Activated and tunneling transitions between the two forced-oscillation regimes of an anharmonic oscillator

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(Submitted 18 November 1985)
Zh. Eksp. Teor. Fiz. **90**, 1430–1439 (April 1986)

We investigate transitions between two-steady-state regimes of forced oscillation (one large-amplitude and one small-amplitude) of an anharmonic oscillator, due to thermal activation and quantum tunneling. We calculate the transition times for those cases where these times are exponentially large. In such cases, the exponents are of the same order of magnitude as the ratio of the oscillation energy to the thermal energy (for activated transitions) or to the quantum energy (for tunneling transitions). The transition probabilities increase sharply near the shutoff point of the small-amplitude oscillation.

INTRODUCTION

In the presence of anharmonicity, an oscillator can have in a certain frequency region forced oscillations with two distinct amplitudes.[1] Such an oscillator is an example of a dynamically bistable system subject to hysteresis.

Any such system is in a certain sense analogous to a particle moving in a potential with two unequal-depth wells separated by a barrier. One of the two stable stationary states of the particle is metastable: after a long enough time the particle always finds itself in the deeper well. This transition can result either from thermal activation or from tunneling through the barrier.

In view of this analogy it is obvious that one of the two regimes of forced oscillation for an anharmonic oscillator is metastable, and, strictly speaking, there is no hysteresis. The transition to the stable regime can be either by activation or by tunneling.

It is of fundamental interest to calculate the time for such a transition. This question can in addition have some practical value when one is discussing resonance in nonlinear systems of atomic scale,[2–4] for which the transition time may turn out to be rather short.

Transitions mediated by thermal fluctuations were investigated by Dykman and Krivoglaz[5] by functional-integral methods. They showed that depending on the parameters of the problem, either the small-amplitude or large-amplitude regimes could be metastable.

The quantum-mechanical investigation of the forced oscillations of an anharmonic oscillator has been the subject of numerous papers (see, e.g., Refs. 6–11 and the citations therein). The possibility of tunneling transitions was first mentioned by Sazonov and Finkel'shteĭn[7]; however, the corresponding transition time calculated by these authors turns out to be incorrect.

Sazonov[9] developed an original quasiclassical method of calculating the tunneling time. However, his demarcation of limiting cases and actual calculations were not correct, except for the case of a rather small driving force. This result agrees essentially with the expression obtained earlier by Larsen and Bloembergen[2] for the time of coherent multiphoton transitions between vibrational levels of a molecule.

In the present work we find the time for transitions between two forced-oscillation regimes of an anharmonic oscillator, for both the activation and tunneling mechanisms. We will investigate only the cases when this time is exponentially large, and limit ouselves to calculating only the exponent. We assume that the characteristic relaxation time for dissipative processes is long enough for these processes to leave the steady-state oscillation amplitude practically unchanged. At the same time, this characteristic time will be assumed small compared to the transition times of interest to us.

Our investigation of both activated and tunneling transitions is based on the use of an effective Hamiltonian which does not contain time explicitly; this is introduced in Section 2. Section 3 is devoted to activated transitions, in which the results of Dykman and Krivoglaz[5] (which pertain to the case of low friction) are rederived by the transport-equation method. In addition, we obtain a stationary distribution function for the oscillator and an explicit expression for the transition time close to the cutoff point for the small-amplitude oscillation. The time for tunneling transitions is calculated in Section 4. These transitions are dominant, except in special cases when the condition $T \ll \hbar\omega_0$ is satisfied, where T is the temperature and ω_0 is the oscillator small-amplitude frequency. We show that, unlike activated transitions, tunneling always leads to metastability of the small-amplitude oscillations. This circumstance destroys the analogy with the example, investigated earlier, of a particle in a two-well potential, in which a state in the shallow well is metastable relative to transitions of both types.

