besides (25) it is necessary to take the nonlinear permittivity $\varepsilon^N_{\omega,\mathbf{k}}$ into account as well, since it has the same order of magnitude as (25). By the standard method for the interaction of resonant and nonresonant fields (see Ref. 11) we obtain

$$\operatorname{Im}\varepsilon^{N}_{\omega,\mathbf{k}}=\frac{4\pi^2e^4}{k^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{dk_1}{k_1^2}\frac{|E_{k_1}|^2}{\omega-\mathbf{kv}}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega_--\mathbf{k}_-\mathbf{v}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\times\delta(\omega_1-\mathbf{k}_1\mathbf{v})\left(k_1\frac{\partial\Phi_\mathbf{p}}{\partial\mathbf{p}}\right). \quad (26)$$

$$\omega_-=\omega-\omega_1,\quad \mathbf{k}_-=\mathbf{k}-\mathbf{k}_1.$$

With allowance for the nonlinear interaction, in place of (17) we now obtain

$$\tfrac{1}{2}|\mathbf{E}_\mathbf{k}^{(0)}(t)|^{-2}\partial|\mathbf{E}_\mathbf{k}^{(0)}(t)|^2/\partial t=\gamma_\mathbf{k}=\gamma_\mathbf{k}^L+\gamma_\mathbf{k}^N, \quad (27)$$

where $\gamma_\mathbf{k}^L$ is defined in analogy with (17), and, if we allow for the difference between (20) and (8), has the form

$$\gamma_\mathbf{k}^L=\left\{\frac{1}{2}\frac{\partial^2\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega^2}\frac{\partial\varepsilon_\mathbf{k}(\omega,t)}{\partial t}\left(\frac{\partial\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega}\right)^{-2}-\frac{\partial^2\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega\,\partial t}\left(\frac{\partial\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega}\right)^{-1}\right\}_{\omega=\omega_\mathbf{k}(t)}, \quad (28)$$

$$\gamma_\mathbf{k}^N=-\{\operatorname{Im}\varepsilon_{\omega,\mathbf{k}}(\partial\varepsilon_\mathbf{k}(\omega,t)/\partial\omega)^{-1}\}_{\omega=\omega_\mathbf{k}(t)}. \quad (29)$$

For the change of (18), i.e., of the number $N_\mathbf{k}(t)$ of quanta, we now obtain

$$\frac{dN_\mathbf{k}(t)}{dt}=-\pi^2|\mathbf{E}_\mathbf{k}^{(0)}(t)|^2\left\{2\operatorname{Im}\varepsilon^N_{\omega,\mathbf{k}}+\frac{\partial^2\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega\,\partial t}\right\}_{\omega=\omega_\mathbf{k}(t)}. \quad (30)$$

We shall perform the following transformations of (26). We expand the difference denominator

$$\frac{1}{\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v}}=\frac{1}{\omega-\mathbf{kv}}+\frac{\omega_1-\mathbf{k}_1\mathbf{v}}{(\omega-\mathbf{kv})^2}+\frac{(\omega_1-\mathbf{k}_1\mathbf{v})^2}{(\omega-\mathbf{kv})^3}+\dots. \quad (31)$$

The use of the expansion (31) is legitimate because (26) contains the factor $\delta(\omega_1-\mathbf{k}_1\cdot\mathbf{v})$. The second term of (31) drops out of (26) because it is odd under the replacement $\omega_1\to-\omega_1$, $\mathbf{k}_1\to-\mathbf{k}_1$ (since $|E_{k_1}|^2$ does not change under this replacement), while the third term of the expansion (31) (and, analogously, all the subsequent terms) gives zero, since commuting $(\omega_1-\mathbf{k}_1\cdot\mathbf{v})^2$ with $(\mathbf{k}\cdot\partial/\partial\mathbf{p})$ in (26) preserves the first power of $(\omega_1-\mathbf{k}_1\cdot\mathbf{v})$, which upon multiplication by $\delta(\omega_1-\mathbf{k}_1\cdot\mathbf{v})$ gives zero. Accordingly we now transform (26):

$$\operatorname{Im}\varepsilon^{N}_{\omega,\mathbf{k}}=\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega-\mathbf{kv}}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega-\mathbf{kv}}\dots=-\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega-\mathbf{kv}}\right]\frac{1}{\omega-\mathbf{kv}}\dots$$

$$=-\frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{(\omega-\mathbf{kv})^2}\right]\dots=\frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{(\omega-\mathbf{kv})^2}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\dots=-\frac{1}{2}\frac{\partial^2\varepsilon_\mathbf{k}(\omega,t)}{\partial\omega\,\partial t} \quad (32)$$

