Strange attractors and the spatial development of turbulence in flow systems

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A theory is constructed for the spatial development of turbulence in one-dimensional flow systems described by a generalized Ginzburg-Landau equation. It is shown that it is possible to represent the turbulent motion at arbitrary distance from the boundary by means of a finite-dimensional attractor. A novel class of transitions of the "order-chaos" type is exhibited in the form of moving transition fronts.

I. INTRODUCTION

The question as to whether strange (stochastic) attractors—finite-dimensional attracting sets—have any relation to turbulence in unbounded nonequilibrium systems, which is of exceptional and fundamental importance for hydrodynamics (turbulent jets, shear layers and other exterior flows), is one of the most difficult problems in the present-day theory of stochastic motions. This problem is nontrivial even in the way it is posed. Indeed, when one talks about turbulence in bounded spatial regions (in hydrodynamics these are the so-called interior flows), a finite-dimensional description is justified in a quite natural way: The spectrum of spatial excitations in a finite volume is finite, and the high-frequency dissipation (viscosity) depriving the small-scale motions of their independence, will cut the spectrum off. As a result of this, the problem reduces, in principle, to choosing correctly the required number of spatial functions which describe the motion in question. The mean characteristics of turbulence in such systems (the dimension,[1] the power spectrum,[2] etc.) can, generally speaking, be determined in terms of a single physical variable, i.e., as a result of treating a sufficiently long realization of the motion considered at a single point of the volume.[3,4] For unbounded and semi-unbounded flows (in hydrodynamics these are the external flows) the problem of the finite-dimensional description of turbulence must be posed in a completely different manner. In such systems the spectrum of spatial excitations is continuous, and the turbulent pulsations are, generally, inhomogeneous, even in their mean characteristics. This inhomogeneity may be related to the presence of sources of a regular external field, boundaries on which there are no turbulent pulsations, etc. The turbulent regime which establishes itself in the medium is either "tangent" to the laminar periodic regime, or develops out of the latter via a sequence of spatially unfolding "bifurcations." It is probably impossible to obtain an idea of such a turbulent motion only in terms of a single observable: even the average characteristics of such a variable will vary in space along the flow.

From a more formal point of view the picture of the appearance and development of turbulence along the system seems to us to be the following. As is well known, a criterion for the appearance of dynamical chaos is the positivity of the Lyapunov exponent, or the nonvanishing of the Kolmogorov-Sinai entropy.[4] For a semi-infinite medium with a flow, generally speaking, the number of Lyapunov exponents turns out to be infinite even at a finite distance from the boundary. However, in the beginning portion of the flow, where it is still regular (e.g., periodic), all the exponents are negative, with the exception of a finite number of vanishing ones.[1)] As spatial bifurcations develop, one or several exponents may become positive.[2)] The point along the flow where this first happens corresponds exactly to the transition to chaotic (turbulent) motion. It seems natural that just beyond such a transition point the dimension of the realization[3] of such a turbulent flow must be small and the time evolution of the field at a point close to the critical one will correspond to motion on a low-dimensional strange attractor. The growth along the system of already existing positive exponents and the vanishing of new exponents corresponds to the development of turbulence downstream along the flow. The dimension of the strange attractor will increase from point to point along the flow. Whether the dimension of the realization (and of the corresponding attractor) may become infinite at a finite distance (an explosive increase in dimension) is not clear *a priori*; however the inverse effect, in which the dimension of the turbulent motion stabilizes along a flow, is possible, as will be shown.

Using as examples several models reflecting a wide circle of physical phenomena, in the present paper we investigate the spatial bifurcations through which occurs the motion along the flow becomes complicated prior to the appearance of chaos. It is shown that under certain conditions it is admissible to use a finite-dimensional description of turbulence arbitrarily far from the boundary and a new class of transitions of the type "order-chaos" is found, having the form of transition fronts.

2. FUNDAMENTAL EQUATIONS; SPATIAL BIFURCATIONS GIVING RISE TO CHAOS

2.1. To analyze the evolution of disturbances in flow systems far above criticality one usually makes use of the nonlinear parabolic equation

$$\frac{\partial u}{\partial t}=u(1-\delta|u|^2)+\gamma\frac{\partial u}{\partial x}+\varkappa\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u, γ, δ, $\varkappa$ are complex coefficients describing the space-time evolution of the amplitude $\exp i(\omega t-kx)$ of the wave. This is the generalized Ginzburg-Landau equation.[1–6] In a certain range of the parameters its solutions are indeed chaotic in time and space, and the Lyapunov exponents (the measure of chaos) depend on the choice of the reference

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system,[6] a fact which implies that Eq. (1) describes "convective chaos" (cf. the convective instability of Ref. 5).