2. EFFECTIVE HAMILTONIAN

We write down the Hamiltonian of an anharmonic oscillator in the presence of a periodic external force in the form:

$$H = \frac{P^2}{2m} + \frac{m\omega_0^2 X^2}{2} + b\frac{X^4}{4} + FX\cos\omega t. \qquad (1)$$

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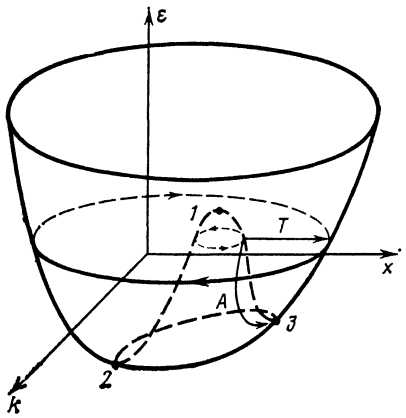

FIG. 1. Plot of $\varepsilon(k,x)$ for $\alpha<\alpha_c$. Points 1 and 2 correspond to stable oscillations with small and large amplitudes respectively, and point 3 is unstable. The internal and external trajectories are shown. The arrows mark activated (A) and tunneling (B) transitions.

We assume that the anharmonicity is small and that the frequency ω of the driving force is close to the resonant frequency ω_0, with $\omega>\omega_0$ (it is just in this case, that two oscillation regimes are possible at $b>0$).

If we make in the Hamilton equations which follow from (1) the substitution

$$X=(m\omega_0)^{-1/2}(q\cos\omega t+p\sin\omega t),$$
$$P=(m\omega_0)^{1/2}(-q\sin\omega t+p\cos\omega t), \tag{2}$$

we obtain

$$\dot q=-\delta p+\beta p(p^2+q^2), \tag{3}$$

$$\dot p=\delta q-\beta q(p^2+q^2)-f, \tag{4}$$

where $\delta=\omega-\omega_0\ll\omega_0$, $\beta=(3b/8)(m\omega_0)^{-2}$, $f=(F/2)(m\omega_0)^{-1/2}$.

In Eqs. (3) and (4) we neglect as usual nonresonant terms which oscillate at frequencies 2ω and 4ω.

Let us recall the well-known results of solving the problem in the classical-mechanics framework. If we set $p=0$ in (4), we obtain a cubic equation[1] for the steady-state amplitude of oscillation q:

$$x^3-x+\alpha=0, \tag{5}$$

where we have introduced a new variable $x=q(\beta/\delta)^{1/2}$. This equation contains only one parameter

$$\alpha=f\beta^{1/2}/\delta^{3/2}. \tag{6}$$

At $\alpha<\alpha_c=2\cdot3^{-3/2}$ we have three solutions to (6), of which one is unstable; at $\alpha>\alpha_c$ there is only one solution. The critical value $\alpha=\alpha_c$ thus corresponds to the small-oscillation shutoff point.

We have disregarded friction in Eqs. (3) and (4). Adding a frictional force $-m\gamma\dot X$ to the original equations of motion adds the terms $-\gamma q/2$ and $-\gamma p/2$ to the right-hand sides of Eqs. (3) and (4). These additions are responsible for the steadying of the oscillations. We will further assume that $\gamma\ll\delta$, $\alpha\gg\gamma/\delta$. For these conditions the steady-state amplitude of oscillation is as previously given by Eq. (5).

For further investigations, it is convenient to go from the Hamiltonian (1) to a new Hamiltonian H' with the help of the canonical transformation (2). Thus,

$$H'=H+\partial G/\partial t, \tag{7}$$

where the generating function G takes in our case the form

$$G(X,\ q,\ t)=[(q^2+m\omega_0X^2)\cos\omega t-2(m\omega_0)^{1/2}qX]/2\sin\omega t. \tag{8}$$

In the same way as in the derivation of Eqs. (3) and (4), in order to go from H to H' it is necessary to discard nonresonant terms which oscillate at the frequencies 2ω and 4ω. We then obtain for H' the expression

$$H'=-\frac{\delta}{2}(p^2+q^2)+\frac{\beta}{4}(p^2+q^2)^2+fq. \tag{9}$$

Hamilton's equations derived from (9) coincide with Eqs. (3) and (4). The effective Hamiltonian (9), which does not contain the time explicitly, was used earlier in Refs. 9 and 12.