[we have integrated by parts and have used the fact that $(\mathbf{k}\cdot\partial/\partial\mathbf{p})(\mathbf{k}_1\cdot\partial/\partial\mathbf{p})=(\mathbf{k}_1\cdot\partial/\partial\mathbf{p})(\mathbf{k}\cdot\partial/\partial\mathbf{p})$]. Substituting (32) into (30), we obtain $dN_\mathbf{k}(t)/dt=0$; i.e., the number of quanta is an adiabatic invariant.

We note that with the aim of simplifying the formulas we have assumed everywhere that the resonant waves (ω_1, $\mathbf{k}_1$) are also longitudinal. However, all the results obtained, including formulas (30), (32) and the conclusion that an adiabatic invariant—the number $N_\mathbf{k}(t)$ of quanta—is conserved, also remain valid for transverse resonant waves (of arbitrary modes, in general). Thus, the result $dN_\mathbf{k}(t)/dt=0$ does not depend on the nature of the resonant waves and is general.

We have proved the statement that an adiabatic invariant (the number of quanta) is conserved under the assumption that the system varies only in time but is spatially uniform. However, the mathematical apparatus developed in Secs. 3 and 4 can be generalized to the case of spatially nonuniform systems. Thus, e.g., it is possible to show, in analogy with Secs. 3 and 4, that in the case of a stationary spatially varying closed one-dimensional system the number of quanta of nonresonant waves is also conserved.

It can be seen from the above derivation that 1) allowance for the nonlinear permittivity is essential, 2) an adiabatic invariant is conserved (naturally) only in closed systems, in the absence of external sources.

5. NONLINEAR INTERACTIONS AND THE POSSIBILITY OF AMPLIFICATION (DAMPING) OF NONRESONANT WAVES IN OPEN SYSTEMS

The question of the nonlinear interactions describable by $\operatorname{Im}\varepsilon^N_k$ [see (26) and (57)] has recently been widely discussed in the literature.[12] The starting point in the very first papers[11] was the assumption that stationary distributions of particles ($\Phi_\mathbf{p}$) and of turbulent fields ($|E(k_1)|^2$) exist. From the analysis performed it is clear that under these conditions $\operatorname{Im}\varepsilon^L_k=0$, and (26) correctly describes the interaction of the nonresonant and resonant waves. Here the most interesting effects are phenomena of conversion upward in frequency, which can be manifested in the generation of Langmuir ion-acoustic waves[11] and in the generation of electromagnetic waves of very high frequencies.[12] In this connection we point out the inaccuracy of Ref. 13, in which the authors attempted to prove from general considerations that $\operatorname{Im}\varepsilon^N_k=0$.

Under real conditions one often observes time-independence. A special role is played by open systems in which there is a source or sink of energy or particles. In those cases when the source $Q_\mathbf{p}$ has a purely regular character, there remains only the effect due to nonlinear conversion (26). Particles moving in the field of a nonresonance wave pump energy over into high-frequency waves, the number of quanta of which can grow in an avalance-like manner with time (if the anisotropy of $|E_{k_1}|^2$ remains constant and if $\Phi_{\mathbf{p}}$ is stationary and, possibly, even isotropic). In conditions when $Q_{\mathbf{p}}$ depends on the distribution of particles [e.g., is determined by (4)], a contribution from $Q_{\mathbf{p}}$ arises in the equations for both the regular part and the turbulent part of the distribution function. In this case the regular distribution is stationary:

$$\pi e^2 \int dk_1\, k_1^{-2} |E_{k_1}|^2 \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right)$$

$$= \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{p}) \Phi_{\mathbf{p}}, \tag{33}$$