As the system goes more and more above criticality, the equation (1) no longer describes processes in fluid systems (flows): superposed on the primary wave there appear excitations of other scales and directions, the motion becomes strongly nonlinear, etc. These complications make it extremely difficult to obtain models of fluid systems which are amenable to analysis directly from the original field equations (e.g., the Navier-Stokes equations). Therefore one is forced to construct phenomenological models based on experimental results. We shall keep in mind that as the supercriticality increases, the flow becomes usually more structured. For example, in boundary layers or submerged jets there appear longlived large-scale vortices which situate themselves across the flow and exhibit their own nontrivial dynamics. At the same time the flow system decomposes naturally into mutually interacting discrete elements. Taking into account only a linear coupling between neighboring structures, we represent the starting equations in the form

$$du_j/dt = F(u_j, \delta) + \gamma(u_j - u_{j-1}) + \varkappa(u_{j+1} - 2u_j + u_{j-1}), \quad j = 1, 2, \dots . \tag{2}$$

Here $du_j/dt = F(u_j, \delta)$ describes the dynamics of one of the elements of the point system, and γ and $\varkappa$ characterize the linear coupling between them: γ is responsible for the nonmutual coupling determined by the presence of the flow and $\varkappa$ can be interpreted as a diffusion coefficient. The physical nature of the vector u_j may differ: these may be, for example, the complex amplitudes of azimuthal modes on transverse vortices in translational hydrodynamic flows,[7] concentrations of substances participating in an autocatalytic chemical reaction,[8] etc. We shall be interested in the process of spatial evolution of chaos along the flow in the absence of external stochastizing (noise) interactions; therefore we shall write the boundary conditions in the form $u_0(t) \equiv 0$ or in the form $u_1(t) - u_0(t) \equiv 0$ consistent with $u_{j+1} - 2u_j + u_{j-1} \equiv 0$).

Self-generation of turbulence along a flow in a system of type (2) was first reported in Ref. 9, where a numerical experiment was used to investigate spatial bifurcations which caused chaos to propagate down along a chain, the dynamics of the elements of which is described by a Landau equation[5] and diffusion was absent. In the same paper the phenomenon of "stabilization of turbulence" (partial synchronization of chaos) was discovered, a phenomenon which consists in the fact that starting with some element of the chain the mean characteristics of turbulence (power spectrum, dimension) become independent of the spatial coordinate j. This phenomenon does not seem to depend on the character of spatial bifurcations leading to the appearance of chaos. In particular, it was observed experimentally[10] in a chain of unidirectionally coupled oscillators, in which chaos appeared on account of spatial period-doubling bifurcations, rather than on account of transition though quasiperiodicity, as was the case in the numerical simulation cited.

2.2. In the analytic description of spatial bifurcations we shall first neglect diffusion. Then for a regime which is spatially inhomogeneous along the direction j we have

$$u_{j-1}^0 = u_j^0 + F(u_j^0, \delta)/\gamma. \tag{3}$$

If for specified initial parameters γ and δ this regime is stable for any j, then no restructuring occurs along the flow. In the general case the solution u_j^0 is stable only up to a certain value j^*. The dynamics of evolution of the disturbances $\xi_j(t)$ superposed on u_j^0 is described by the system

$$d\xi_j/dt = F_u'(u_j^0, \delta)\xi_j - \gamma\xi_{j-1}(t). \tag{4}$$

The matrix of this system is block-triangular, so the characteristic exponents of Eq. (4) coincide with those of the system of partial equations:

$$d\xi_j/dt = F_u'(u_j^0, \delta)\xi_j. \tag{5}$$

This relationship is physically obvious: the convective components $-\xi_{j-1}(t)$ play the role of external forces in the partial equations and thus can naturally have no influence on the exponential growth of the disturbances. This circumstance is extremely important. It implies, in particular, that in the chain of elements the solution changes in time as it passes through the point j^* (i.e., the stability of the initial motion is lost in the same way stability is lost when a parameter is changed in the corresponding point system (the parameter in this case is the value of u_j^0 which changes along the chain). Thus, in passing through the point j^* the motion is restructured in a way which can be predicted if one knows the possible bifurcations in the point system.

The new stationary motion which appears for $j > j^*$ (this may consist, e.g., of modulated oscillations) can also be described by means of the system (3), but with different variables (amplitudes of the modulation).[3)] Repeating the analysis one can then determine the coordinate of the next spatial bifurcation, etc. In view of the discreteness of the spatial coordinate, the number of spatial bifurcations preceding the appearance of chaos must be finite (see Appendix 1).

3. THE GENERALIZED GINZBURG-LANDAU EQUATION; TRANSITION THROUGH QUASI-PERIODICITY; STABILIZATION OF THE TURBULENCE

3.1. When the point system is described by the equation[8]

$$du_j/dt = u_j(1 - \delta|u_j|^2), \tag{6}$$

where $\delta = 1 - i\beta$, the original system (2) takes the form of the Ginzburg-Landau equation complemented by the additional convective term $\gamma(u_j - u_{j-1})$. Since u_j and δ are complex, in what follows the parameter γ will be considered complex. In the analysis of the spatial development of chaos we shall treat the diffusion ($\sim\varkappa$) perturbatively.