Let us introduce the new variables

$$x=q(\beta/\delta)^{1/2},\quad k=p(\beta/\delta)^{1/2},\quad \varepsilon=H'\beta/\delta^2. \tag{10}$$

Then we will have in place of (9)

$$\varepsilon(k,\ x)=-\tfrac12(k^2+x^2)+\tfrac14(k^2+x^2)^2+\alpha x. \tag{11}$$

In Fig. 1 we show the function $\varepsilon(k,x)$ for $\alpha<\alpha_c$. The extrema of this surface correspond to steady-state oscillations at $k=0$ in the presence of small friction, and to the values of x determined by (5). Points 1 and 2 correspond to stable oscillations with small and large amplitudes, respectively, while point 3 corresponds to unstable ones. At $\alpha=\alpha_c$ points 1 and 3 coalesce, and for $\alpha>\alpha_c$ there remains one extremum, point 2.

Obviously, the phase trajectories are intersections of the surface $\varepsilon(k,x)$ with the plane $\varepsilon=$ constant. From Fig. 1 it is seen that in the energy interval $\varepsilon_3<\varepsilon<\varepsilon_1$ there are two types of trajectory between points 1 and 3, internal and external, separated by a classically inaccessible region of phase space (Fig. 2a; see also Ref. 8). The phase trajectories with account taken of a small amount of friction are shown in Fig. 2b. If we do not take into account thermal fluctuations or quantum tunneling, point 1 as well as point 2 correspond to a stable stationary state.

3. TRANSITIONS INDUCED BY THERMAL FLUCTUATIONS

Let us place an anharmonic oscillator driven by an external force in a heat bath of temperature T. The kinetic equation for the oscillator distribution function ρ takes the form

$$\frac{\partial\rho}{\partial t}+\{H,\rho\}-\gamma S(\rho)=0, \tag{12}$$

where (H,ρ) denotes Poisson brackets, and the Hamiltonian H is given by formula (1). We write the collision integral $S(\rho)$ in a differential form

$$S(\rho)=\frac{\partial}{\partial P}\left(P\rho+mT\frac{\partial\rho}{\partial P}\right), \tag{13}$$

which is valid under the assumption that the relative change

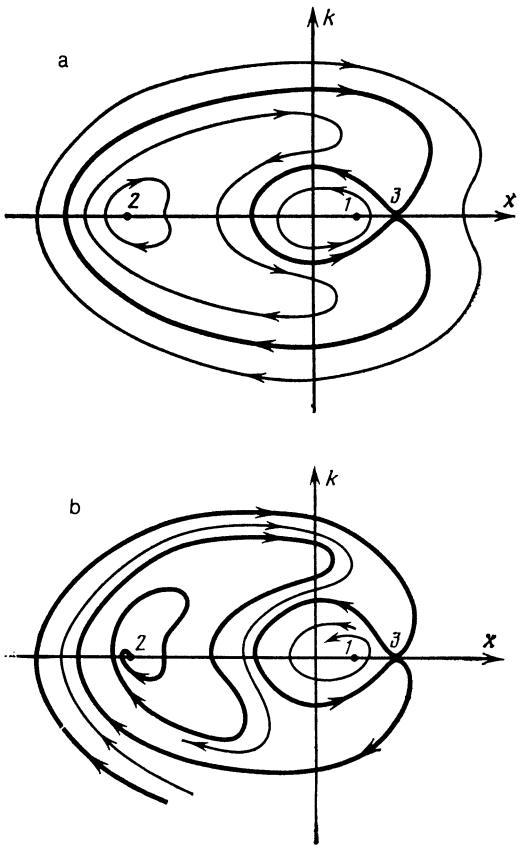

FIG. 2. Phase trajectories determined by Eq. (11): a) in the absence of friction, b) when friction is present. The thick curve is the separatrix. The internal trajectories are located near point 1.

in momentum in an elementary interaction with the heat bath is small. This assumption correpsonds to describing dissipation in the classical equations of motion by introducing a frictional force—$\gamma m\dot{X}$.

Let us transform from the variables P, X to the dimensionless variables k, x, by using the canonical transformation (2) and Eqs. (10). Then, discarding as before the nonresonant oscillating terms, we obtain

$$\frac{\partial \rho}{\partial t}+\delta\{\varepsilon,\rho\}-\gamma\hat{S}(\rho)=0, \tag{14}$$

$$\hat{S}(\rho)=\frac{1}{2}\frac{\partial}{\partial k}\left(k\rho+\frac{T}{\omega_0}\frac{\beta}{\delta}\frac{\partial\rho}{\partial k}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(x\rho+\frac{T}{\omega_0}\frac{\beta}{\delta}\frac{\partial\rho}{\partial x}\right), \tag{15}$$

where $\varepsilon(k, x)$ is given by formula (11).