and therefore $\operatorname{Im} \varepsilon_k^L$, which arises as a consequence of time-dependence vanishes and only $\operatorname{Im} \varepsilon_k^N$ remains. However, because of the presence of the term $\delta Q_{\mathbf{p}} = -(\partial/\partial \mathbf{p}) \cdot \mathbf{F}(\mathbf{p}) \delta f_{\mathbf{p}}$, ε_k^L is modified as follows:

$$\varepsilon_k^L = \varepsilon_k^{L(0)} + \delta\varepsilon_k^L, \quad \varepsilon_k^{L(0)}$$

$$= 1 + \frac{4\pi e^2}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right), \tag{34}$$

$$\delta\varepsilon_k^L = -i \frac{4\pi e^2}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{p}) \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right). \tag{35}$$

We shall make the following transformations:

$$\frac{1}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{p}) \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right) = \frac{1}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial p_i} \frac{1}{\omega - \mathbf{k}\mathbf{v}}$$

$$\times \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) F_i(\mathbf{p}) \Phi_{\mathbf{p}} - \frac{1}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial p_i} \frac{\Phi_{\mathbf{p}}}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) F_i(\mathbf{p}). \tag{36}$$

After we have discarded the total derivative with respect to the momenta, which does not appear in (35), the first term in (36) acquires the form

$$\frac{1}{2(\omega - \mathbf{k}\mathbf{v})^2} \frac{\partial}{\partial p_i} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) F_i(\mathbf{p}) \Phi_{\mathbf{p}}$$

$$= \frac{1}{2(\omega - \mathbf{k}\mathbf{v})^2} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) \left(\frac{\partial}{\partial \mathbf{p}} \mathbf{F}(\mathbf{p}) \Phi_{\mathbf{p}}\right)$$

$$= \frac{\pi e^2}{2} \int \frac{dk_1}{k_1^2} \frac{|E_{k_1}|^2}{(\omega - \mathbf{k}\mathbf{v})^2} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right)$$

$$\times \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right). \tag{37}$$

In the last step we have used the relation (33). On the basis of (32) and (33) we find that the contribution of (37) to $\delta\varepsilon_k^L$ will be equal in magnitude but opposite in sign to ε_k^N. Thus, in the sum $\varepsilon_k^N + \delta\varepsilon_k^L$ there remains only the contribution of the second term in (36):

$$\operatorname{Im}(\delta\varepsilon_k^L + \varepsilon_k^N)$$

$$= \frac{4\pi e^2}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial p_i} \frac{\Phi_{\mathbf{p}}}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) \dot{F}_i(\mathbf{p}). \tag{38}$$

For the excitation of Langmuir waves by ion-acoustic waves under conditions of stationary ion-acoustic turbulence generated by a constant electric field $\mathbf{E}_0$ (see Ref. 7), we have

$$\mathbf{F}(\mathbf{p}) = e\mathbf{E}_0, \quad (\mathbf{k}, \partial/\partial \mathbf{p}) F_i(\mathbf{p}) = 0 \tag{39}$$

and, consequently, $\operatorname{Im}(\delta\varepsilon_k^L + \varepsilon_k^N) = 0$, i.e., nonresonant Langmuir waves cannot be generated. Evidently, the authors of Ref. 13 were edging intuitively toward a proof of this statement. However, their route (an attempt to prove that $\operatorname{Im} \varepsilon_k^N = 0$) was, as already noted, fundamentally erroneous. It is clear that the absence of upward conversion in frequency in the given case is a consequence of the very special form of the function $\mathbf{F}(\mathbf{p})$ ($\mathbf{F}$ does not depend on $\mathbf{p}$). In all other cases in open systems (e.g., when two-particle collisions of electrons and ions are taken into account) upward conversion in frequency is possible, even if we ignore the fact that in actual experiments there are always losses and a stationary distribution is established.

6. PROOF THAT THE CONTRIBUTION OF SECOND-ORDER CURRENTS TO THE NONLINEAR INTERACTION VANISHES

The expression (26) that was used above for the nonlinear permittivity ε_k^N was obtained under the assumption that a nonzero contribution to it arises only from nonlinear currents of third order in the field. We shall demonstrate the erroneous nature of Refs. 14, in which the nonlinear permittivity was found to have a nonzero contribution arising from second-order currents.

