Slow modes in the spin hydrodynamics of ^{3}He-*B*

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The nonlinear interaction between acoustic and spin modes in ^{3}He-*B* [R. Combescot, J. Phys. C. **14**, 1619 (1981); R. Combescot and T. Dombre, Phys. Lett. **76A**, 293 (1980)] is studied with the aim of finding a way of detecting second sound, considered in the pumping regime. A special type of textured spin waves is used. These waves are long-wavelength, low-frequency modulations of the WP mode of magnetic ringing, and are associated with the propagating magnetic disturbance (PMD) reported by R. A. Webb, R. E. Sager, and J. C. Wheatley [J. Low Temp. Phys. **26**, 439 (1977)]. These waves, which have a magnon spectrum $\omega \propto q^2$, are oscillations of the unit vector **w** (the axis of the spin precession at frequency $\omega_{\rm WP}$), with $\omega \ll \omega_{\rm WP}$ [V. L. Golo, Sov. Phys. JETP **59**, 1221 (1984)]. It is shown that in a thin layer near the sound source the second sound generates a dephasing $\delta\psi$ of this precession. According to Golo (*op. cit.*) the dephasing is a rapidly damped mode that interacts nonlinearly with the oscillations of the vector **w**, which are weakly damped. Thus, pumping of second sound generates dephasing of the precession. According to Golo (*op. cit.*), the dephasing dies out in a thin layer near the sound source, while the oscillations of the vector **w** propagate beyond the layer. The latter oscillations are evidently the PMD excitations of Webb, Sager, and Wheatley (*op. cit.*), which are readily observed experimentally. This situation is also of interest in the study of surface phenomena in superfluid ^{3}He-*B*, since the very small penetration depth of second sound could make it a probe for elementary excitations in thin surface layers. The numerical estimates performed indicate that for temperatures $1 - T/T_c \cong 0.01$, pressure $p = 21.7$ bar, and second-sound frequency and power of the order of 100 Hz and 10^{-3} erg/sec, respectively, the dephasing $\delta\psi$ of the spin precession can reach 0.1 rad and lead to appreciable swinging of the precession axis. For these values, the second-sound penetration depth is of the order of 0.001 cm.

1. INTRODUCTION

The aim of this paper is to determine whether perceptible interaction may be expected between the acoustic and spin modes in superfluid ^{3}He-*B*. The calculations performed in this paper using the model of nonlinear spin hydrodynamics of Refs. 1 and 2 and with allowance for Leggett-Takagi dissipation show that for second sound in the pumping regime this interaction can be appreciable if we make use of the spin waves discovered by Webb, Sager, and Wheatley[3] (for a theoretical interpretation of these, see Ref. 4). This is an unusual type of spin waves, with no known analog for other magnets. Unlike ordinary spin waves, the direct experimental observation of which involves considerable difficulties, the waves found in Ref. 3 (which, after their discoverers, we shall call WSW waves) are clearly detected.

We recall that in Ref. 3 the WSW waves were generated as follows. The ^{3}He-*B* sample was placed in a rectangular cavity, and a constant external field H_B, somewhat smaller in magnitude than the dipole field, was applied in a small part of the sample; after the system had come to equilibrium, the field was switched off sharply, i.e., in a time much shorter than the relaxation time. At a certain distance from the coils that produced the field that was switched off were placed the coils of a detector, which registered the arriving magnetic excitation after a certain time interval. It was found in Ref. 3 that the velocity of propagation of the excitation is anomalously small—of the order of 10 cm/s, which is an order of magnitude smaller than the velocity of ordinary spin waves in superfluid ^{3}He (Refs. 5, 6). To detect the WSW waves the authors of Ref. 3 employed an original technique based on specific properties of ^{3}He-*B*. The propagation of an excitation from the coils producing the field that was switched off was detected from its effect on the so-called WP (wall-pinned) mode of magnetic ringing, which was destroyed by the action of the arriving excitation.

In the present work the WP mode plays a fundamental role. This mode of magnetic ringing was discovered by Webb, Sager and Wheatley,[3] a theoretical interpretation of it was given by Brinkman[7] in the dissipationless approximation, and its relaxation properties were explained by Leggett and Takagi.[8] The theory of the WP mode is very important for confirmation of the two-fluid model, proposed in Ref. 8, of the spin dynamics of superfluid ^{3}He. To generate the WP mode, a constant external magnetic field H_M, somewhat smaller in magnitude than the dipole field H_D, was applied throughout the volume of the rectangular sample of ^{3}He-*B*; the system was given time to reach equilibrium, after which the field was switched off sharply. As a result, a long-lived (with a lifetime of the order of 10 ms and longer), sharp, very clearly detectable signal arose. Its frequency coincided to within 6% with that given by the theoretical formula[7]

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$\omega_W = \sqrt{0.4}\gamma H_M$. It was shown by Brinkman and Cross[9] that in the regime of the WP mode the spin vector **S** rotates with frequency ω_W about a certain constant axis **w**, the position of which in space is determined by the initial configuration of the spin and order parameter. For the purposes of the present work it is very important that, in the presence of external perturbations, e.g., spatial gradients, the direction of the axis **w** can change.