For $\varkappa = 0$, assuming that steady-state oscillations have established themselves in the first element ($u_1(t) = Ae^{i\beta t}$), where A characterizes the amplitude of the external force in the frequency β, and after the substitutions

$$a_j = u_j(1 - \operatorname{Re}\gamma)^{-1/2} \exp\{i[j(\arg\gamma) - (\beta - \operatorname{Im}\gamma/\operatorname{Re}\gamma)\, t_{st}]\},$$
$$\omega_0 = (\beta - \operatorname{Im}\gamma/\operatorname{Re}\gamma)(1 - \operatorname{Re}\gamma)^{-1},$$
$$\gamma' = |\gamma|(1 - \operatorname{Re}\gamma)^{-1}, \quad t = t_{st}(1 - \operatorname{Re}\gamma)$$

the system under investigation can be written in the form

$$da_j/dt = a_j(1-i\omega_0) - (1-i\beta)|a_j|^2 a_j - \gamma' a_{j-1}, \quad j=1, 2, \ldots \qquad (7)$$

with the boundary condition $a_1^0 = A$. Then the equation (3) for the stationary regime takes on the form

$$a_j^0(1-i\omega_0) - (1-i\beta)|a_j^0|^2 a_j^0 + \gamma' a_{j-1}^0 = 0. \qquad (8)$$

According to Eqs. (4) and (5) the stability of this regime is determined by the behavior of the disturbances in the system

$$d\xi_j/dt = [1 - i\omega_0 - 2i(1-i\beta)|a_j^0|^2]\xi_j - (1-i\beta)a_j^{02}\xi_j^*. \qquad (9)$$

This yields the expressions for the characteristic exponents ($|a_j^0|^2 = I_j$)

$$\lambda_j^{(1,2)} = 1 - 2I_j \pm [(1+\beta^2)I_j^2 - (\omega_0 - 2\beta I_j)^2]^{1/2}. \qquad (10)$$

The stationary values of the intensity I_j are determined from Eq. (8):

$$I_j[1+\omega_0^2 + (1+\beta^2)I_j^2 - 2(1+\beta\omega_0)I_j] = \gamma'^2 I_{j-1}, \quad I_0 = 1/(1 - \mathrm{Re}\,\gamma). \qquad (11)$$

When this regime loses stability at $j = j^*$ a new periodic motion arises in the system [only this type of bifurcation is possible in the point system (6)], i.e., the initial single-frequency regime for $j > j^*$ turns into a quasiperiodic, two-frequency regime. By means of scale transformations and subsequent averaging one again obtains the initial equation (7) for this modulated regime, and so on (cf. Appendix 2).

Let us carry out a stability analysis of the solution (11). The dependence of I_j on I_{j-1} for various values of the parameters is shown in Fig. 1. It is clear that this mapping may have up to three fixed points (corresponding to a spatially-homogeneous periodic regime). It follows from the analysis of (10) that the values of I_j situated where $dI_j/dI_{j-1} < 0$ in the region of the graph of $I_j(I_{j-1})$ (Fig. 1) are unstable. Indeed, in order to have a negative real part of λ_j it is necessary and sufficient that the intensities of the oscillations in each element satisfy the relations

$$I_j \geqslant 1/2, \quad I_j \notin [I_1^*, I_2^*],$$
$$I_{1,2}^* = [3(1+\beta^2)]^{-1}\{2(1+\beta\omega_0) \pm [(1+\beta\omega_0)^2 - 3(\omega_0-\beta)^2]^{1/2}\}. \qquad (12)$$

In the parameter region $1 + \beta\omega_0 < 0$ we always have $I_{1,2}^* < 0$ and therefore for stability it is sufficient that $I_j \geqslant 1/2$, i.e., the single-period regime with an oscillation intensity that increases monotonically and slowly along the chain is stable. In the parameter regime $1 + \beta\omega_0 > 0$ the situation is more interesting. Here $I_{1,2}^* > 0$. The stability of regimes with intensity which increases monotonically along the paths is again insured by the condition $I_j \geqslant 1/2$. Thus, if the spatially-homogeneous regime which corresponds to the upper branch of the mapping (11) (see Fig. 1), has a small amplitude ($I^0 < 1/2$) then the single-period oscillations are stable only on the first few flow elements (corresponding to those iterations of the mapping which are situated above the line $I_j = 1/2$ on Fig. 1). As soon as the intensities of the stationary pulsations (on the j^*th element) fall below 1/2 the regime of periodic oscillations becomes unstable and changes along the flow path to a beat regime; in the $(j^* + 1)$th subsystem there exists a limit cycle (in the original variables this will be a 2-torus). It follows from Eq. (12) that the stability region of the periodic regime in the space of the parameters ω_0, β, and δ is described by the inequalities

$$(\omega_0-\beta)^2 \leqslant \gamma^2(1+\beta^2), \quad \omega_0 \geqslant (\beta^2-1)/2\beta, \quad (\omega_0 - \beta/2)^2 \leqslant \gamma^2 - 1/4. \qquad (13)$$