Let us find a steady-state solution to Eq. (14), bearing in mind that $\gamma \ll \delta$. For $\gamma = 0$, any arbitrary function of the energy ε is a steady-state solution. This function, like $\varepsilon(k, x)$, is two-valued in the energy interval $\varepsilon_3 < \varepsilon < \varepsilon_1$, where for each energy there are two trajectories, internal and external. We denote the corresponding branches of the distribution function by $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$. Outside this interval there is only one branch $\rho_2(\varepsilon)$.

For γ is different from zero we set $\rho_{1,2} = \rho_{1,2}(\varepsilon) + \rho'_{12}(k, x)$, where $\rho'_{1,2}(k, x)$ is a small correction of order γ/δ, with $\langle \rho'_{12}(k, x)\rangle_{1,2} = 0$. The angle brackets denote averaging over the equal-energetic surface $\varepsilon(k, x) = \varepsilon$:

$$\langle g(k,x)\rangle_{1,2}=\int dk\,dx\,\delta(\varepsilon(k,x)-\varepsilon)\,g(k,x). \tag{16}$$

Here the subscripts 1 and 2 indicate that in the calculation of the integral on the right side of (16) we need include the contribution from either the internal or external trajectories, respectively.

We solve Eq. (14) by iteration. The conditions under which first-order equations are solvable for the function $\rho'_{1,2}$ determine the form of the function $\rho_{1,2}(\varepsilon)$. Since $\langle\{\varepsilon, \rho'_{1,2}\}\rangle_{1,2} = 0$, the condition

$$\langle\hat{S}(\rho_{1,2}(\varepsilon)\rangle_{1,2}=0 \tag{17}$$

must be met. Using (15) and (16) and integration by parts, we can transform (17) into

$$\frac{d}{d\varepsilon}\left(B_{1,2}(\varepsilon)\rho_{1,2}+\frac{T}{\omega_0}\frac{\beta}{\delta}D_{1,2}(\varepsilon)\frac{d\rho_{1,2}}{d\varepsilon}\right)=0, \tag{18}$$

where

$$B_{1,2}(\varepsilon)=\left\langle k\frac{\partial\varepsilon}{\partial k}+x\frac{\partial\varepsilon}{\partial x}\right\rangle_{1,2}$$

$$D_{1,2}(\varepsilon)=\left\langle\left(\frac{\partial\varepsilon}{\partial k}\right)^2+\left(\frac{\partial\varepsilon}{\partial x}\right)^2\right\rangle_{1,2}. \tag{19}$$

The solution to (18) under the condition that is flux in energy space is

$$\rho_{1,2}(\varepsilon)=C\exp\left[-\frac{\omega_0}{T}\frac{\delta}{\beta}\int_{\varepsilon_3}^{\varepsilon}\frac{B_{1,2}(\varepsilon')}{D_{1,2}(\varepsilon')}d\varepsilon'\right], \tag{20}$$

where C is a normalization constant chosen to be the same for the two branches from the condition that the change in the distribution function be continuous as we cross the separatrix $\rho_1(\varepsilon_3) = \rho_2(\varepsilon_3)$.

Expression (20) gives the steady-state distribution function for an anharmonic oscillator driven by an external periodic force. We note that by definition $D_{1,2>0}$. The quantities $B_{1,2}(\varepsilon)$ can be cast, using (16) and (19), in the form

$$B_1(\varepsilon)=-2\oint k_1\,dx,\quad B_2(\varepsilon)=2\oint k_2\,dx, \tag{21}$$

where $k_1(x)$ and $k_2(x)$ are the dimensionless momenta obtained from (11) for the internal and external trajectories, respectively. The difference in sign is due to the fact that the directions of motion along these trajectories are opposed (see Fig. 2a). Thus, $B_{1<0}$ and $B_{2>0}$, and therefore the function $\rho_1(\varepsilon)$ has a maximum at $\varepsilon = \varepsilon_1$, while $\rho_2(\varepsilon)$ is a maximum at $\varepsilon = \varepsilon_2$. A schematic plot of the steady-state distribution function is shown in Fig. 3.