In Ref. 3 the WP mode was used as a detector in the following way. Besides the field H_B, produced by the exciting coils in only a limited region of the sample, a constant field H_M was produced throughout the volume. The magnitudes of the fields H_B and H_M in Ref. 3 were of the order of 10 and 2 G. Like H_B, the field H_M was also switched off, but with a delay Δt. It is known that after the switching off of the uniform external field applied throughout the whole volume of the sample the system arrives at the regime of the WP mode after a time of the order of 1 ms. Therefore, if the delay time is sufficiently short, an excitation propagating after the switching off of the field H_B will not have time to reach the region in which the coils detecting the WP-mode signal are placed, and so the latter signal will be detected. If, however, the delay Δt is such that the perturbation has time to reach the detection coils, the possibility of generation of the WP mode will be lost. On the other hand, for a sufficiently long delay Δt the perturbation has time to move away from the region of the detection coils before the field H_M is switched off and generation of the WP mode begins. In this case too, the WP mode will be detected. This technique has turned out to be very sensitive and capable of recording magnetization perturbations that could not be detected by an ordinary quantum magnetometer.[3]

In Ref. 4 it was postulated that the WSW waves are associated with elementary excitations near the WP mode. The spectrum of elementary excitations near the long-lived magnetization-ringing mode (the so-called Brinkman-Smith mode[10]) was first studied by Fomin.[11] He considered waves in a fairly strong magnetic field (much stronger than the dipole field). These waves are characterized by the presence of a fixed direction (specified by the external field) of the spin-precession axis, whereas the WSW waves discussed in our work are associated with oscillations of this axis.

As shown in Ref. 4, WSW waves evidently correspond to slow modes of elementary excitations near the WP mode. There are two such modes: a propagating mode of oscillations of the vector **w**, with a magnon-type spectrum $\omega \propto q^2$, and a rapidly damped mode of the dephasing $\delta\psi$ of the spin precession about the axis **w**.[2)] Both modes arise when one takes account of spatial gradients in the configuration of the spin and order parameter.

In the present work WSW waves are investigated with allowance for the hydrodynamic corrections introduced by sound waves. It should be noted that in the framework of the general nonlinear theory of spin hydrodynamics[1] estimating physical effects is a complicated problem because of the large number of coefficients, the values of which are not accurately known. In this respect the study of WSW waves has definite advantages, since their low frequencies make it possible to average the spin-dynamics equations over the high frequencies, as a result of which most, but not all, of the nonlinear terms drop out. Finally, the spectrum of the system consists of the above-mentioned modes of oscillation of **w** and of the dephasing $\delta\psi$. Here the acoustic modes play the role of a source for the dephasing $\delta\psi$. The estimates performed show that one can expect a perceptible interaction of WSW waves and second sound. This circumstance makes it possible to use WSW waves to study second sound in ^{3}He-B. As is well known, direct observation of second sound is made difficult by the enormous damping, of the order of 10^5 cm^{-1} at frequency 1 kHz (Ref. 13). However, precisely this circumstance can be used for the generation of WSW waves, and, hence, for the indirect observation of second sound. At frequencies of the order of 100 Hz the penetration depth of second sound is, apparently, on the order of 0.001 cm. Therefore, on the one hand, it is possible to use a hydrodynamic description, while, on the other hand, the spatial gradients turn out to be sufficiently large for excitation of WSW waves by means of second sound.

2. INTERACTION OF SECOND SOUND AND WSW WAVES

The nonlinear spin-hydrodynamics model proposed in Refs. 1 and 2 is based on systematic computation of the orders of the conserved hydrodynamic quantities and on allowance for symmetry under time reversal, spatial reflection, rotations in the spin and orbital variables, and Galilean invariance. For details we refer the reader to Ref. 1, and give here only the final results needed for the following analysis.

In Ref. 1 it is assumed that the normal and superfluid velocities, the spin, and the momentum flux are first-order quantities, and that taking the gradient raises the order by unity. The density fluctuation $\delta\rho$ and entropy fluctuation $\delta\sigma$ are first-order quantities of smallness, and their gradients are of second order. The hydrodynamic expressions—in particular, the energy density, are expanded to third-order quantities, including the gradients of the order parameter.

The order parameter for ^{3}He-B has the form

$$A_{ij}=(\Delta/\sqrt{3})\, e^{i\varphi}R_{ij},$$

where R_{ij} is the 3-dimensional rotation matrix, parametrized by the axis **n** and rotation angle θ:

$$R_{ij}=\delta_{ij}\cos\theta+(1-\cos\theta)\, n_i n_j-\varepsilon_{ijk}n_k\sin\theta,$$

and Δ and φ are the modulus and phase of the order parameter. The superfluid velocity v_s^i and spin superfluid velocity $v_{\alpha i}^s$ are given by the formulas

$$v_s{}^i=\frac{\hbar}{2m}\,\partial_i\varphi,\qquad v_{\alpha i}{}^s=-\frac{\hbar}{4m}\,\varepsilon_{\alpha\beta\gamma}R_{\beta j}\partial_i R_{\gamma j},$$

where $\hbar$ is Planck's constant and is the mass of the ^{3}He atom.