3.2. The establishment of a chaotic regime with power spectrum and dimension D which do not vary with j (for sufficiently large j) along the chain (7), as observed in the numerical simulation[7], can be explained in the following manner. We have seen that to steady one- and two-period regimes corresponds a smooth increase of the intensity of pulsations along the chain, up to the mean value I^0 (the fixed point of the mapping). It is natural to assume (and this is confirmed in the numerical experiment) a similar variation of the mean intensity of the pulsations also for the steady chaotic regime. This assumption is satisfied, in particular, by a solution of the form

$$a_j(t) = e^{i\varphi} a_{j-1}(t+\tau) \equiv \Psi(t),$$

where the phase φ and the retardation time τ are determined by the parameters γ and β. Substitution of this solution into Eq. (7) leads to the differential equation with retarded argument

$$d\Psi/dt = (1-i\omega_0)\Psi - (1-i\beta)|\Psi|^2\Psi + \gamma e^{-i\varphi}\Psi(t-\tau). \qquad (14)$$

With respect to general solutions of such an equation (see, e.g., Ref. 12) it is known that there always exist only a finite number of eigenvalues for the corresponding linearized problem, eigenvalues which have a positive real part. Thus, among the solutions of the infinite-dimensional system (7) there exist stochastic solutions which have a finite dimension.[4)] However, it is in general not clear whether such solutions are stable.

The rigorous result on stabilization of turbulence (time-independence of the average properties of the chaotic regime) for sufficiently large j may be obtained from an analysis of the Lyapunov characteristic exponents. The number

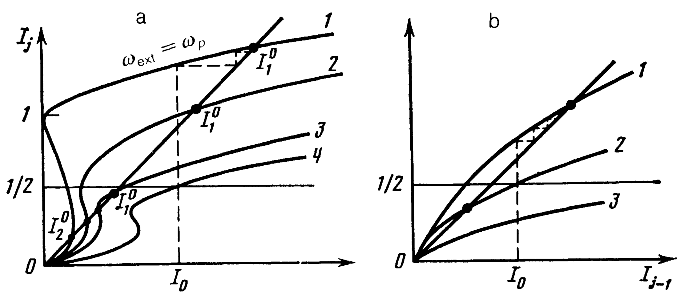

FIG. 1. Graphs of the map (11) for different values of the parameters. a) $(1+\beta\omega_0) \geqslant 0$, $(1+\omega_0^2)(1+\beta^2) \geqslant (\omega_0-\beta)^2$ the map has two nontrivial fixed points I_1^0 and $I_2^0 < 1/2$ (on the curves 1, 2 $I_1^0 > 1/2$, on the curves 3, 4 we have $I_1^0 \leqslant 1/2$); b) $\gamma^2 > (1+\omega_0^2)$ one nontrivial fixed point (curves 1,2); for $\gamma^2 < (1+\omega_0^2)$ there is only the trivial fixed point (curve 3).

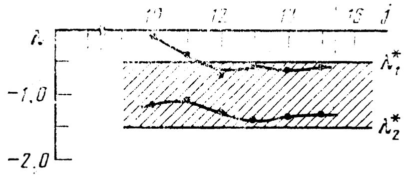

FIG. 2. The position of the newly appearing Lyapunov exponents for $j \geqslant 10$ in the (λ, j) plane; $|\lambda^*_{1,2}| \gg \lambda_{max} = 0.05$.

of Lyapunov exponents of the realization taken at the $(j+1)$st element of the chain exceeds by two the number of exponents of the realization taken from the jth element. For increasing j the mean characteristics of the turbulent regime will obviously remain unchanged only in the case when the newly produced exponents (as one moves along the chain) will be negative and large. This circumstance was verified numerically. The corresponding results are shown in Fig. 2. As j increases the number of positive exponents and exponents which are close to zero does not change and pairs of negative exponents appearing along j are situated in the strip $\lambda_j^{(1,2)} \in [\lambda^*_1, \lambda^*_2]$ where $\lambda^*_{1,2} \ll 0$.

Thus, in effect, neither the Kolmogorov-Sinai entropy

$$H = \sum_{l=1}^{n} \lambda_l \, (\lambda_1, \ldots, \lambda_n \geqslant 0),$$

nor the dimension $D = M + d$ (M is the number of first exponents for which the sum is still positive, and

$$d = \left(\sum_{l=1}^{M} \lambda_l \right) \Big/ \, |\lambda_{M+1}|,$$

cf. Ref. 13) will change along the flow for sufficiently large j.

3.3. We now return to the original equation with diffusion

$$da_j/dt = a_j(1-\delta|a_j|^2) + \gamma(a_j - a_{j-1}) + \varkappa(a_{j+1} - 2a_j + a_{j-1}). \quad (15)$$

It is obvious that even a weak coupling which transfers information on the spatial disturbances "upstream", i.e., from the $(j+1)$st element to the jth, may strongly influence the way chaos evolves along the chain. However, different characteristics of the behavior of the dynamical system under investigation will depend differently on the magnitude of the diffusion. Indeed, assume that far enough downstream along the chain a chaotic regime has established itself, characterized by dimension D and a power spectrum $S(\omega)$. On account of feedback the chaotic pulsations of the jth element will be transmitted back to the $(J-1)$st, etc., up to the left end of the chain—the first element. The smaller the magnitude of the coupling $\varkappa$, the smaller will obviously be the reaction of the preceding oscillator to the feedback from the succeeding one, and this action can be considered as an outside noise which will somewhat distort the proper nonlinear dynamics of the initial portion of the chain, and moreover add to it a chaotic component of dimension D depending on the total number of elements of the chain.[13] Thus, speaking of the dimension of a realization along a path, no matter how small the feedback, it must in general remain constant along j and be equal to D. Nevertheless the character of the pulsations along the path will differ substantially, as well as the power spectrum of the pulsations in different elements. One may continue to speak about a spatial evolution of chaos and of bifurcations along a path for moderately large j, having in mind the smooth part of the solution of the system (15), a solution which must be represented in the form

