Stability of a Bennett pinch

B. É. Meĭerovich

Institute of Physical Problems, Academy of Sciences of the USSR
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The two-fluid approximation in relativistic electromagnetic hydrodynamics is used to analytically calculate the behavior of small isothermal z-independent perturbations in an axisymmetric Bennett pinch near the stability threshold. The results confirm the previous numerical findings that the equilibrium configurations are unstable against shearing of the system as a whole but are stable to filamentation. The spectrum of the acoustic Goldstone branch of the long-wave pinch oscillations is calculated.

1. INTRODUCTION

The problem of hot plasma confinement has stimulated work on equilibrium plasma configurations[1] and their stability[2] in the framework of magnetohydrodynamics (MHD). An energy principle for analyzing plasma MHD stability was developed in Ref. 3 and applied successfully to several complicated systems in Refs. 4–6.

The MHD equations neglect the relative motion of the electrons and ions in the plasma, which is regarded as a continuous, electrically neutral, conducting fluid. Yet it is clear that the passage of current through the plasma necessarily involves a relative motion of the electron and ion subsystems. Their relative velocity v_0 (drift velocity) is related to the current I by $I = eN_e v_0$, where e is the electron charge and N_e is the number of electrons per unit length of the current channel.

The applicability of the MHD approximation to high-current plasma systems is discussed in detail by Kadomtsev[7]. The relative motion of the electron and ion subsystems can be neglected if the drift velocity is small compared to the characteristic velocities of the problem. For time-dependent flows, v_0 must be small compared to the MHD velocity of the plasma. On the other hand, because an equilibrium plasma is at rest, the thermal velocities of the electrons and ions are the only characteristic velocities relevant to studies of the equilibrium plasma configurations.

Trubnikov[8] was the first to carry out an MHD analysis of the stability of a fully ionized cylindrical plasma column carrying a longitudinal current uniformly distributed over a cross section. He found that such a system is destabilized by axisymmetric ($m = 0$) longitudinally periodic perturbations, and sausage formation and pinching occur. The need to avoid the sausage instability is one of the reasons why the more complicated toroidal plasma configuration was preferred.

However, pinch research over the last three decades has shown that "plasma in which a strong sausage formation occurs may be more efficient neutron emitters than systems specifically designed to avoid the sausage instability."[9] Experiments with dense pinched plasmas revealed that the plasma column is stable against axisymmetric perturbations during the entire lifetime of the discharge.[10,11] The MHD analysis of the nonlinear stage of the sausage instability carried out in Ref. 12 shows that the current is not cut off even though abrupt constriction of the current channel may occur.

Pinch systems are currently not restricted to devices in which the plasma current is high and the drift velocity low. For example, for relativistic electron beams partially neutralized by ions, v_0 is comparable to the speed of light and is certainly much greater than the thermal velocities of the particles. It has become clear that the one-fluid MHD model cannot be used to fully analyze even the macroscopic stability of pinch systems.

If the drift velocity is large compared to the thermal velocities, the pinch system can be described macroscopically by the two-fluid MHD equations.[13] In this case one can neglect the mutual drag exerted by the electrons and ions, which can be regarded as equilibrium subsystems that interact through the electromagnetic field induced by the charged particles themselves. The quasineutrality approximation is also invalid for relativistic drift velocities. Furthermore, the analysis has been limited to small oscillations of the pinch system[14,15] about the equilibrium state because no general methods for analyzing stability (in particular, no analog of the Bernstein-Frieman-Kruskal-Kulsrud energy principle[3]) has yet been derived for the two-fluid electromagnetic hydrodynamic (EMHD) model.