The expression found in Ref. 1 for the energy density ε has the form

$$\varepsilon=\varepsilon_0(\rho,\sigma)+{}^1/_2\rho_n v_n{}^2+{}^1/_2\rho_s v_s{}^2+\gamma^2S^2/2\chi$$
$$+{}^1/_2\rho_{ij\alpha\beta}v_{\alpha i}{}^s v_{\beta j}{}^s-\gamma\rho_n v_s{}^i(\lambda^{e}_{ij\alpha\beta}S_\alpha v_{\beta j}{}^s$$
$$+\mu^e\varepsilon_{ijk}R_{\alpha k}\partial_j S_\alpha)+c_\rho\varepsilon_{ijk}R_{\alpha k}v_{\alpha j}{}^s\partial_j\rho+c_s\varepsilon_{ijk}R_{\alpha k}v_{\alpha i}{}^s\partial_j\sigma. \qquad (1)$$

Here v_n^i and v_s^i are the normal and superfluid velocities, ρ_n and ρ_s are the densities of the normal and superfluid components, S_a is the spin, and $\rho_{ija\beta}, \lambda^c_{ja\beta}, \mu^c, c_\rho$, and c_s are phenomenological parameters. It should be noted that the expression $(1/2)\rho_{ija\beta}v^s_{ai}v^s_{\beta j}$ is none other than the usual gradient energy that one introduces into the Leggett Hamiltonian when one wishes to take spatial variation of the order parameter into account.

The spin flux can be expressed in terms of ε and has the form

$$g_{\alpha i}=v_i{}^n S_\alpha+\varepsilon_{\alpha\beta\gamma}\left[\frac{\partial\varepsilon}{\partial(\partial_i S_\beta)}S_\gamma+\frac{\partial\varepsilon}{\partial(\partial_i R_{\beta j})}R_{\gamma j}\right]. \quad (2)$$

The equations of motion of the order parameter have the form

$$\partial_t\varphi=-\frac{2m}{\hbar}(\mu+v_n{}^i v_s{}^i), \quad \partial_t R_{\alpha i}=\varepsilon_{\alpha\beta\gamma}\omega_\beta R_{\gamma i},$$

$$\omega_\alpha=h_\alpha+\frac{2m}{\hbar}v_n{}^j v_{\alpha j}{}^s-\frac{1}{2}R_{\alpha i}(\operatorname{rot}\mathbf{v}_n)_i, \quad h_\alpha=\frac{\partial\varepsilon}{\partial S_\alpha}-\partial_i\frac{\partial\varepsilon}{\partial(\partial_i S_\alpha)}$$

(μ is the chemical potential). The spin-conservation equation has the form

$$\partial_t S_\alpha+\partial_i g_{\alpha i}=F_\alpha^{dip}, \quad (3)$$

where $F_\alpha^{\rm dip}$ is the source due to the dipole moment.

The equation of the theory of Ref. 1 are dissipationless. In the present paper, dissipation is taken into account in the minimal way—by means of the Leggett-Takagi mechanism[8] (see below). Since we are interested only in the qualitative features of the processes under consideration, we shall confine ourselves to the case when the quantities depend only on one spatial variable z.

It is convenient to go over to dimensionless variables

$$S_R=\Omega^{-1}\gamma^2\chi^{-1}S, \quad t_R=\Omega t, \quad z_R=L^{-1}z,$$

where γ is the gyromagnetic ratio, χ is the susceptibility, Ω is the Leggett frequency, and L is the spatial scale, for which we take the wavelength of the WSW waves, equal to 0.1 cm. At a pressure of 20.7 bar and near T_c the Leggett frequency is given by the formula[14]

$$\Omega=2\pi\sqrt{12}\cdot10^5(1-T/T_c)^{1/2},$$

in view of which we can take a scale of $\Omega = 5\cdot10^5$ rad/s, corresponding to dipole fields of 15–20 G. It should be noted that in the region of low temperatures the values of the dipole fields can be two to three times larger.

In this paper we investigate a system in the acoustic pumping regime, and therefore it is necessary to consider only the equations for the spin and order parameter, with the sound entering these equations in the form of an external field. The corresponding equations follow from the spin-conservation equation (3) and the above equations of motion for the order parameter. In the one-dimensional case and with allowance for Leggett-Takagi dissipation they have the form

$$\partial_t\theta=\omega_\alpha n_\alpha+{}^{16}/_{15}(\Gamma_\parallel/\Omega)\sin\theta(\cos\theta+{}^1/_4), \quad (3a)$$

$$\partial_t n_\alpha={}^1/_2\varepsilon_{\alpha\beta\gamma}\omega_\beta n_\gamma+{}^1/_2\operatorname{ctg}(\theta/2)n_\alpha-{}^1/_2\operatorname{ctg}(\theta/2)\omega_\beta n_\beta n_\alpha, \quad (3b)$$

$$\partial_t S_\alpha=-{}^1/_2 k\varepsilon_{\alpha\beta\gamma}(R_{\beta i}R_{\gamma i}')'-k\varepsilon_{\alpha\beta\gamma}(R_{\beta 3}R_{\gamma 3}')'$$
$$-\left[\left(v_n-\frac{\gamma\hbar}{2m}\rho_n\lambda^c v_s\right)S_\alpha\right]'+{}^{16}/_{15}\sin\theta(\cos\theta+{}^1/_4)n_\alpha. \quad (3c)$$