$$a_j(t) = U_j(t) + \varepsilon_j w(t), \quad \varepsilon_j \ll 1. \quad (16)$$

Here $\varepsilon_j w(t)$ is a small chaotic addition of dimension D, which decreases together with the number j, and $U_j(t)$ is the smooth part of the solution, determined by the nonlinear dynamics of the initial portion of the chain for $\varkappa = 0$ (on this portion the intensity of fully developed chaos of dimension D is still small). The possibility of such a description was tested in a numerical experiment carried out in a path of 150 elements, where the last 50 elements served as an "absorbing wall"—their dissipation increased with the number j; at the boundary of the last element we set $a_{150} - a_{149} \equiv 0$. As can be seen from Fig. 3, even on the first few elements of the chain, which exhibit almost periodic oscillations, one detects signs of random pulsations of the amplitude. Such pulsations increase slowly along the flow, and in parallel with this process and on its background there takes place another one: periodic oscillations of large intensity (with a weak modulation amplitude) are replaced by quasiperiodic ones and then by chaotic motions. The transition to intense chaos along the path turned out to be the smoother, the larger the chosen feedback (the constant $\varkappa$). For sufficiently large values of $\varkappa$ intense chaotic pulsations were observed even on the first element of the path.

For weak diffusion ($\varkappa \ll 1$) the transition from single-period to quasiperiodic pulsations and subsequent loss of stability of the quasiperiodic regime along the chain can be investigated analytically, constructing a perturbation theory on the basis of solutions of the system (7). Within the framework of Eq. (15) the single-period regime is described by the mapping

$$\varkappa\bar{x} = -[x(1-i\omega_0) - (1-i\beta)|x|^2 x - \gamma y], \quad \bar{y} = x, \quad (17)$$

where $x = a_j^0$, $y = a_{j-1}^0$. For $\varkappa \ll 1$ this mapping has fixed points which are close in x to the fixed points of the map (8), with the difference, however, that the fixed points of (8) in

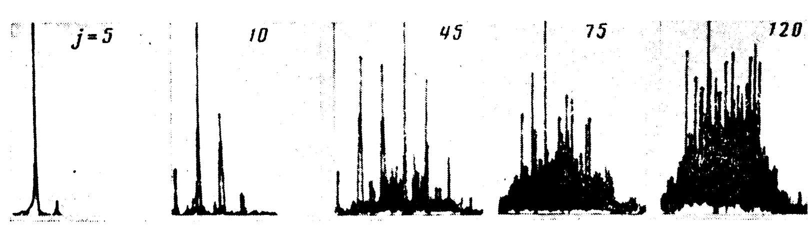

FIG. 3. The spatial distribution of turbulence in a system with diffusion [$\varkappa = 0.5$; $\gamma = 1 + 1.71i$; $\beta = 5$; $a_1(t) = a_0(t)$].

Eq. (17) turn out to be saddle points: two new directions with multipliers $\sim 1/|\varkappa|$ appear. Moreover, all the trajectories of (17) go off very quickly to infinity with a speed $\geqslant 1/|\varkappa|$ (we recall that this is a spatial divergence, and not one in time). The only surface which satisfies the boundary conditions $a_0 = A$ (for $j = 1$) and $|a_j| < \infty$ is a stable separatrix surface for the saddle-point equilibrium state.[5] For $\varkappa \ll 1$ the coordinates of the points on this surface are approximately (to $O(|\varkappa|)$) determined by the mapping (8). Indeed, since $y = \bar{x}$, and for $\varkappa = 0$

$$\gamma y = -[\bar{y}(1-i\omega_0) - (1-i\beta)|\bar{y}|^2\bar{y}] = f(\bar{y}),$$

then it follows from Eq. (17) that for $\varkappa \neq 0$

$$\varkappa f^{-1}(\gamma\bar{y}) = f(\bar{y}) + \gamma y + O(|\varkappa|).$$

This mapping is close to (8), i.e., the previous considerations are valid for it.