The density, pressure, electromagnetic field, and velocities of the charged fluids are all coupled and oscillate together in a pinched system. These oscillations have so far been considered only in a few limiting cases for which an analytic treatment is possible. It was shown in Ref. 15 that in the two-fluid EMHD model, the pinch system is particularly susceptible to destabilization by short-wave ion-sound oscillations which are uniform in the direction of the current and cause the current channel to break up into parallel structures or jets. A condition for pinch systems to be stable against current channel stratification was derived there by analyzing the behavior near the stability threshold. In general, numerical methods are needed to verify if this condition is satisfied; however, an analytic treatment is possible for a Bennett pinch,[16] in which the electron and ion densities decay radially in the same way. In the present paper we show that for Bennett pinches the two-fluid EMHD model can be used to find the linear response in addition to permitting an analytic treatment of the stability problem.

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2. BENNETT EQUILIBRIUM

The crucial assumption in the two-fluid EMHD description of pinching is that the thermal velocities of the electrons and ions are small compared to the drift velocity v_0. Because the Coulomb cross sections fall off rapidly as the relative velocity of the colliding particles increases, this assumption enables one to neglect the mutual drag exerted by the electron and ion subsystems. If we neglect radiation, each of these subsystems can be regarded as in thermal equilibrium in the average force field associated with their collective interaction. As far as the individual particles are concerned, the collective electromagnetic field behaves like an external field, even though it is generated by the charged particles themselves. When a current flows, the system cannot be in overall thermodynamic equilibrium because the relative velocity of the electron and ion subsystems is nonzero.

Provided the electrons and ions can be treated as two subsystems in thermodynamic equilibrium, no further approximations are needed to describe the pinching. The equilibrium states of an axisymmetric system which does not vary in the direction of the current can be classified in terms of six parameters—the number of particles N_α per unit length, the temperature T_α, and the velocities v_α of the subsystems ($\alpha = e,i$); moreover, one of the v_α can be made to vanish by choosing a suitable reference frame. Four dimensionless parameters can be constructed from the remaining five dimensional variables. If we assume a Boltzmann distribution, the energy of magnetic compression must balance the energy of expansion (thermal and electrostatic) of the particles[17] in order for the pinch system to be in equilibrium. This energy balance condition imposes a constraint on the parameters of the system, so that only three dimensionless variables remain independent; we may take them to be $\beta = v_0/c, \Lambda = T_i/ZT_e$, and $\varepsilon = e^2N_e\beta^2/T_e$, where c is the speed of light, T_i and T_e are the ion and electron temperatures, Z is the ion charge, and ε is the energy of magnetic compression divided by the electron temperature. In general, if one assumes a Boltzmann distribution then an axisymmetric pinch system uniform in the direction of the current will have a three-parameter family of equilibrium configurations.

The equations describing the equilibrium configurations in the classical Boltzmann approximation can be expressed in the form[18]

$$\frac{1}{z}\frac{d}{dz}z\frac{d\psi_e}{dz} = \Lambda[e^{-\psi_i}-(1-\beta^2)e^{-\psi_e}],$$

$$\frac{1}{z}\frac{d}{dz}z\frac{d\psi_i}{dz} = e^{-\psi_e}-e^{-\psi_i}, \qquad (1)$$

where the functions ψ_α satisfy the boundary conditions

$$\psi_e(0)=\psi_e'(0)=\psi_i'(0)=0, \quad \psi_i(0)=y_0. \qquad (2)$$

The parameter y_0 appears here in place of ε, and β, Λ, and y_0 completely specify all the possible configurations. The electron and ion densities $n_{\alpha 0}(r)$ are given in terms of $\psi_\alpha(z)$ by

$$n_{e0}(r)=\frac{T_i}{4\pi Ze^2r_0^2}\exp\left\{-\psi_e\left(\frac{r}{r_0}\right)\right\},$$

$$n_{i0}(r)=\frac{T_i}{4\pi Z^2e^2r_0^2}\exp\left\{-\psi_i\left(\frac{r}{r_0}\right)\right\},$$

where the discharge radius r_0 is chosen as a convenient scaling factor (the condition that the various forces be in equilibrium for an isolated pinch system does not alone suffice to determine r_0).