Here $\Gamma_\parallel$ is the longitudinal-NMR linewidth, specifying a single relaxation term, the external field is assumed to be absent, and the prime denotes differentiation with respect to the spatial coordinate z. The coefficient k multiplying the gradient terms can be calculated in the weak-coupling approximation:

$$k=[c_\parallel/L\Omega]^2,$$

where $c_\parallel$ is the velocity of a longitudinal spin wave.[5] In the one-dimensional case under consideration,

$$\omega_\alpha=S_\alpha+(2m/\hbar)\Omega^{-1}v_n{}^j v_{\alpha j}{}^s.$$

All the dynamical variables are understood to be dimensionless.

In the equation for the spin the first two terms are the same as in the Laggett-Takagi theory with allowance for the spatial gradients of the order parameter, and the third term is due to the motion of the superfluid and normal components. The last term corresponds to the dipole moment. In the usual way, the Leggett-Takagi dissipation appears only in one equation—namely, that for the angle θ.

For the purpose of the present work it is very important that the regime of the WP mode, as is well known,[9] is characterized by the vector introduced by Brinkman and Cross:

$$\mathbf{J}=\sin^2(\theta/2)\{\operatorname{ctg}(\theta/2)[\mathbf{Sn}]-\mathbf{S}+(\mathbf{Sn})\mathbf{n}\}. \quad (4)$$

The vector J is an integral of motion of the dissipationless Leggett equations. The regime of the WP mode with neglect of dissipation is characterized by the following constraints:

1) The vector **n** of the axis of the order parameter remains perpendicular to the vectors **S** and **J**;

2) the angle θ does not change, and the angle between **S** and **J** is equal to $\frac{1}{2}(\theta+\pi)$;

3) the vectors **S**, **J**, and **n** are connected by the relation

$$\mathbf{S}=-\mathbf{J}-\operatorname{ctg}(\theta/2)[\mathbf{Sn}]; \quad (5)$$

4) the modulus of the vector **J** and the angle θ are connected by the relation

$$\mathbf{J}^2+{}^{64}/_{15}\sin^4(\theta/2)(\cos\theta+{}^1/_4)=0; \quad (6)$$

5) the vectors **S** and **n** rotate about **J** with a constant angular velocity equal to

$$\boldsymbol{\omega}=-{}^1/_2\sin^{-2}(\theta/2)\mathbf{J}. \quad (7)$$

It follows from the above formulas that in the regime of the WP mode the vector **n** of the axis of the order parameter can be represented in the form

$$\mathbf{n}=\mathbf{u}\cos\psi+\mathbf{v}\sin\psi, \quad (8)$$

where the unit vectors **u** and **v**, together with **w**, form an orthonormal basis. The vector **w** mentioned in the Introduction is the normalized vector **J**: $\mathbf{w}=\mathbf{J}/J$.

From the above equations for the spin and order parameter follows the equation for the vector **J**:

$$\partial_t J_\alpha={}^1/_2(\delta_{\alpha\beta}-R_{\beta\alpha})g_\beta'+{}^1/_2\varepsilon_{\beta\xi\eta}\omega_\xi R_{\eta\alpha}S_\beta$$
$$+{}^8/_{15}(\Gamma_\parallel/\Omega)\sin\theta(\cos\theta+{}^1/_4)\{\cos\theta[\mathbf{Sn}]-\mathbf{S}\sin\theta\}_\alpha. \quad (9)$$

Here we have used the fact that the vector J can be represented in the form

$$\mathbf{J}=-{}^1/_2\mathbf{S}+{}^1/_2\hat{R}^{-1}(\mathbf{n},\theta)\mathbf{S}.$$

The term due to the dipole moment is absent here, since the vector **J** is an integral of the dissipationless Leggett equations and commutes with the dipole energy for suitably chosen Poisson brackets.[15]

Thus, the vector **J** can change in time as a consequence of the following factors: 1) Leggett-Takagi dissipation; 2) the presence of spatial graidents; 3) interaction with $\mathbf{v}_s$ and $\mathbf{v}_n$. Dissipation does not take the system out of the regime of the WP mode, but the last two factors generate excitations of the system near the WP mode. In the limit of low frequencies and long wavelengths these excitations are waves of spatial modulation of the WP mode such that, over sufficiently small length scales, the configuration of the spin and order parameter can be assumed to be spatially uniform, while in the large, on the scale of the whole sample, there is no spatial uniformity.