Let us now determine the exponential factors in the expressions for the times τ_{12} and τ_{21} required to go from the vicinity of point 1 to that of point 2 and back. If the initial distribution is localized, for example, near point 1, then in a time on the order of γ^{-1} a quasi-equilibrium distribution for which $\rho_2(\varepsilon) = 0$ sets in while the function $\rho_1(\varepsilon)$ is determined by a formula similar to (20):

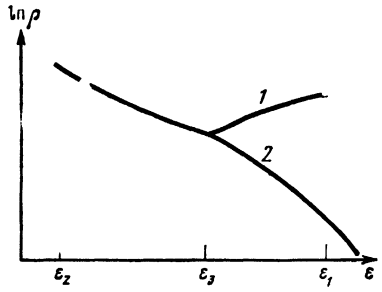

FIG. 3. Steady-state distribution function for $\tau_{12} < \tau_{21}$. The function is double-valued in the energy range $\varepsilon_3 < \varepsilon < \varepsilon_1$.

$$\rho_1(\varepsilon) = C_1 \exp\left(-\frac{\omega_0}{T}\frac{\delta}{\beta}\int_{\varepsilon_1}^{\varepsilon}\frac{B_1}{D_1}d\varepsilon'\right). \quad (22)$$

The steady-state character of this distribution is disturbed only by an exponentially small flux thorugh the activation barrier. The transition time τ_{12} is determined by the value of the quasi-equilibrium function (22) at $\varepsilon = \varepsilon_3$. Thus,

$$\tau_{12} \sim \exp\left(\frac{\omega_0}{T}\frac{\delta}{\beta}I_1(\alpha)\right); \quad I_1(\alpha) = -\int_{\varepsilon_3}^{\varepsilon_1}\frac{B_1}{D_1}d\varepsilon'. \quad (23)$$

We recall that $B_1 < 0$. By analogy, we can obtain

$$\tau_{21} \sim \exp\left(\frac{\omega_0}{T}\frac{\delta}{\beta}I_2(\alpha)\right), \quad I_2(\alpha) = \int_{\varepsilon_2}^{\varepsilon_3}\frac{B_2}{D_2}d\varepsilon'. \quad (24)$$

Equations (23) and (24) coincide with the results of Dykman and Krivoglaz,[5] who calculated the quantities I_1 and I_2 numerically and showed that $I_1 < I_2$ for $\alpha > 0.11$ and $I_1 > I_2$ for $\alpha < 0.11$. Thus, for $\alpha > 0.11$ the time τ_{12} is shorter than τ_{21}, and the small-amplitude oscillation regime (point 1) is metastable. In the opposite case, the large-amplitude oscillation regime is metastable. The steady-state distribution (20) is accordingly localized at $\alpha > 0.11$ near point 2 (see Fig. 1), while at $\alpha < 0/11$ near point 1.

For $\alpha \sim 1$, the dimensionless quantities $K_{1,2}$ are also of order unity. In this case, the exponents in (23) and (24) are in order of magnitude equal to the ratio of the oscillation energy to the thermal energy. For $\alpha \ll 1$, we have[5] $I_1 = 1/2$, $I_2 = 2\alpha$,

Let us evaluate the quantity $I_1(\alpha)$ near the small-oscillation shutoff point, where $\Delta = \alpha_c - \alpha \ll 1$. In this case, $\tau_{21} \gg \tau_{12}$. As α approaches α_c, points 1 and 3 in Fig. 1 come together, and at $\alpha = \alpha_c$ we have $x_1 = x_3 = x_c = 3^{-1/2}$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_c = 1/12$. Setting $x = x_c + y$, $\varepsilon = \varepsilon_c + \eta$ and assuming that $k, y, \eta \ll 1$, we find from (11) that

$$\eta = -(k_1^2/3) + y(3^{-1/2}y^2 - \Delta) - 3^{-1/2}\Delta. \quad (25)$$

Points 1 and 3 correspond to the values $\eta_{1,2} = -3^{-1/2}\Delta \pm 2\cdot 3^{-5/4}\Delta^{1/2}$.