In general, Eqs. (1) with the boundary conditions (2) must be solved numerically for arbitrary β,Λ,y_0. The value of the physical parameter ε corresponding to specified β, Λ, and y_0 is then calculated from the solution.

For fixed β and Λ, there exists a value

$$y_0=y_0^B=-\ln[1-\beta^2\Lambda/(1+\Lambda)] \qquad (3)$$

for which the radial density distributions for the electrons and ions are identical. The solution in this case was first derived by Bennett[16]; it depends on two parameters (β and Λ are arbitrary) and is given by

$$\psi_e^B(z)=2\ln[1+(z/z_0)^2], \quad \psi_i^B(r)=\psi_e^B(r)+y_0^B,$$

where $z_0^2 = 8(1+\Lambda)/\Lambda\beta^2$.

It is remarkable that for a Bennett distribution, the two-fluid EMHD model can also be solved analytically to find the linear response (assumed to be uniform along the current).

3. STABILITY THRESHOLD

Oscillations in a pinch system were studied analytically in Ref. 15 in the two-fluid EMHD model for the extreme case when the wavelength is very small or very large compared to the radius of the pinch. The long-wave oscillations were associated with net displacements of individual regions of the current channel. Owing to the "elasticity" of the electromagnetic field, the pinch system is stable against long-wave perturbations.

For axisymmetric plasma structures that are uniform in the direction of the current (i.e., z-independent), the perturbations are of the form

$$f(r,\varphi,z,t)=f(r)\exp(i\omega t-ikz-im\varphi).$$

After linearization, the two-fluid EMHD equations for a nonrelativisitc, nondissipative system have the following form. The equation of continuity for the electron ($\alpha = e$) and ion ($\alpha = i$) subsystems:

$$i\omega_\alpha' n_{\alpha 1}+\frac{1}{r}\frac{d}{dr}rn_{\alpha 0}v_{\alpha 1r}-in_{\alpha 0}\left(kv_{\alpha 1z}+\frac{m}{r}v_{\alpha 1\varphi}\right)=0;$$

the Euler equations:

$$i\omega_\alpha' m_\alpha v_{\alpha 1r}+\frac{1}{n_{\alpha 0}}\frac{dp_{\alpha 1}}{dr}-\frac{n_{\alpha 1}}{n_{\alpha 0}^2}\frac{dp_{\alpha 0}}{dr}$$
$$+e_\alpha\left[\frac{d\varphi_1}{dr}-\frac{v_{\alpha 0z}}{c}\left(ikA_{1r}+\frac{dA_{1z}}{dr}\right)-\frac{v_{\alpha 1z}}{c}\frac{dA_{0z}}{dr}\right]=0, \qquad (4)$$

$$i\omega_\alpha' m_\alpha v_{\alpha 1z}-\frac{ikp_{\alpha 1}}{n_{\alpha 0}}+e_\alpha\left(-ik\varphi_1+\frac{v_{\alpha 1r}}{c}\frac{dA_{0z}}{dr}\right)=0, \qquad (5)$$

$$i\omega_\alpha' m_\alpha v_{\alpha 1\varphi}$$
$$-\frac{imp_{\alpha 1}}{rn_{\alpha 0}}+e_\alpha\left[-\frac{im\varphi_1}{r}+\frac{v_{\alpha 0z}}{c}\left(-ikA_{1\varphi}+\frac{imA_{1z}}{r}\right)\right]=0; \qquad (6)$$

the entropy conservation equations

$$i\omega_\alpha' S_{\alpha 1}+v_{\alpha 1r}\frac{dS_{\alpha 0}}{dr}=0;$$

the magneto- and electrostatic equations:

$$\Delta A_{1i}=-\frac{4\pi}{c}\sum_\alpha e_\alpha(n_{\alpha 1}v_{\alpha 0 i}+n_{\alpha 0}v_{\alpha 1 i}),\qquad \Delta\varphi_1=-4\pi\sum_\alpha e_\alpha n_{\alpha 1}.$$

Here $\omega_\alpha' = \omega - kv_{\alpha 0z}$ is the Doppler-shifted oscillation frequency for particles of species α, and

$$\Delta=\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-k^2-\frac{m^2}{r^2}$$

is the Laplace operator.