In order to give these arguments a quantitative form, we shall average Eq. (9) over the basis solution specified by Eqs. (5) and (8). The averaging procedure itself is the same as that in Ref. 4, where only the gradients of the order parameter but not $\mathbf{v}_s$ and $\mathbf{v}_n$ were taken into account; the essentially new point is that in the averaging process we take account of the ordering of the quantities. Besides the rule given in Ref. 1, we must keep in mind that the vector **J**, and θ, and phase ψ can be represented in the form

$$\mathbf{J}=\mathbf{J}_{SH}+\delta\mathbf{J},\quad \theta=\theta_{SH}+\delta\theta,\quad \psi=\psi_{SH}+\delta\psi, \tag{10}$$

where $\mathbf{J}_{SH}$, θ_{SH}, and ψ_{SH} are spatially uniform quantities that vary in time only as a result of the Leggett-Takagi dissipation, and $\delta\mathbf{J}$, $\delta\theta$, and $\delta\psi$ are their spatially nonuniform fluctuations. In accordance with Ref. 1 the fluctuations are one order smaller than the spatially uniform quantities. This condition, of course, is an extremely strong requirement, but, evidently, near the regime of the WP mode it is reasonable to adopt it. The justification for this is the very existence of the sharp WP-mode signal, which indicates that the fluctuations are not too large. Since J has the same order as the spin, $\mathbf{J}^2$ is of order two. In this case, it follows from the relation (6) that $\theta_{SH}-\theta_0$ (where $\theta_0=\arccos(-1/4)$ is the equilibrium value of the angle θ) is a second-order quantity, and, consequently, $\delta\theta$ is a third-order quantity.

It follows from this that the gradients $\partial\delta\theta$ are at least fourth-order quantities and are therefore eliminated from all the equations, which, throughout, are considered with an accuracy up to and including third-order quantities. Another very important point is that the gradients of the vector **w** are quantities of at least second order, since they are gradients of the fluctuations $\delta\mathbf{w}$, and the latter are first-order quantities; because of this, in the averaging process a large number of terms turn out to be of fourth order and should be discarded.

As a result of the averaging we obtain the following three equations (specifying the basis solution—for details, see Ref. 4) for the modulus of the vector **J** and the projections of the vector **J** onto the unit vectors **u** and **v**:

$$J\partial_t w_u=A_{uu}w_u''+A_{uv}w_v''+{}^5/_{32}kw_3u_3\delta\psi'', \tag{11}$$

$$J\partial_t w_v=A_{vu}w_u''+A_{vv}w_v''+{}^5/_{32}kw_3v_3\delta\psi'', \tag{12}$$

$$\partial_t J=-B\delta\psi''-\frac{5}{32}kw_3(v_3w_u''-u_3w_v'')-\frac{6}{25}\frac{\Gamma_\parallel}{\Omega}J^3-\left(v_n-\frac{\gamma\hbar}{2m}\lambda^c\rho_n v_s\right)'J_{SH}, \tag{13}$$

where

$$A_{uv}=k\left[\frac{5}{8}+\frac{515}{1024}v_3^2+\frac{365}{1024}u_3^2+\frac{25}{64}w_3^2+\frac{25}{1024}(v_3^2-u_3^2)\right], \tag{14a}$$

$$A_{vu}=-k\left[\frac{5}{8}+\frac{515}{1024}u_3^2+\frac{365}{1024}v_3^2+\frac{25}{64}w_3^2+\frac{25}{1024}(u_3^2-v_3^2)\right], \tag{14b}$$

$$A_{uu}=-A_{vv}=\frac{75}{512}ku_3v_3,$$

$$B=k\left[\frac{5}{4}+\frac{65}{64}(u_3^2+v_3^2)+\frac{15}{32}w_3^2\right]. \tag{15}$$

The dissipative term that we have taken into account is formally fourth-order, as we should expect, since the dissipative part of the hydrodynamic equations is of higher order than the reactive part. However, the coefficient of this dissipative term turns out to be much larger that the coefficients of the reactive terms, even though the latter are of third order. We note also that all the coefficients in Eq. (11)–(13) are calculated for $\theta=\arccos(-1/4)$.

Equation (13) can be transformed to a more convenient form. For this we note first of all that it follows from Eqs. (3b) that

$$[\mathbf{n}\partial_t\mathbf{n}]=-{}^1/_2\sin^{-2}(\theta/2)\mathbf{J}-\mathbf{X}(\mathbf{v}_n,\ \mathbf{v}_s), \tag{16}$$

where the coordinates of the vector $\mathbf{X}(\mathbf{v}_n, \mathbf{v}_s)$ are given by the relation

$$X_\alpha=\frac{m}{\hbar}\Omega^{-1}v_n{}^j\left[\varepsilon_{\alpha\beta\gamma}v_{\beta j}{}^s n_\gamma\operatorname{ctg}\frac{\theta}{2}-v_{\alpha j}{}^s+v_{\beta j}{}^s n_\beta n_\alpha\right].$$

From the equations for the basis solution (5) and (8) follows the equality

$$\mathbf{w}[\mathbf{n}\partial_t\mathbf{n}]=\partial_t\psi+\mathbf{v}\partial_t\mathbf{u}. \tag{17}$$