The contribution from the second-order currents to the nonlinear permittivity has the form

$$\operatorname{Im} \delta\varepsilon_k^N = -\frac{16\pi^2 e^6}{k^2} \int \frac{dk_1}{k_1^2} |E_{k_1}|^2 \frac{\operatorname{Im}(S_{k,k_1,k-k_1} S_{k-k_1,k,-k_1})}{(\mathbf{k}-\mathbf{k}_1)^2 \varepsilon_{k-k_1}^L}, \tag{40}$$

$$S_{k,k_1,k_2} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \mathbf{k}\mathbf{v} + i0} \left\{ \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_2 - \mathbf{k}_2\mathbf{v} + i0} \left(\mathbf{k}_2 \frac{\partial}{\partial \mathbf{p}}\right) + \left(\mathbf{k}_2 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_1 - \mathbf{k}_1\mathbf{v} + i0} \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \right\} \Phi_{\mathbf{p}}. \tag{41}$$

In the given nonlinear interaction the scattering condition is assumed not to be fulfilled, i.e., $\omega - \omega_1 - (\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{v} \neq 0$, $\operatorname{Im} \varepsilon_{k-k_1}^L = 0$. Therefore,

$$\operatorname{Im} S_{k,k_1,k-k_1} = -\pi \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k} - \mathbf{k}_1, \frac{\partial}{\partial \mathbf{p}}\right)$$

$$\times \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right)$$

$$= -\pi \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{\omega - \omega_1 - (\mathbf{k} - \mathbf{k}_1)\mathbf{v}} \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right)$$

$$= -\operatorname{Im} S_{k-k_1,k,-k_1}. \tag{42}$$

For the real parts of $S_{k,k,k-k_1}$ and $S_{k-k_1,k,-k_1}$ we have

$$\operatorname{Re} S_{k,k_1,k-k_1} = \int \frac{d\mathbf{p}}{(2\pi)^3} \left\{ \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_- - \mathbf{k}_-\mathbf{v}} \left(\mathbf{k}_- \frac{\partial}{\partial \mathbf{p}}\right) + \left(\mathbf{k}_- \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_1 - \mathbf{k}_1\mathbf{v}} \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \right\} \Phi_{\mathbf{p}}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3}\Big\{\Big[\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)(\mathbf{kv})\Big]$$

$$\times\Big[\frac{1}{(\omega-\mathbf{kv})^2(\omega_--\mathbf{k}_-\mathbf{v})}\Big(\mathbf{k}_-\frac{\partial}{\partial\mathbf{p}}\Big)+\Big(\frac{1}{(\omega-\mathbf{kv})^2(\omega_1-\mathbf{k}_1\mathbf{v})}$$

$$+\frac{1}{(\omega_1-\mathbf{k}_1\mathbf{v})^2(\omega-\mathbf{kv})}\Big)\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)\Big]$$

$$+\frac{1}{(\omega-\mathbf{kv})(\omega_1-\mathbf{k}_1\mathbf{v})}\Big(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Big)\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)\Big\}\,\Phi_\mathbf{p}, \tag{43}$$

$$\mathrm{Re}\,S_{k-k_1,k,-k_1} = \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega_--\mathbf{k}_-\mathbf{v}}\Big\{\Big(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Big)\frac{1}{\omega_1-\mathbf{k}_1\mathbf{v}}\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)$$

$$-\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)\frac{1}{\omega-\mathbf{kv}}\Big(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Big)\Big\}\,\Phi_\mathbf{p} = \int\frac{d\mathbf{p}}{(2\pi)^3}\Big\{\Big[\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)(\mathbf{kv})\Big]$$

$$\times\Big[\frac{1}{(\omega_--\mathbf{k}_-\mathbf{v})(\omega_1-\mathbf{k}_1\mathbf{v})^2}\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)-\frac{1}{(\omega_--\mathbf{k}_-\mathbf{v})(\omega-\mathbf{kv})^2}$$

$$\times\Big(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Big)\Big]+\frac{1}{(\omega-\mathbf{kv})(\omega_1-\mathbf{k}_1\mathbf{v})}\Big(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\Big)\Big(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\Big)\Big\}\,\Phi_\mathbf{p}. \tag{44}$$