For short-wave perturbations the radial dependence $f(r)$ may be taken to have the form

$$f(r)=\exp\left\{-i\int^r q(r')\,dr'\right\},\qquad qr_0\gg 1.$$

In this case the oscillations are localized and the linearized equations reduce to an algebraic system. If $\Omega_e \gg \omega$ and $m\sim qr_0\gg 1$, the spectrum for longitudinally uniform oscillations is given by[15]

$$\omega^2=\omega_i^2(Q^2r_d^2-\beta^2),$$

Here $\Omega_e = (e_e/m_e c)dA_{z0}/dr$ is the local Larmor frequency of the electrons, r_0 is the pinch radius, ω_i is the ion Langmuir frequency, $Q^2 = q^2 + m^2/r^2$, and r_d is the Debye radius.

The localization condition $qr_0\gg 1$ for the perturbations breaks down near the stability threshold $Q_{cr}r_d^2 = \beta^2$, and their wavelength becomes comparable to the radius of the discharge. Thus the local approximation cannot be used near threshold, and in general the equations must be solved numerically. However, the Bennett case $y_0 = y_0^B(3)$ is exceptional in that the existence of a stability threshold can be demonstrated analytically. The following analytic treatment can be used to simplify the numerical analysis of stability for all values of y_0 for which equilibrium is possible.

The problem simplifies substantially because the instability threshold (at which the current channel breaks up into small pinches occurs when ω^2 vanishes. We can therefore let $\omega^2\to -0$ to see whether or not a stability threshold exists. The system will be unstable if and only if the system of homogeneous linear equations admits a nontrivial solution with zero boundary values.

For $\omega = k = 0$, the Euler equations (5) imply that $v_{\alpha 1r} = 0$. Since the entropy conservation equation holds identically, we need only consider isothermal perturbations: $p_{\alpha 1} = T_\alpha n_{\alpha 1}, T_\alpha = \text{const}$. The continuity equation with $v_{\alpha 1r} = 0$ implies that $v_{\alpha 1\varphi} = 0$ if $m\neq 0$. Equations (4) and (6) then imply that the z-projection $v_{\alpha 1z}$ of the velocity vanishes, and we get the following system of equations for a pinch near the stability threshold:

$$\Delta\nu_\alpha-\frac{4\pi e_\alpha}{T_\alpha}\sum_\beta e_\beta n_{\beta 0}(r)(1-\beta_\alpha\beta_\beta)\nu_\beta=0,\qquad \alpha,\beta=i,e. \qquad (7)$$

Here

$$\Delta=\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{m^2}{r^2}\right)$$

is the Laplace operator, the subscripts α,β take the values i, e for ions and electrons, $\nu_\alpha = n_{\alpha 1}/n_{\alpha 0}$ are the relative density perturbations, and $\beta_\alpha = v_\alpha/c$, where v_α is the velocity of subsystem α and c is the speed of light.

For a Bennett pinch, the change of variable

$$x=\ln(r/r_0z_0)$$

reduces (7) to the form

$$(d^2/dx^2-m^2)\nu_\alpha=a_{\alpha\beta}\nu_\beta\,\mathrm{ch}^{-2}x, \qquad (8)$$

where summation over the repeated subscript β is understood. In a reference frame moving with the ion subsystem ($\beta_i = 0, \beta_e = \beta$), the tensor $a_{\alpha\beta}$ has the components

$$a_{ee}=2(1+\Lambda)(1-\beta^2)/\beta^2,\quad a_{ei}=-2(1+\Lambda-\Lambda\beta^2)/\beta^2,$$
$$a_{ie}=-2(1+\Lambda)/\Lambda\beta^2,\quad a_{ii}=2(1+\Lambda-\Lambda\beta^2)/\Lambda\beta^2. \qquad (9)$$