From Eqs. (16) and (17) it follows that

$$\partial_t\psi=-{}^1/_2\sin^{-2}(\theta/2)J-\mathbf{v}\partial_t\mathbf{u}-\mathbf{w}\mathbf{X}(\mathbf{v}_n,\ \mathbf{v}_s). \tag{18}$$

From Eq. (18) we obtain an expression for the averaged value $\langle J\rangle$:

$$\langle J\rangle=-2\sin^2(\theta/2)\partial_t\psi-\langle\mathbf{v}\partial_t\mathbf{u}\rangle-\langle\mathbf{w}\mathbf{X}(\mathbf{v}_n,\mathbf{v}_s)\rangle.$$

We note that in the case of a spatially uniform configuration and with neglect of v_n^i and v_s^i we have the relation

$$J_{SH}=-2\sin^2(\theta_{SH}/2)\partial_t\psi_{SH},$$

from which, in particular, it follows that $\partial_t\psi_{SH}$ is of first order. Gathering together all these facts, we obtain the following equation for δJ and $\delta\psi$, in which the last two terms are of the third order:

$$\delta J=-2\sin^2(\theta/2)\partial_t\delta\psi-2\sin^2(\theta_0/2)\langle \mathbf{v}\partial_t\mathbf{u}\rangle -2\sin^2(\theta_0/2)\langle \mathbf{w}\mathbf{X}(\mathbf{v}_n,\mathbf{v}_s)\rangle.$$

It follows from the latter equality that, to within small terms of third order inclusive, we have the relations

$$J_{SH}\delta J=-2\sin^2(\theta_0/2)\partial_t\delta\psi,\quad \partial_t\delta J=-2\sin^2(\theta_0/2)\partial_t^2\delta\psi,$$

using which we can rewrite Eq. (13) in the form of an equation for $\delta\psi$ alone:

$$\partial_t^2\delta\psi={}^1/_{16}(29-7w_3^2)k\delta\psi''+{}^1/_8 kw_3(v_3w_u''-u_3w_v'') -{}^{18}/_{25}(\Gamma_\parallel/\Omega)J_{SH}^2\partial_t\delta\psi-{}^4/_5 J_{SH}(v_n-(\gamma\hbar/2m)\lambda^c\rho_n v_s)'. \quad (19)$$

The change of J_{SH} in time and the order of coefficients in Eqs. (11), (12), and (19) can be estimated using the data of Ref. 3. It is known that Leggett–Takagi dissipation leads to a change of the frequency of the WP mode in time, in accordance with the law[3] $f^{-2}=f_0^{-2}+\alpha t$. According to Ref. 3, near T_c and at a pressure of 20.7 bar the coefficient α is a quantity of order 10^{-5} s. An analogous law

$$J_{SH}^{-2}=J_0^{-2}+\beta t,$$

governs the variation of J_{SH}, with β given by the formula[4]

$$\beta={}^1/_2(\Gamma_\parallel/\Omega)\cos^2(\theta/2)\sin^{-4}(\theta/2), \quad (20)$$

where J_{SH} and t are understood to be dimensionless. For application to the data of Ref. 3, $J_{SH}=0.3$ and $\beta\cong 0.2$ (in dimensional variables, $\beta\cong 10^{-4}$ s), i.e.,

$$J_{SH}\approx J_0(1+0.02t)^{-1/2}.$$

The characteristic time of the WSW wave is estimated from its frequency, which is given by the formula[4]

$$\nu=\varkappa(2\pi/\gamma H)(c_\parallel/\lambda)^2, \quad (21)$$

where $\varkappa$ is a dimensionless quantity of order unity, H is the external field that is to be switched off, $c_\parallel$ is the velocity of the longitudinal spin wave,[5] and λ is the wavelength of the WSW wave. In the following, in accordance with Ref. 3, we shall assume that $1-T/T_c=0.02$, the pressure $p=20.7$ bar, and $H=5$ G. From the data of Ref. 3, it follows that $\Gamma_\parallel/\Omega=0.3$. In accordance with the microscopic theory of Ref. 5, taking the temperature and pressure into account we take the value of $c_\parallel$ to be 200 cm/s. In the region of lower temperatures the value of $c_\parallel$ grows (see Ref. 6). We take the scale L to be equal to 0.1 cm, which corresponds to the characteristic wavelengths of the WSW waves under the conditions of Ref. 3.

Under these conditions we have, according to formula (21), a frequency value $\nu\cong 250$ Hz; i.e., the characteristic time for the WSW waves is of the order of a few milliseconds. It follows from formula (20) that the change of J_{SH} in this time will be insignificant (twofold or threefold in all), and, taking into account that we are interested only in orders of magnitude, we can neglect it.

The dimensionless coefficient of the gradient terms in Eq. (19) turns out to be of order 10^{-5}, and that of the dissipative term is found to be of order 10^{-2}.