Using (16) and (19), we can represent D_1 in the form

$$D_1 = -\oint (\partial\varepsilon/\partial k_1)(1 + (dk_1/dy)^2)dy. \quad (26)$$

From formula (25) it follows that in the energy interval $\eta_3 < \eta < \eta_1$ the quantity $(dk_1/dy)^2$ is of order $\Delta^{1/2} \ll 1$. Neglecting this quantity in expression (26) and recognizing that $\partial\varepsilon/\partial k_1 = -2k_1/3$, according to (25), we obtain $D_1 = -B_1/3$. We get ultimately

$$I(\alpha) = (4/3^{1/4})\Delta^{3/2}. \quad (27)$$

Thus, near the shutoff point the activation barrier is lowered and the transition time is significantly decreased.

We emphasize that (23) and (24) can be used when the exponent is large compared to one and at $\gamma\tau \gg 1$.

4. TUNNELING TRANSITIONS

For sufficiently low temperatures the transition 1→2 between the two oscillation regimes takes place via quantum tunneling from an internal trajectory near point 1 to an external trajectory with the same energy ε (see Fig. 1).

Let us calculate the transition probability in the quasiclassical approximation. To do this we need to find the function $k(x)$; from (11) we obtain for the internal and external trajectories

$$k_{1,2}^2(x) = 1 - x^2 \mp (1 + 4(\varepsilon - \alpha x))^{1/2}. \quad (28)$$

There are four turning points, where $k = 0$, in the energy interval $\varepsilon_3 < \varepsilon < \varepsilon_1$ of interest to us. Typical plots of $k^2(x)$ are shown in Fig. 4. The points a_1, b_1 and a_2, b_2 are the respective turning points for the internal and external trajectories. Besides these we have one more turning point $c = (1 + 4\varepsilon)/4\alpha$, where $k_1(c) = k_2(c)$. At this point, although $k \neq 0$, the velocity $\dot{q}$ vanishes, as is clear from (3). In the case depicted in Fig. 4, point c lies in the classically inaccessible region.

From Fig. 4 it can be seen that in the classically inaccessible region of phase space, where $k^2 < 0$, there is a path that connects the internal and external classical trajectories. We know[13] that it is the action calculated along this path which

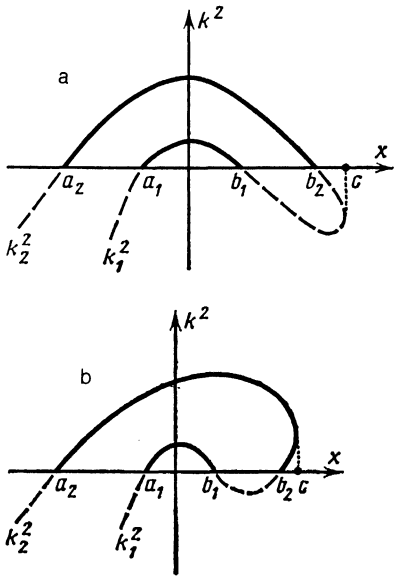

FIG. 4. The function $k^2(x)$ according to equation (11). a_1, b_1, a_2, b_2, and c are turning points. The dashed lines correspond to classically inaccessible trajectories. The dashed line b_1b_2 is a trajectory along which a tunneling transition takes place.

determines the tunneling probability. Limiting ourselves to calculating the exponential factor in the expression for the transition time τ_{12}, we obtain

$$\tau_{12} \sim \exp\left(\frac{2}{\hbar}\int_{q_1}^{q_2} |p|dq\right), \tag{29}$$

where $q_{1,2} = (\delta/\beta)^{1/2} b_{1,2}$. Changing from p and q to the variables k and x in Eqs. (10), we obtain

$$\tau_{12} \sim \exp\left(\frac{2\delta}{\hbar\beta} J(\alpha)\right), \quad J(\alpha) = \int_{b_1}^{b_2} |k|dx. \tag{30}$$

The dimensionless quantity J in (30) depends on the value of the classical parameter α defined by (6) and on the dimensionless energy ε. We will be interested in the transition time at the energy $\varepsilon = \varepsilon_1$ corresponding to steady-state small-amplitude oscillations (point 1 of Fig. 1). Actually, in the absence of dissipation, the closed internal trajectories in Fig. 1 are stationary and tunneling transitions are possible from these trajectories to the external ones (and back). However, in the presence of friction (see Fig. 2b), any internal trajectory leads the system to point 1 that corresponds to the steady-state small-amplitude oscillations before tunnel transition occurs. Thus, it is interesting to study the tunneling from point 1 to an external trajectory with the same energy. The reverse tunneling transition can be ignored, for owing to friction any external trajectory takes the sytem to point 2 that corresponds to steady-state large-amplitude oscillations. In order to calculate the energy ε_1 we must use Eq. (11), in which we need to set $k = 0$ and take for x that root of (5) which has the smallest absolute value.