They depend only on the two parameters $\beta = v_0/c$ and $\Lambda = T_i/ZT_e$, as is to be expected since Bennett equilibrium corresponds to a two-parameter family of equilibrium structures. We will need the following properties of the tensor $a_{\alpha\beta}$ (9):

$$a_{ee}+a_{ei}=a_{ie}+a_{ii}=-2,$$
$$S=\mathrm{Sp}\,a=a_{ee}+a_{ii}=2[(1+\Lambda)(1+\Lambda-\Lambda\beta^2)/\Lambda\beta^2-1],$$
$$\Delta=\det a=-4(1+\Lambda)(1+\Lambda-\Lambda\beta^2)/\Lambda\beta^2.$$

The trace S and determinant Δ of the matrix $a_{\alpha\beta}$ are related by

$$S+2=-\Delta/2.$$

If Eqs. (8) have a nontrivial solution satisfying the zero boundary conditions

$$\nu_\alpha(-\infty)=\nu_\alpha(\infty)=0,\qquad \alpha=e,i$$

then the system must be unstable.

We have derived Eqs. (8) by using the nonrelativistic two-fluid EMHD model; however, their validity is not limited to $\beta\ll 1$. Equations (8) can be derived from the electro- and magnetostatic equations, as was done in Ref. 19. In this case one uses only the relativistic invariance of the distribution function[20] and the fact that when $k = 0$, the perturbation leaves the system uniform along the direction of the original current.

The general solution of system (8) can be found if we note that the associated Legendre functions satisfy the equation

$$(d^2/dx^2-m^2)P_\nu^m(\mathrm{th}\,x)=-\nu(\nu+1)(\mathrm{ch}\,x)^{-2}P_\nu^m(\mathrm{th}\,x). \qquad (10)$$

Assuming a solution of (8) of the form

$$\nu_\alpha(x)=b_\alpha P_\nu^m(\mathrm{th}\,x) \qquad (11)$$

we then find a homogeneous system of algebraic equations

$$(\lambda\delta_{\alpha\beta}+a_{\alpha\beta})b_\beta=0 \qquad (12)$$

for the coefficients b_α, where $\lambda = \nu(\nu + 1)$. The solvability condition $\det(\lambda\delta_{\alpha\beta}+a_{\alpha\beta})=0$ for (12) reduces to the quadratic equation $\lambda^2+\lambda S+\Delta=0$ for λ. Noting that $S^2/4-\Delta=(S+4)^2/4$, we find that there are two values $\lambda_1=2$ and $\lambda_2=\Delta/2$ for which the system (12) [and hence also (8)] has a nontrivial solution. These values correspond to four values of the index ν; however, only the two values

$$\nu_1=1,\quad \nu_2=-{}^1/_2+i(S+{}^3/_2)^{1/2}$$

are actually needed to construct the general solution of (8). If we note that for nonintegral ν the functions $P_\nu^m(\operatorname{th} x)$ and $P_\nu^m(-\operatorname{th} x)$ give linearly independent solutions of Eq. (10), we can write the general solution as

$$\nu_\alpha(x)=b_{\alpha1}P_1^m(\operatorname{th} x)+b_{\alpha2}Q_1^m(\operatorname{th} x)+b_{\alpha3}P_{\nu_2}^m(\operatorname{th} x)+b_{\alpha4}P_{\nu_2}^m(-\operatorname{th} x). \quad (13)$$

Here the coefficients $b_{\alpha1}$ and $b_{\alpha2}$ satisfy Eqs. (12) with $\lambda=\lambda_1=2$, while $b_{\alpha3}$ and $b_{\alpha4}$ satisfy (12) with $\lambda=\lambda_2=\Delta/2$.