Before estimating the last term in Eq. (19), we shall convince ourselves that it makes sense to consider only the pumping of second sound. Indeed, in order that the corresponding term in Eq. (19) not be negligible, the gradient contained in it should be rather large, i.e., the wavelength should be not greater than 0.1 cm; but this corresponds to frequencies of the order of 10^5 Hz (the velocity of first sound[16] in ^{3}He is of the order of 440 m/s), whereas the averaged equations that we have obtained are valid up to frequencies of the order of 10^3 Hz inclusive. The remaining possibility is that this is second sound, the velocity of propagation of which, according to theoretical estimates based on the two-fluid model of an isotropic superfluid liquid, is of the order of 1 cm/s (Ref. 13). Since we are interested only in order-of-magnitude estimates, we shall not elucidate the relative roles of the terms with v_n and v_s in Eq. (19), especially since the term with v_s is evidently small near T_c (Ref. 17).

Taking into account the small penetration depth, we take the field of the sound wave in the form

$$v\propto Ae^{-\alpha z}\cos qz\cos\omega t$$

and consider pumping at a frequency of 100 Hz, with amplitude $A=0.1$ cm/s. According to Ref. 13 the penetration depth will be of the order of 0.001 cm. For these quantities the order of the last term in (19), which corresponds to the pumping, is found to be 10^{-4}. It should be noted that the term corresponding to pumping in Eq. (19) is nonzero only in the thin layer penetrated by the second sound.

The estimates given show that the term corresponding to Leggett-Takagi dissipation should be kept, since it is two to three orders of magnitude greater than the reactive terms.

The equations (11), (12), and (19) indicate that the process of generation of WSW waves by pumping of second sound occurs as follows. In a thin layer near the sound source the pumping leads to the appearance of a dephasing $\delta\psi$ which, in turn, induces oscillations of the vector $\mathbf{w}$. The latter propagate beyond the layer penetrated by the second sound, while the dephasing mode dies out on account of the Leggett-Takagi dissipation.

Equation (19) can be simplified substantially. In fact, the change $\delta\psi$ is induced primarily by the pumping, which, in comparison with the dissipation represented by the term with the first derivative with respect to the time, is small. Therefore, we can assume that the second derivative $\partial_t^2\delta\psi$ will be of second order with respect to the pumping, i.e., will be extremely small and can be neglected. As a result, we obtain the following equation for the dephasing:

$${}^{18}/_{25}(\Gamma_\parallel/\Omega)J_{SH}^2\partial_t\delta\psi={}^1/_{16}(29-7w_3^2)k\delta\psi'' +{}^1/_8 kw_3(v_3w_u''-u_3w_v'')-{}^4/_5 J_{SH}[v_n-(\gamma\hbar/2m)\lambda^c\rho_n v_s]'. \quad (22)$$

In the limiting case $\mathbf{w}\parallel\mathbf{z}$ we have $u_3=v_3=0$ and $w_3=1$, Eq. (22) reduces to an equation of one-dimensional diffusion with a source:

$${}^{18}/_{25}(\Gamma_\parallel/\Omega)J_{SH}^2\partial_t\delta\psi={}^{11}/_8 k\delta\psi''-{}^4/_5 J_{SH}[v_n-(\gamma\hbar/2m)\lambda^c\rho_n v_s]'. \quad (23)$$

From the estimate given above for the coefficients of Eq. (22) it follows that in a time of the order of a few milliseconds (the characteristic time for a WSW wave with wavelength of the order of 0.1 cm) the dephasing produced by pumping of second sound becomes of the order of unity. This result is indicated by Eq. (23). In its turn, the presence of the dephasing $\delta\psi$, concentrated in a thin surface layer and, consequently, having large gradients, entails, according to Eqs. (11) and (12), pumping along the direction of the vector **w**.

Thus, pumping of second sound will lead to the excitation of WSW waves.

3. CONCLUSION

The estimates given in this paper appear to show that, under favorable conditions, the interaction between second sound and WSW waves can be sufficiently appreciable to make it possible to detect second sound using the technique of Ref. 3. The important point here is the use of a special type of spin waves, corresponding to elementary excitations near a certain long-lived magnetic-ringing mode. This is due to the fact that for interaction with second sound one needs slow low-frequency ($\sim$10–100 Hz) spin-dynamics modes whose propagation can be clearly detected in experiment. Precisely these properties are possessed by the WSW waves[3] associated with the WP mode. To detect second sound one could also use other long-lived modes and the associated low-frequency elementary excitations, which can differ substantially from the WSW waves in the configuration of the spin and order parameter and in the presence of applied external fields.

Apart from the well known Brinkman-Smith modes[10] and the WP mode (see Ref. 18, in which a method is given for describing such modes by means of a "three-dimensional window" into the phase space of the Leggett-Takagi equations), in the spin dynamics of ^{3}He-B there are many other types of long-lived modes. It is useful to note that the majority of these modes cannot be obtained by simply switching off an external magnetic field (either completely or to a certain residual value) or by the action of a rotating radio-frequency pulse (see Ref. 18). For example, it is known[10,18] that in the presence of an external field H the Leggett-Takagi equations have a stable stationary solution, i.e., a long-lived mode, which goes over into the WP mode as $H\to 0$. However, the numerical analysis performed in Ref. 18 shows that when an initial external field H_i is switched off sharply, leaving a residual field H parallel to H_i, an entirely different mode is generated, which, however, also goes over into a WP mode $H\to 0$. This circumstance can be important when one is choosing a long-lived spin-dynamics mode with the aim of investigating second sound, since a mode that is suitable in all respects may be difficult to realize.