The stability of the regime of pulsations which are periodic in time for $\varkappa \neq 0$ is determined by the Lyapunov exponents of the system (15), linearized near the stationary regime (17). A perturbative calculation for ($\varkappa \ll 1$) of the correction λ_j' to the Lyapunov exponents λ_j^0 in Eq. (10) yields

$$\lambda_j'^{(1)} \sim -\gamma|\varkappa| E[(\lambda_j^{0(1)} - \lambda_{j+1}^{0(1)})(\lambda_j^{0(1)} - \lambda_{j+1}^{0(2)})]^{-1} \quad (18)$$

(and similarly for $\lambda_j'^{(2)}$), where $E \sim 0$, Eq. (1). It is clear that the influence of weak diffusion at the beginning of the path is insignificant: for $j \gtrsim 1$ $(\lambda_j^{0(k)} - \lambda_{j+1}^{0(k)}) \sim \lambda_j^{0(k)}$ ($k = 1,2$), and therefore $\lambda_j'^{(k)} \sim |\varkappa|$. At the same time, sufficiently far downstream this influence becomes noticeable: for large j all $\lambda_j^{0(k)} \to \lambda^{0(k)}$ and the corrections $\lambda_j'^{(k)}$ increase.[6] Thus, a weak feedback, without changing the qualitative character of the transition from the periodic regime to the quasiperiodic one, may shift the transition point by one element down along the path, if the transition occurs at its beginning, and by several elements if the transition occurs at large j. Numerical experiments confirm these results.

4. WAVES OF ORDER-CHAOS TRANSITIONS

A chaotic regime which is close to spatially homogeneous which has established itself in a flow system for sufficiently large j may be unstable against the synchronizing action of an external periodic field fed to the boundary of the system. The phenomena which should be observed in this situation are to a certain degree analogous to those which occur in potentially unstable (excitable) media when an external pulse is started in them. In this situation, as is well known, in an excitable medium there appears a transition wave, corresponding to the passage of the medium from the unstable state into the stable one. A similar transition from the unstable (chaotic) state of the nonequilibrium medium into a stable (synchronized) state must be observed in the flow systems under consideration here. To prove this possibility it suffices to convince oneself that in that region of parameter space where in the autonomous chain one observes a regime of spatial evolution and establishment of chaos, in the presence of a periodic external force only the regular regime is stable.

We exhibit this for the system (15) with the boundary conditions

$$a_0(t) = A_{\text{ext}} \exp(i\omega_{\text{ext}} t). \quad (19)$$

The chain must be most sensitive to synchronization at an external frequency ω_{ext} close to the partial frequencies of its component elements, i.e. (taking into account the complex coupling γ), at the frequency

$$\omega_{\text{ext}} \approx \omega_p = \beta(1 - \text{Re}\,\gamma) + \text{Im}\,\gamma/\text{Re}\,\gamma.$$

A stationary spatially-homogeneous regime of synchronization is described by the mapping (cf. Eq. (11); we consider the case $\varkappa = 0$)

$$I_j(1-I_j)^2 = I_{j-1}\gamma^2/(1+\beta^2), \quad I_0 = A_{\text{ext}}^2. \quad (20)$$

For arbitrary values of the parameters β and γ this mapping has a stable fixed point $I_1^0 = 1 + |\gamma|/(1+\beta^2)^{1/2}$, and, as can be seen from Fig. 1a (curve 1), for an arbitrary amplitude of the external field there exists a trajectory of the mapping completely situated in the stability region. Moreover, the regime of complete synchronization (20) also turns out to be globally stable, i.e., in the infinite-dimensional phase space of the system (7), (19) for $\omega_{\text{ext}} = \omega_p$ there do not exist any other attracting sets.

In order to prove this rather strong statement it suffices only to assume that on the first k elements of the chain as $t \to \infty$ a single-period regime establishes itself at the frequency of the external force $\omega_{\text{ext}} = \omega_p$. Since the $(k+1)$st element of the path will then be described by a second-order autonomous equation, whose phase space admits in general only attractors of two types: stable equilibria and limit cycles, it will suffice to prove the absence of limit cycles. This implies the global stability of the synchronization regime on the $(\mathbf{k}+1)$st element of the chain, and by induction—on all other elements. After the substitution

$$b_j(t) = a_j(t) \exp\{i \arg(1-i\beta)\}, \quad \gamma' = \gamma(1+\beta^2)^{-1/2}$$

in agreement with Eq. (7) the oscillation equation of the $(k+1)$st element is written in the form

$$db_{k+1}/dt = (1-i\beta)(b_{k+1} - |b_{k+1}|^2 b_{k+1} + \gamma' b_k^0). \quad (21)$$

Making use of the function

$$H_{k+1} = |b_{k+1}|^2 - {}^1/_2|b_{k+1}|^4 + \gamma'(b_k^0 b_{k+1}^* + b_k^{0*} b_{k+1}),$$

we can write this complex equation in the generalized gradient form:

$$\frac{db_{k+1}}{dt} = (1-i\beta)\frac{\partial H_{k+1}}{\partial b_{k+1}^*}, \quad \frac{db_{k+1}^*}{dt} = (1+i\beta)\frac{\partial H_{k+1}}{\partial b_{k+1}}. \quad (22)$$

The absence of periodic motions in the system (21) follows from the fact that the sign of the time derivative of H_{k+1} does not change sign:

$$\frac{dH_{k+1}}{dt} = \frac{\partial H_{k+1}}{\partial b_{k+1}}\frac{db_{k+1}}{dt} + \frac{\partial H_{k+1}}{\partial b_{k+1}^*}\frac{db_{k+1}^*}{dt} = 2\frac{\partial H_{k+1}}{\partial b_{k+1}}\frac{\partial H_{k+1}}{\partial b_{k+1}^*}$$

$$= 2\left|\frac{\partial H_{k+1}}{\partial b_{k+1}}\right|^2 \geqslant 0$$

(here use was made of the fact that $H_{k+1} = H^*_{k+1}$). Since, as is easily seen, the infinity in Eq. (21) is absolutely unstable in view of the absence of limit cycles, the only stable equilibrium state of this equation is globally stable. Taking into account that the action of the external field (at the frequency $\omega_{ext} = \omega_p$) on the first element of the path is stationary, i.e., $b_0(t) = \text{const}$, we find by induction for $t \to \infty$ that a synchronization regime establishes itself along the path for arbitrary initial conditions.