Of the four linearly independent Legendre functions in (13), only the first satisfies the zero boundary conditions, and then only if $m=1$: $P_1^1(\operatorname{th} x)=1/\operatorname{ch} x$,

$$\nu_\alpha(x)=b_{\alpha1}/\operatorname{ch} x,\quad m=1. \quad (14)$$

There are no other nontrivial solutions of (8) satisfying zero boundary conditions for the range of parameters $0<\beta\leqslant1, 0<\Lambda<\infty$ of physical interest. This confirms the conclusion reached previously in Ref. 15, where a numerical analysis suggested that Bennett pinches are unstable to perturbations with $m=1$ but are stable for $m>1$.

The instability for $m=1$ has a straighforward interpretation. Under isothermal conditions it is present for all values of β and Λ and reflects the invariance of the system with respect to global transverse displacements. When $k\neq0$ the system is no longer uniform along the current, but the pinch oscillation spectrum must contain an acoustic Goldstone branch which becomes a static shear as $k\to0$ and describes continuous bending of the pinch similar to the vibrations of a string.

4. ACOUSTIC BRANCH OF THE LONGWAVE OSCILLATION SPECTRUM

The Goldstone branch of the spectrum can be calculated by the technique used by Pitaevskiĭ[21] in his analysis of the oscillations of a vortex filament in a nonideal Bose gas. Let ω and k be nonzero but small enough so that

$$\Omega_\alpha\gg\omega,\ kv_{0\alpha},\quad \Omega_\alpha=e_\alpha H/m_\alpha c.$$

Then the motion of both the electrons and the ions is "frozen out" in the rz plane normal to the magnetic field, and we can take $v_{\alpha1r}=v_{\alpha1z}=0$. Thus only the motion of the electron and ion fluids along the magnetic lines of force must be considered, $v_{\alpha1\varphi}\neq0$.

We consider the case when the number N_e of electrons per unit length of the current channel satisfies

$$e^2N_e/m_ic^2\ll1.$$

We can then neglect the terms containing the azimuthal component A_φ of the perturbation of the vector potential in the equations of motion. Since $m_e\ll m_i$, we may also neglect the inertia of the electrons. The acoustic branch of the pinch oscillation spectrum is then the one associated with the azimuthal displacements of the ion component. In a reference frame for which the ion subsystem is stationary at equilibrium, we recover Eqs. (7) with the term

$$\Delta\nu_\alpha=\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{1}{r^2}\right)\nu_\alpha$$

replaced by

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\varkappa^2-\frac{1}{r^2}\right)\left(1-\frac{m_i\omega^2r^2}{T_i}\right)\nu_i,$$

and

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\varkappa^2-\frac{1}{r^2}\right)\nu_e$$

in the equations for the ions ($\alpha=i$) and electrons ($\alpha=e$), respectively. Passing from r to the variable $x=\ln(r/r_0z_0)$, we get the following equations for a Bennett pinch:

$$\left(\frac{d^2}{dx^2}-\varkappa^2e^{2x}-1\right)(1-\Omega^2e^{2x})\nu_i-\frac{a_{i\beta}\nu_\beta}{\operatorname{ch}^2 x}=0,$$
$$\left(\frac{d^2}{dx^2}-\varkappa^2e^{2x}-1\right)\nu_e-\frac{a_{e\beta}\nu_\beta}{\operatorname{ch}^2 x}=0, \quad (15)$$
$$\Omega^2=\omega^2m_ir_0^2z_0^2/T_i,\quad \varkappa^2=k^2r_0^2z_0^2,$$

where Ω and $\varkappa$ are the dimensionless frequency and wave vector of the oscillations. Since $\Omega^2\ll1$ and $\varkappa^2\ll1$, Eqs. (15) can be solved separately for $x\gg1$ and for $\varkappa^2e^{2x}\ll1$; these two regions overlap when $1\ll x\ll\ln(1/\varkappa)$.