If we take into account the small penetration depth of second sound, it is desirable to have a long-lived magnetic-ringing mode such that the associated waves of elementary excitations have as small a wavelength as possible at the given frequency. The character of this frequency dependence can be estimated in the following, most general way. The long-lived mode is characterized by the magnitude of the magnetization associated with it, to which corresponds a certain field H. For example, for the WP mode this is the external field that is to be switched off, and for the Brinkman-Smith mode[10] it is the constant external field. The elementary excitations with wavelength λ are modulations of the long-lived mode, and are determined by the coefficients of the gradient energy (see Sec. 2). The size of these coefficients corresponds to the velocity of ordinary spin waves in superfluid ^{3}He (Ref. 5) and can be comparable to the velocity $c_\parallel$ of a longitudinal spin wave.[5] We note that the velocities of the longitudinal and transverse spin waves are of the same order.[5] Thus, we have three fundamental dimensional quantities: λ, H, and $c_\parallel$. From these we can construct two dispersion laws: 1) a linear dispersion law $\nu\sim c_\parallel/\lambda$ which corresponds to longitudinal oscillations of the spin and is damped in accordance with the Leggett-Takagi mechanism (in the case of WSW waves, this also holds for the dephasing ($\delta\psi$) mode); 2) a quadratic dispersion law (for the WSW waves this is the mode of oscillations of the vector of the spin-precession axis; see Eq. (21))

$$\nu[\text{Гц}] \sim (2\pi/\gamma H)\,(c_\parallel/\lambda)^2.$$

The method developed in this paper makes it possible to study the interaction of second sound with the spin waves associated with an arbitrary long-lived magnetic-ringing mode. The choice of the WP mode and WSW waves is predicated on the fact that they can be detected beautifully in experiment and admit a simple theoretical treatment that makes it possible to exhibit all the important aspects of the phenomenon:

1) Pumping of second sound creates, in a thin layer near the loudspeaker, a gradient of the dynamical variable corresponding to a longitudinal (with respect to the spin) slow mode (in the present work, this mode is the dephasing $\delta\psi$);

2) Leggett-Takagi dissipation leads to the result that the longitudinal slow spin mode is damped, and its damping can be comparable to the damping of second sound in the considered range of frequencies ($\sim$10–100 Hz);

3) the interaction of the longitudinal slow mode and a transverse mode (in the present work the latter mode consists in oscillations of the vector **w** of the precession axis) inside a thin layer, in which the longitudinal mode does not have time to become damped, excites a wave of the a transverse mode that is propagating and can be detected far from the source of the second sound.

In accordance with the quadratic dispersion law indicated above, we expect that for wavelengths $\lambda\sim 0.01$ cm and fields $H\sim 100$ G there can be a resonance between the second sound and the transverse component of the slow mode of elementary excitations, and that the presence of this resonance should make it possible to ascertain the penetration depth of the second sound from the velocity of propagation of a wave of the slow mode. The damping longitudinal mode serves as an intermediary mode effecting an interaction between the second sound and the transverse propagating mode.

In the present work we have not considered high-frequency (greater than 1 kHz) second sound. It would be very interesting to ascertain whether it is possible to obtain something similar to parametric excitation of spin waves on account of surface effects. In the pumping regime the latter can be extremely large in view of the very small penetration depth. If in this case there is some kind of interaction with the spin–order-parameter variables, in the same way as occurs in the present work, the excitation of spin waves is possible. Parametric excitation of spin waves by an external magnetic field is studied in Ref. 19.

The interaction of spin modes and acoustic modes for phases other than ^{3}He-B has already been studied earlier. The presence of coupling of second sound with a spin wave in the A1 phase was studied theoretically in Refs. 20 and 21 and experimentally in Ref. 13. However, the character of the phenomenon there is completely different from that considered in the present work. The important point is that in the A1 phase we have a situation in which the second sound interacts directly with spin waves, and not with slow modes such as those considered in the present paper.

Fishman and Sauls[22] have shown that zero sound in ^{3}He in the normal phase in the presence of a magnetic field can excite oscillations of the longitudinal magnetization superposed on the Larmor precession. This resembles the generation (considered in the present paper) of the dephasing $\delta\psi$ by means of second sound.

It is worthy of note that in superfluid ^{3}He-B there are further possibilities for the study of second sound using the second-sound-induced emission of WSW waves. It is evident that in a more general context second sound can serve as a probe for the investigation of surface effects in superfluid ^{3}He; e.g., for the recently widely debated problem of the role of surface currents in the formation of the angular momentum in ^{3}He-B (Ref. 23).

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[2)]We note that an analogous situation obtains in nematic liquid crystals in a state with rapid rotation of the director.[12]

Note added in proof (6 February 1986): When our article was already in press, the paper by S. T. Lu and H. Kojima [Phys. Rev. Lett. **55**, 1677 (1985)] appeared, in which second sound was detected in ^{3}He-B.

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