However, it may take quite long to establish a synchronization regime: it is determined not only by the system parameters, but also by the initial level of excitation of the elements forming the system. Since the oscillations of the elements are not isochronous, a spread in the initial data must soon lead to the establishment of a chaotic regime (for instance, for values of the parameters discussed in the preceding section). The synchronization of this regime, starting from the first element, will gradually take over the elements downstream, i.e., a propagating synchronization front is formed. When the mismatch between the frequency of the external field and the partial oscillation frequency of the elements of the chain is larger than the width of the synchronization band, a regular beat regime will form at the left edge of the path. This regime, like the single-period regime, will take the place of the chaos: a "beat-chaos" transition wave is formed (see Fig. 4). If the duration of the periodic driving signal is finite, then the spatial region of synchronization (in j) will also be finite, i.e., we are dealing with a regular spot on a turbulent background.

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APPENDIX 1

Here we prove the finiteness of the number of period-doublings along the path to chaos on a chain consisting of elements for which the dynamics is described by one-dimensional parabolic mappings[7)]

$$x_j(n+1)=2\delta x_j(n)+2x_j^2(n)+\gamma x_{j-1}(n+1),$$
$$j=1,\ 2,\dots,\quad x_0(n)=0. \qquad \text{(A1)}$$

here n is a discrete time and $\delta > 0$ characterizes the degree of nonequilibrium of the system. Let $\delta \in [0.5, 1.5]$, then on the first element a regular regime of oscillations of period T will be realized [corresponding for $\gamma = 0$ to a fixed point of the mapping (A1)]. In this case the analog of Eq. will be $[x_j^0(n+1) = x_j^0(n)]$

$$x_j^0=2\delta x_j^0+2x_j^{02}+\gamma x_{j-1}^0. \qquad \text{(A2)}$$

For sufficiently small coupling γ and $\delta \lesssim 1.5$ the single-frequency oscillation regime becomes unstable for some j^* (Ref. 9) and along the chain a "period-doubled" regime establishes itself, which later loses its stability, and so on. In order to determine the number of spatial bifurcations of the period-doubling type we make use of the approximate renormalization group scheme of Ref. 11.

The stationary oscillation regimes (of period 2^mT) which are established between the spatial bifurcation points will be close to homogeneous in the case of weak coupling γ. Keeping this in mind, we consider in the framework of Eq. (A1) the behavior of disturbances $\zeta_j(n)$ superposed on a spatially homogeneous $[x_j(n) = x_{j-1}(n)]$ regime of doubled period $[x_j(n+2) = x_j(n)]$ described by the equations

$$x(n+1)=2\delta x(n)+2x^2(n)+\gamma x(n+1),$$
$$x(n)=2\delta x(n+1)+2x^2(n+1)+\gamma x(n). \qquad \text{(A3)}$$

The deviations in which we are interested are described by the equations

$$\zeta_j(n+1)=d\zeta_j(n)+2\zeta_j^2(n)+\gamma\zeta_{j-1}(n+1),$$
$$\zeta_j(n+2)=p\zeta_j(n+1)+2\zeta_j^2(n+1)+\gamma\zeta_{j-1}(n+2), \qquad \text{(A4)}$$

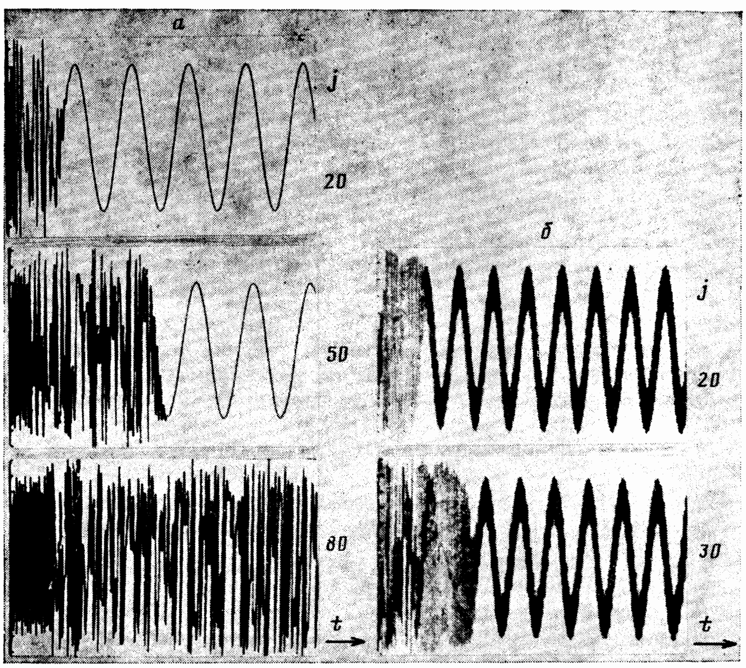

FIG. 4. Waves of phase transitions: a—regular oscillations into chaos [$\gamma = 0.7(1 - 1.71i)$; $\rho = 3.42$, $\varkappa = 0.001$; $a_0 = 0$, $I_0 > 1/2$]; b)—beats into chaos (same parameters, $I_0 < 1/2$).