For $x\gg1$, the solutions of Eqs. (15) tending to zero as $x\to\infty$ are expressible in terms of modified Bessel functions:

$$\nu_e=CK_1(\varkappa e^x),\quad \nu_i=DK_1(\varkappa e^x)(1-\Omega^2e^{2x})^{-1},\quad x\gg1$$

(the terms proportional to $\operatorname{ch}^{-2}x$ are negligible for $x\gg1$). In the intermediate region $1\ll x\ll\ln(1/\varkappa)$ we have

$$\nu_e=C\left(\frac{e^{-x}}{\varkappa}+\frac{\varkappa\ln\varkappa}{2}e^x\right),$$
$$\nu_i=D\left[\frac{e^{-x}}{\varkappa}+\left(\frac{\varkappa\ln\varkappa}{2}+\frac{\Omega^2}{\varkappa}\right)e^x\right]. \quad (16)$$

For $x\ll\ln(1/\varkappa)$ the terms proportional to $\varkappa^2$ and Ω^2 may be treated as perturbations, and we seek a solution of the form

$$\nu_\alpha=\nu_{\alpha0}+\nu_{\alpha1}. \quad (17)$$

For $\nu_{\alpha0}$ we obtain system (8) with $m=1$. The solution tending to zero as $x\to\pm\infty$ is given by (14), where the factors $b_{\alpha1}$ satisfy Eqs. (12) with $\lambda=2$. Since

$$\det(2\delta_{\alpha\beta}+a_{\alpha\beta})=0,$$

we can set one of the amplitudes equal to unity, $b_{e1}=1$, say. Equations (12) then imply that we also have $b_{i1}=1$, and therefore $\nu_{\alpha0}=1/\operatorname{ch}x$.

For $\nu_{\alpha1}$ we get the system

$$(d^2/dx^2-1)\nu_{\alpha1}-a_{\alpha\beta}\nu_{\beta1}\operatorname{ch}^{-2}x=f_\alpha, \quad (18)$$

where

$$f_e=\varkappa^2e^{2x}/\operatorname{ch}x,\quad f_i=\varkappa^2e^{2x}/\operatorname{ch}x+2\Omega^2/\operatorname{ch}^3x.$$

We are interested in the solution of (18) that tends to zero as $x\to-\infty$ and matches the asymptotic solution (16) for $x\gg1$. We use the method of variation of constants and seek a solution of the form

$$\nu_{\alpha1}=\sum_{i=1}^{2}A_iu_i+g_\alpha\sum_{i=3}^{4}A_iu_i, \tag{19}$$

where the u_i are the four linearly independent solutions of the homogeneous system:

$$u_1=\frac{1}{\operatorname{ch}x},\quad u_2=\frac{2x+\operatorname{sh}2x}{\operatorname{ch}x},\quad u_3=P_{\nu_2}{}^1(\operatorname{th}x),$$
$$u_4=P_{\nu_2}{}^1(-\operatorname{th}x),$$

the functions $A_i(x)$ are to be found, $g_e=1$, and $g_i=-g$, where

$$g=\frac{1}{\Lambda}\left(1-\frac{\Lambda\beta^2}{1+\Lambda}\right)^{-1}.$$

We impose the two additional constraints

$$\sum_{i=1}^{2}A_i'u_i=0,\qquad \sum_{i=3}^{4}A_i'u_i=0 \tag{20}$$

on $A_i(x)$, where the primes denote derivatives with respect to x. Substitution of (19) in (18) together with (20) leads to the equations

$$\sum_{i=1}^{2}A_i'u_i'+g_\alpha\sum_{i=3}^{4}A_i'u_i'=f_\alpha. \tag{21}$$

Equations (20) and (21) yield two uncoupled systems of algebraic equations for the pairs A_1', A_2' and A_3',A_4':

$$\sum_{i=1}^{2}A_i'u_i=0,\ \sum_{i=1}^{2}A_i'u_i'=\frac{f_i+gf_e}{1+g}=\frac{2\Omega^2}{(1+g)\operatorname{ch}^3x}+\frac{\varkappa^2e^{2x}}{\operatorname{ch}x},$$

$$\sum_{i=3}^{4}A_i'u_i=0,\qquad \sum_{i=3}^{4}A_i'u_i'=\frac{f_e-f_i}{1+g}=-\frac{2\Omega^2}{(1+g)\operatorname{ch}^3x}.$$