With allowance for the equality $(\mathbf{k},\partial/\partial\mathbf{p})(\mathbf{k}_1\cdot\mathbf{v}) = (\mathbf{k}_1,\partial/\partial\mathbf{p})(\mathbf{k}\cdot\mathbf{v})$ we find from (43) and (44) that

$$\mathrm{Re}\,S_{k-k_1,\,k,\,-k_1}=\mathrm{Re}\,S_{k,\,k_1,\,k-k_1}.$$

Thus, we have given a general proof that, for the interaction of resonance and nonresonance waves,

$$\mathrm{Im}\,\delta\varepsilon_k{}^N=0. \tag{45}$$

This assertion was made without detailed proof in Refs. 11 and 12. Thus, the use of the expression (26) for the nonlinear permittivity is correct.

7. CONSERVATION OF THE NUMBER OF QUANTA FOR ARBITRARY MODES OF OSCILLATION

The results of Sec. 4 can also be generalized to the case of arbitrary modes of oscillation. Taking into account only the time dependence, we obtain (for the spatially uniform problem) in place of (21)

$$\delta f_{\mathbf{p},\mathbf{k}}(t)=e\int_0^\infty\frac{d\tau\,d\omega\,e^{-i\omega\tau}}{2\pi i(\omega-\mathbf{kv})}\Big\{\mathbf{E}_\mathbf{k}(t-\tau)+\Big[\frac{\mathbf{v}}{c}\mathbf{H}_\mathbf{k}(t-\tau)\Big]\Big\}$$

$$\times\frac{\partial\Phi_\mathbf{p}(t-\tau)}{\partial\mathbf{p}}. \tag{46}$$

Here $\mathbf{H}_\mathbf{k}(t)$ is expressed in terms of $\mathbf{E}_\mathbf{k}(t)$ using the Maxwell equations:

$$\mathbf{H}_\mathbf{k}(t)=-ic\int_{-\infty}^t dt_1[\mathbf{k}\mathbf{E}_\mathbf{k}(t_1)]. \tag{47}$$

Substituting (47) into (46) and changing the order of integration, we obtain

$$\delta f_{\mathbf{p},\mathbf{k}}(t)=e\int_0^\infty\frac{d\omega\,d\tau}{2\pi i(\omega-\mathbf{kv})}E_{\mathbf{k},i}(t-\tau)\Big\{e^{-i\omega\tau}\frac{\partial\Phi_\mathbf{p}(t-\tau)}{\partial p_i}$$

$$-i(k_jv_i-\delta_{ij}(\mathbf{kv}))\int_0^\tau d\tau_1\,e^{-i\omega\tau_1}\frac{\partial\Phi_\mathbf{p}(t-\tau_1)}{\partial p_j}\Big\}. \tag{48}$$

Next, in analogy with (22), we must express the induction $\mathbf{D}_\mathbf{k}(t)$ in terms of $\delta f_{\mathbf{p},\mathbf{k}}(t)$:

$$\mathbf{D}_\mathbf{k}(t)=\mathbf{E}_\mathbf{k}(t)+4\pi e\int_{-\infty}^t dt_1\,\mathbf{v}\delta f_{\mathbf{p},\mathbf{k}}(t_1)\frac{d\mathbf{p}}{(2\pi)^3}.$$

Omitting the calculations, we give the final expression [obtained in analogy with (23)] for the instantaneous dielectric permittivity $\varepsilon_{\mathbf{k},ij}(\omega,t)$ in the quasilinear approximation:

$$\varepsilon_{\mathbf{k},ij}(\omega,t)=\delta_{ij}+\varepsilon^{(1)}_{\mathbf{k},ij}(\omega,t)+\varepsilon^{(2)}_{\mathbf{k},ij}(\omega,t). \tag{49}$$