where $d = 2\delta + 4x(n+1)$, $p = 2\delta + 4x(n)$. To order ζ^2 one can write in place of (A4)

$$\zeta_j(n+2) \approx pd\zeta_j(n) + 2(p+d^2)\zeta_j^2(n) + \gamma[\zeta_{j-1}(n+2) + p\zeta_{j-1}(n+1)]. \quad (A5)$$

Carrying out a scale transformation $y = (p + d^2)\zeta$ and considering ζ_j two time intervals later we are led for y from the equation (A5) again to the original equation (A1), with renormalized constants

$$\delta_{NEW} = pd/2, \gamma_{NEW} = \gamma\{1 - [2\delta + 4x(n)]\} \approx 2\gamma + O(\gamma^2). \quad (A6)$$

Here we have used the solution of Eq. (7):

$$x^0 \approx -{}^1/_2 \left[\left(\frac{1-\gamma}{2} + \delta\right) + \left(\frac{1-\gamma}{2} + \delta\right)^{1/2}\left(\delta - \frac{3(1-\gamma)}{2}\right)^{1/2}\right].$$

Thus, after the doubling the parameters γ and δ cannot change by arbitrarily small increments, accumulating at the critical point—the point where chaos appears. The change of γ and δ according to (A6) means that the accumulation region which contains an infinite number of bifurcations (as well as the point of transition itself) will be "jumped over," i.e., the number of spatial bifurcations where the period doubles along the flow will indeed be finite.

APPENDIX 2

In order to describe the motions (7) after the first spatial bifurcation (for $i > j^*$) we choose the parameters ω_0 and β so that $I_1^0 = \varepsilon/2$ (see Fig. 1), where $0 < \varepsilon \ll 1$. According to Eq. (10) we have

$$\lambda^0 = 2\varepsilon \pm i\Omega, \quad \Omega = [(\omega_0 - 2\beta I_2^0)^2 - (1+\beta^2) I_0^2]^{1/2},$$

i.e., the solution of Eq. (7) will be close to an oscillation at the beat frequency Ω, with a slowly varying amplitude $C_j(t)$:

$$(a_j - a_j^0) = (1+iB)C_j(t)\exp(i\Omega t) + (1+iB^*)C_j^*(t)\exp(-i\Omega t),$$
$$B = (I_1^0 + i\Omega)/(1 - I_1^0)\beta. \quad (A7)$$

After substituting (A7) into Eq. (7) and averaging we obtain

$$\frac{dC_j}{dt} = 2\varepsilon C_j - 2(1 + |B^2|)(1 - i\beta')C_j|C_j|^2 + \gamma C_{j-1} + O(\varepsilon^2),$$
$$\beta' = \beta\left[\frac{3}{2i}\frac{1+|B|^2}{B-B^*} - \frac{i}{2}(B-B^*)/(1+|B|^2)\right].$$

After the substitution $a_{j\mathrm{NEW}} = [(1+|B|^2)/\varepsilon]^{1/2} C_j$, $\gamma_{\mathrm{NEW}} = \gamma/2\varepsilon$, $y_{\mathrm{NEW}} = 2\varepsilon t$ this equation reduces to the original equation (7), i.e., each time it passes through "quasiperiodicity" the coupling parameter increases by a factor $1/2\varepsilon$.

1) We deal only with flow systems (flows) in which there are no external random sources. The number of vanishing exponents corresponds here to the number of incommensurate frequencies in the flow spectrum.[4]

2) On Lyapunov exponents in infinite-dimensional systems, see Ref. 4.

3) For spatial bifurcations there occurs a scaling similar to the one encountered for bifurcations in a point system when a parameter is changed (see, e.g., Ref. 11).

4) The finite dimension of the solution (realization) follows directly from the finiteness of the number of roots in the right half-plane and the boundedness of their absolute value.[4]

5) All non-finite spatial distributions turn out to be unstable on account of the strict dissipativity of the system.

6) For $j \to \infty$ all exponents of the system (9) are infinitely degenerate (an infinite-dimensional system has only two distinct exponents). Taking into account a weak diffusive coupling $\sim \varkappa$ lifts this degeneracy: each exponent $\omega^{0(k)}$ defined by Eq. (10) for $\varkappa \neq 0$ splits into a countable set of exponents which in the complex plane are situated in a strip of width $|\varkappa|^{1/2}$ near $\lambda^{0(k)}$.

7) Parabolic mappings describe the dynamics of the most varied physical systems which exhibit upon variation of a parameter a chain of period-doubling bifurcations.

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