The solution is readily found to be

$$A_1'=-\frac{f_i+gf_e}{1+g}\frac{u_2}{w_{12}},\qquad A_2'=\frac{f_i+gf_e}{1+g}\frac{u_1}{w_{12}},$$
$$A_3'=-\frac{f_e-f_i}{1+g}\frac{u_4}{w_{34}},\qquad A_4'=\frac{f_e-f_i}{1+g}\frac{u_3}{w_{34}}, \tag{22}$$

where the Wronskians $w_{ik}=u_iu_k'-u_ku_i'$ are nonzero and independent of x for the linearly independent pairs of functions u_i and u_k; in fact, $w_{12}=4$.

We can use (22) to find the functions $\nu_{\alpha1}$ that vanish at $x=-\infty$:

$$\nu_{\alpha1}(x)=-\int_{-\infty}^{x}dx'\left\{\left[\frac{2\Omega^2}{(1+g)\operatorname{ch}^3x'}+\frac{\varkappa^2e^{2x'}}{\operatorname{ch}x'}\right]\frac{u_1u_2'-u_2u_1'}{w_{12}}+\frac{2\Omega^2g_\alpha}{(1+g)\operatorname{ch}^3x'}\frac{u_3u_4'-u_3'u_4}{w_{34}}\right\}+A_{10}u_1+g_\alpha A_{40}u_4. \tag{23}$$

We set the constant A_{10} equal to zero, since a term proportional to u_1 is already contained in $\nu_{\alpha0}$. The primes in (23) indicate functions of the argument x' over which the integration is performed: $u_i'=u_i(x')$, $u_i=u_i(x)$.

Since Eqs. (16) are accurate only up to logarithmic terms, we may neglect $\varkappa^2e^{2x}/\operatorname{ch}x'$ in (23); with (17) and (23), we then obtain

$$\nu_\alpha(x)=u_1(x)+\frac{2\Omega^2I_1}{(1+g)w_{12}}u_2(x)+g_\alpha\left(A_{40}+\frac{2\Omega^2I_3}{(1+g)w_{34}}\right)u_4(x) \tag{24}$$

for $1\ll x\ll\ln(1/\varkappa)$. Here

$$I_i=\int_{-\infty}^{+\infty}u_i(x)\frac{dx}{\operatorname{ch}^3x},\qquad I_1=\frac{4}{3}.$$

The function $u_1(x)\approx2e^{-x}$, while $u_2(x)$ and $u_4(x)$ grow exponentially: $u_2(x)=e^x$, $u_4(x)=u_{40}e^x$, $x\gg1$. Writing

$$\mathcal{A}=\left(A_{40}+\frac{2\Omega^2I_3}{(1+g)w_{34}}\right)u_{40},$$

we see that in order for expressions (16) and (24) to coincide, we must have $C=D=2\varkappa$ and

$$2\Omega^2I_1/(1+g)w_{12}+\mathcal{A}=\varkappa^2\ln\varkappa,$$
$$2\Omega^2I_1/(1+g)w_{12}-g\mathcal{A}=\varkappa^2\ln\varkappa+2\Omega^2.$$

Eliminating $\tilde{A}$, we find that $\varkappa$ and Ω are related by

$$2\Omega^2(I_1/w_{12}-1)=(1+g)\varkappa^2\ln\varkappa.$$

Substituting $I_1/w_{12}=1/3$, we get the spectrum

$$\omega^2=\frac{3}{4}(1+g)\frac{T_i}{m_i}k^2\ln\frac{\gamma}{kr_0z_0},\qquad kr_0\ll1, \tag{25}$$

for the acoustic Goldstone branch of the pinch oscillations, where the constant γ is ~1. A more detailed analysis including the logarithmic terms gives the value

$$\gamma={}^1/_2e^{-C-{}^1/_2}=0.681\ldots$$

for γ in (25).

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