Here $\varepsilon^{(1)}_{\mathbf{k},ij}$ and $\varepsilon^{(2)}_{\mathbf{k},ij}$ are the contributions from the first and second terms in (48):

$$\varepsilon^{(1)}_{\mathbf{k},ij}(\omega,t)=\frac{4\pi e^2}{\omega}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{v_i}{\omega-\mathbf{kv}}\frac{\partial\Phi_\mathbf{p}(t)}{\partial p_j} \tag{50}$$

$$\varepsilon^{(2)}_{\mathbf{k},ij}(\omega,t)=\frac{4\pi e^2}{\omega^2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{v_i(k_lv_j-\delta l_j(\mathbf{kv}))}{\omega-\mathbf{kv}}\frac{\partial\Phi_\mathbf{p}(t)}{\partial p_l}. \tag{51}$$

The first term in (49) is analogous in structure to the longitudinal dielectric permittivity $\varepsilon_\mathbf{k}(\omega,t)$ in Sec. 4. But the second term in (49) has a somewhat different form. Therefore, in place of formula (24), describing the longitudinal waves, we have

$$\mathrm{Im}\,\varepsilon^L_{\mathbf{k},ij}=\frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\varepsilon^{(1)}_{\mathbf{k},ij}(\omega,t)+\frac{1}{\omega}\frac{\partial}{\partial\omega}\omega\frac{\partial}{\partial t}\varepsilon^{(2)}_{\mathbf{k},ij}(\omega,t)$$

$$=\frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\varepsilon_{\mathbf{k},ij}(\omega,t)+\frac{1}{\omega}\frac{\partial\varepsilon^{(2)}_{\mathbf{k},ij}(\omega,t)}{\partial t}.$$

The dispersion equation now has the form $[\mathbf{E}(t)=\mathbf{e}(t)E(t)]$

$$[k^2\delta_{ij}-k_ik_j-\omega_\mathbf{k}{}^2(t)c^{-2}\varepsilon_{\mathbf{k},ij}(\omega_\mathbf{k}(t),\,t)]e_j(t)=0. \tag{52}$$

The simplest case for analysis is that of unpolarized electromagnetic radiation, which is transverse:

$$\langle e_i(t)e_j(t)\rangle={}^1/_2(\delta_{ij}-k_ik_j/k^2). \tag{53}$$

Introducing

$$\varepsilon_k^{(t)}={}^1/_2(\delta_{ij}-k_ik_j/k^2)\varepsilon_{\mathbf{k},ij} \tag{54}$$

and following the method developed in Secs. 3 and 4 of the present paper, we now obtain for the change in the number $N_\mathbf{k}(t)$ of quanta

$$N_\mathbf{k}(t)=2\pi^2|E_\mathbf{k}^{(0)}(t)|^2\Big(\frac{1}{\omega^2}\frac{\partial}{\partial\omega}\omega^2\varepsilon_\mathbf{k}^{(t)}(\omega,t)\Big)_{\omega=\omega_\mathbf{k}(t)}, \tag{55}$$

$$\frac{dN_\mathbf{k}(t)}{dt}$$

$$=-2\pi^2|E_\mathbf{k}^{(0)}(t)|^2\Big\{2\,\mathrm{Im}\,\varepsilon_\mathbf{k}^{N(t)}+\frac{\partial^2\varepsilon_\mathbf{k}^{(t)}(\omega,t)}{\partial\omega\,\partial t}$$

$$+\frac{2}{\omega}\frac{\partial\varepsilon_\mathbf{k}^{(2)(t)}(\omega,t)}{\partial t}\Big\}_{\omega=\omega_\mathbf{k}(t)}, \tag{56}$$

where $\varepsilon_\mathbf{k}^{(t)}$ and $\varepsilon_\mathbf{k}^{(2)(t)}$ are defined in (49)–(51) and (54), the quantity $\partial\Phi_\mathbf{p}(t)/\partial t$ appearing in (56) is found from the quasilinear equation (1), and $\mathrm{Im}\,\varepsilon_k^{N(t)}$, obtained by the standard method, has the form

$$\operatorname{Im}\varepsilon_k^{N(t)}=\frac{2\pi^2e^4}{\omega}\int\frac{dk_1}{k_1^2}|E_{k_1}|^2\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)$$
$$\times\frac{d\mathbf{p}}{(2\pi)^3}\frac{v_i}{\omega-\mathbf{kv}}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)$$
$$\times\frac{1}{\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v}}\left\{\delta_{lj}\left(1-\frac{\mathbf{kv}}{\omega}\right)+\frac{k_lv_j}{\omega}\right\}$$
$$\times\frac{\partial}{\partial p_l}\delta(\omega_1-\mathbf{k}_1\mathbf{v})\left(\mathbf{k}_1\frac{\partial\Phi_\mathbf{p}}{\partial\mathbf{p}}\right) \quad (57)$$

(here, to simplify the formulas, without loss of generality we again choose the resonant field to be longitudinal).