The classical approximation which we have used to find the quasiclassical tunneling probability is valid, as usual, under the condition that the exponent in (30) is large compared with unity.

Thus, the problem was reduced to calculating the integral $J(\alpha)$ for $\alpha < \alpha_c$, when there are two steady-state oscillation regimes. As in the preceding section, we will investigate limiting cases $\alpha \ll 1$ and $\alpha_c - \alpha \ll 1$. It is obvious that in the intermediate region $(\alpha_c - \alpha)/\alpha \sim 1$ the quantity $j(\alpha)$ is of order unity.

Let us first investigate the case $\alpha \ll 1$. The energy ε_1 here is of order α^2; thus, we can set $\varepsilon_1 = 0$. Neglecting terms of order α, for the turning points b_1, b_2 and c we have $b_1 = 0$, $b_2 = \sqrt{2}$, $c = (4\alpha)^{-1}$, so that $c \gg b_2$ and the function $k^2(x)$ corresponds to Fig. 4a. The integral in (30) in this case the form

$$J(\alpha) = \int_{b_1}^{c} |k_1|dx - \int_{b_2}^{c} |k_2|dx. \tag{31}$$

Let us break up each of the integration intervals into two parts: $x < x_0$ and $x > x_0$, where $1 \ll x_0 \ll \alpha^{-1}$. In the region $x < x_0$ we have $|k_1| \approx x$, $|k_2| \approx (x^2 - 2)^{1/2}$. For $x > x_0$ the difference $|k_1| - |k_2|$ is approximately equal to the expression $x^{-1}(1 - 4\alpha x)^{1/2}$. When we evaluate these integrals, we find, up to small terms of order α, that

$$J(\alpha) = \ln\frac{\sqrt{2}}{\alpha} - \frac{3}{2}. \tag{32}$$

Let us compare our results with those of Ref. 9, in which tunneling transitions were studied for the case $\alpha \ll 1$. We note first that the quasiclassical method proposed in Ref. 9 in fact coincides with the one we have used: if we introduce a new integration variable z according to the formula $x = (\varepsilon + z - z^2)/\alpha$ and carry out a few transformations, we reduce the general formula (30) to the form obtained in Ref. 9

However, the author of Ref. 9 erroneously distinguished between limiting cases $f \gg \hbar^{3/2}\beta$ and $f \ll \hbar^{3/2}\beta$, regarding only the first as quasiclassical and the second as quantum-mechanical. In Ref. 9 the result of calculating the integral in the first case differed from (32) and seems wrong to us. For the quantum case $f \ll \hbar^{3/2}\beta$, the expression cited in Ref. 9 was derived earlier[2] by another method. In fact, at $\delta \gg \hbar\beta$ this expression is a quasiclassical result and coincides with (30) and (32). In reality the condition $\delta \gg \hbar\beta$ is the only criterion for applicability of the quasiclassical approximation (except for the region in α-space near α_c; see below) and when it is satisfied relation beween f and $\hbar^{3/2}\beta$ is immaterial. Using the definition (6), this condition can be rewritten in the form $f \gg \hbar^{3/2}\beta\alpha$, from which it is clear that if $\alpha \ll 1$, then the quasiclassical approximation is valid also at $f \ll \hbar^{3/2}\beta$.

Let us turn to the limiting case $\Delta = \alpha_c - \alpha \ll 1$, which corresponds to the region close to the shutoff point for the small-amplitude oscillation. As $\alpha \to \alpha_c$, the turning points a_1, b_1 and b_2 approach one another and all tend to the value $x_c = 3^{-1/2}$. The function $k^2(x)$ now corresponds to Fig. 4b, and the integral in Eq. (30) takes the form

$$J(\alpha) = \int_{b_1}^{b_2} |k_1|dx. \tag{33}$$

As in the preceding section, we set $x = x_c + y$, $\varepsilon = \varepsilon_c + \eta$ and choose the energy correction $\eta = \eta_1$ such that its value corresponds to point 1; we then obtain from (25)

$$|k_1|^2 = 3^{1/2}(y + y_0)^2(2y_0 - y), \quad y_0 = 3^{-1/4}\Delta^{1/2}. \tag{34}$$

Substituting this k_1 into (33) and integrating with respect to y from $-y_0$ to $2y_0$, we finally obtain

$$J(\alpha) = (4 \cdot 3^{9/4}/5)\Delta^{5/4}. \tag{35}$$

Thus, as expected, the probability for tunneling increases sharply in the neighborhood of the shutoff point. For $\Delta \lesssim (\hbar\beta/\delta)^{4/3}$ the transmission coefficient turns out to be of order unity. In this region, however, the quasiclassical approximation cannot be used.

In connection with formula (30), we make the following remark. In the absence of friction, tunneling transitions could occur only when the quantum levels corresponding to the internal and external trajectories are equal (to an accuracy $\hbar/\tau_{12}$). In this case the populations would oscillate with a characteristic period determined by Eq. (30) (without the factor of 2 in the exponent). However, in the presence of friction such that $\gamma\tau_{12} \gg 1$, as a consequence of the broadening of the levels, the tunneling transitions become possible even for unequal levels. In this case the exponential factor is always determined by (30). The level difference is manifest only in the value of the pre-exponential factor, which we do

not calculate.

5. CONCLUSION

Equations (23), (24) and (30) determine respectively the times of activation and tunneling transitions between the two regimes of forced oscillation of an anharmoic oscillator. Far from the shutoff point of the small-amplitude oscillation [at $(\alpha_c - \alpha)/\alpha \sim 1$], the exponents in these formulas are of the order of the ratio of the oscillation energy to the thermal energy T (for the activated case) or to the quantum energy $\hbar\omega_0$ (for tunneling transitions). Consequently, the activated transitions play a fundamental role for $T \gg \hbar\omega_0$ and the tunneling transitions for $T \ll \hbar\omega_0$.

As the shutoff point is approached the probabilities of these and other transitions rise sharply. We call attention to the differences between the dependences of the exponents on the quantity $\Delta = \alpha_c - \alpha$ for activated and tunneling transitions [see Eqs. (27) and (35)]. The height of the activation barrier is proportional to $\Delta^{3/2}$. The tunneling probability is determined by the product of the square root of the barrier height and its width, the latter of order $\Delta^{1/2}$ [see (34)]. This gives a $\Delta^{5/4}$ dependence for the exponent. Thus, sufficiently close to the shutoff point, the main cause of these transitions is always thermal activation.

It is interesting to note that the small-amplitude oscillation regime is always metastable in respect to tunneling transitions. By contrast, if the activated transitions dominate, then at $\alpha < 0.11$ it is the large-amplitude oscillations which are unstable. This difference is thus significant at small values of the parameter α. Comparing the exponents in (24) and (30), we find that for $\alpha \ll 1$ the tunneling transitions play the fundamental role if $\alpha > \alpha_0 = (T/\hbar\omega_0) \ln (\hbar\omega_0/T)$. In this case, the small-amplitude oscillations are metastable. If however $\alpha < \min(\alpha_0; 0.11)$ then the large-amplitude oscillations are metastable.

We note that our assumption, that the relative change in momentum and energy for each scattering event responsible for dissipative processes is small, is quite important. For example, if besides these processes there are infrequent collisions (with frequency $\nu \ll \gamma$) accompanied by large energy loss, then just one such collision is enough to stop large-amplitude oscillation state. Thus, even at $\tau_{12} < \tau_{21}$ this regime is possible only for $\nu\tau_{12} \ll 1$. In the opposite case, the small-amplitude oscillation will be the steady-state regime. In the case of quasi-elastic collisions with large momentum transfer there occurs an effective intermixing of the trajectories and two regimes of oscillation will not exist at all. Just such a situation arises in the case of nonlinear cyclotron resonance in semiconductors, which was investigated in Ref. 4. Thus, the hysteresis predicted in Ref. 4, which is related to nonparabolicity of the electronic spectrum, should not be observed.

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Translated by Frank J. Crowne