Integrating (57) twice by parts over the momenta, we can convince ourselves that the sum in the square brackets in (56) vanishes. Thus, we have shown that an adiabatic invariant (the number of quanta) is conserved in closed systems for transverse waves too.

8. EXAMPLES OF NONLINEAR INTERACTION IN OPEN SYSTEMS IN ASTROPHYSICS; DISCUSSION OF THE RESULTS

We shall consider how nonresonant waves evolve in an open system, using the example of electromagnetic waves of high frequencies ($\omega\gg\omega_{pl}$) in the presence of longitudinal resonant waves. We shall assume that the losses $Q_\mathbf{p}$ in the system completely balance the quasilinear acceleration of the particles, i.e., the relation (33) is valid. This leads [see (35)] to an additional term in the linear (now transverse) dielectric permittivity describing the nonresonant waves:

$$\delta\varepsilon_k^{L(t)}=-i\frac{2\pi e^2}{\omega}\int\frac{d\mathbf{p}}{(2\pi)^3}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)$$
$$\times\frac{v_i}{\omega-\mathbf{kv}}\frac{\partial}{\partial\mathbf{p}}\mathbf{F}(\mathbf{p})\frac{1}{\omega-\mathbf{kv}}$$
$$\times\left\{\left(1-\frac{\mathbf{kv}}{\omega}\right)\delta_{lj}+\frac{k_lv_j}{\omega}\right\}\frac{\partial\Phi_\mathbf{p}}{\partial p_l}. \quad (58)$$

The expression for the nonlinear permittivity is given by formula (57), which, by means of transformations analogous to those performed in formulas (36) and (37) and with the use of the result of Sec. 7, can be brought to the form

$$\operatorname{Im}\varepsilon_k^{N(t)}=-\pi e^2\int\frac{d\mathbf{p}}{(2\pi)^3}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\left\{\frac{\partial}{\partial\omega}\frac{v_i}{\omega(\omega-\mathbf{kv})}\right.$$
$$\times\left[\delta_{lj}\left(1-\frac{\mathbf{kv}}{\omega}\right)+\frac{k_lv_j}{\omega}\right]+\left.\frac{2v_i(k_lv_j-\delta_{lj}(\mathbf{kv}))}{\omega^3(\omega-\mathbf{kv})}\right\}\frac{\partial}{\partial p_l}$$
$$\times\pi e^2\int\frac{dk_1}{k_1^2}|E_{k_1}|^2\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1v)\left(\mathbf{k}_1\frac{\partial\Phi_\mathbf{p}}{\partial\mathbf{p}}\right), \quad (59)$$

in which the quasilinear collision integral has been separated out explicitly. Expressing the latter in terms of a source using Eq. (33), after some transformations we obtain the following expression for the growth rate:

$$\gamma_k=-\frac{\omega}{2}\operatorname{Im}(\delta\varepsilon_k^{L(t)}+\varepsilon_k^{N(t)})=-\pi e^2\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega(\omega-\mathbf{kv})}$$
$$\times\left\{\left(v_i-k_i\frac{(\mathbf{kv})}{k^2}\right)+\frac{k_i}{2(\omega-\mathbf{kv})}\right.$$
$$\times\left.\left(v^2-\frac{(\mathbf{kv})^2}{k^2}\right)\right\}\frac{\partial}{\partial p_j}\Phi_\mathbf{p}\frac{\partial F_j(\mathbf{p})}{\partial p_i}. \quad (60)$$

The results obtained open up the possibility of creating ultrahigh-frequency amplifiers on completely different principles, using time-independent weakly anisotropic resonant waves of relatively low frequencies in open systems.

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