Nonequilibrium distributions of quasiparticles produced by parametric excitation of magnons in an antiferromagnet with a decay spectrum

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A study is made of the quasiparticle interaction processes that limit the amplitude of parametrically excited magnons in an antiferromagnet with a decay spectrum. It is shown that even a small excess above the parametric instability threshold gives rise to a sequence of kinetic instabilities which create several groups of nonequilibrium quasiparticles. A detailed analysis is made of the development of a hierarchy of kinetic instabilities in iron borate under experimental conditions [B. Ya. Kotyuzhanskiĭ and L. A. Prozorova, Sov. Phys. JETP **54**, 1013 (1981); **56**, 903 (1982); **59**, 384 (1984); B. Ya. Kotyuzhanskiĭ, L. A. Prozorova, and L. E. Svistov, Sov. Phys. JETP **59**, 644 (1984)]. Calculations are reported of the dependences of the number and of the spectral width of the distribution of parametrically excited magnons on the pump power and on the intensity of the static magnetic field.

INTRODUCTION

In studies of parametric excitation of waves it is usual to assume that the distribution deviates strongly from equilibrium only well above the threshold. However, in the case of three-wave interactions major changes in the spectral density of waves with frequencies far from the parametric resonance region are possible even near the threshold, and if the excitation is hard—at the very threshold of parametric instability.

These changes are due to secondary instabilities of parametrically excited states above the threshold. A series of such instabilities has the effect that an increase in the pump power results in a transition from a weak nonequilibrium to a highly nonequilibrium (turbulent state) of the wave system when all the degrees of freedom are excited and smooth Kolmogorov-type spectra occur (see Ref. 1 for the spin wave system of a ferromagnet). We shall describe the initial stage of the transition to turbulence in an antiferromagnet with a decay spectrum and show which groups of waves are excited due to parametric magnons and which mechanisms ensure the establishment of a steady state.

The interst in nonlinear processes that appear when spin and acoustic waves are excited in an antiferromagnet is primarily due to the exchange enhancement of the magnetoelastic interaction.[2–4] Anomalously large amplitudes of three-wave magnon-phonon interactions are a consequence of this enhancement.[5,6]

We shall consider an antiferromagnet with an easy-plane anisotropy and a high Néel temperature Θ_N in which the maximum velocity of spin waves s exceeds the velocity c of sound and, therefore, Cherenkov emission of a phonon by a magnon is allowed. The existence of such decay of magnons into phonons gives rise to a number of interesting features of the kinetics of the excited waves. Different aspects of the behavior of magnons and phonons above the threshold under parametric excitation conditions have been investigated experimentally for one of the most striking representatives of high-temperature easy-plane antiferromagnets, namely, iron borate ($FeBO_3$).[7–10]

We shall develop a nonlinear theory of parametric excitation of spin waves in an easy-plane antiferromagnet with a decay spectrum. We shall show that even a slight excess above the parametric instability threshold gives rise to a so-called kinetic instability of parametric spin waves, which results in the creation of intense packets of magnons and of second-generation phonons. We shall discuss in detail a typical case encountered in experiments when the decay into a magnon and a phonon is forbidden in the case of secondary magnons. In this situation such secondary magnons merge with phonons and form a packet of magnons at the parametric spin wave frequency not wider than a packet of parametric spin waves. The interaction of all these packets limits the degree of excitation of waves.

An increase in the pump power makes a packet of secondary magnons unstable against decay into two phonons. The resultant third-generation phonons interact not only with secondary magnons, but also with parametric spin waves, which again limits the number of such waves.

At high pump powers the other types of waves may be excited, the first of these being long-wavelength phonons with wave vectors close to the reciprocal of the size of a sample. Generation of such phonons may be due to an instability either of packets of third-generation phonons or of a packet of parametric spin waves directly.

The positions of various groups of waves excited in an easy-plane antiferromagnet just above the threshold are shown in Fig. 1. Packets of waves of later generations are much wider than the initial packet of parametric spin waves. The transition to fully developed turbulence occurs because of the excitation of new groups of waves and also because of the broadening of the existing packets.

Specific calculations carried out in the present paper for $FeBO_3$ show that because of the hard excitation of parametric spin waves their amplitude at the threshold permits the formation of second-generation waves. An allowance for

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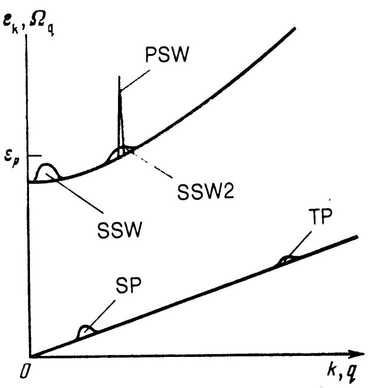

FIG. 1. Nonequilibrium packets of spin waves excited in an easy-plane antiferromagnet. Here, PSW are parametric spin waves, SSW and SP are secondary magnons (spin waves) and phonons; SSW2 is a packet of magnons formed as a result of coalescence of secondary spin waves and secondary phonons; TP are third-generation phonons; $\omega_\mathbf{k}$ and $\Omega_\mathbf{q}$ are the dispersion curves of spin waves and phonons.

this circumstance can explain qualitative features of the state of parametric spin waves above the threshold, viz., linear dependence of the spectral width $\delta\omega$ of a packet of parametric spin waves on the pump amplitude h and a peak in the dependence of $\delta\omega$ on a static magnetic field H; moreover, we can calculate quantitative characteristics of an excited system of spin and acoustic waves. Our theory predicts accumulation of secondary magnons near the bottom of the magnon spectrum in fields $H = H^* \approx 250$ Oe.

The discussion is organized as follows. In § 1 we shall consider the mechanisms of the interaction between parametrically excited magnons and quasiequilibrium magnons and phonons. We shall find smooth deformations of equilibrium magnons and phonon distributions, which result in nonlinear decay of parametric spin waves when the number of these waves is small. In § 2 we shall study the linear and nonlinear stages of the first kinetic instability and determine the susceptibility of a system of parametric spin waves. The second and third kinetic instabilities of nonequilibrium magnons and phonons will be discussed in § 3. The widths of the frequency distributions of excited magnons will be found in § 4.

§ 1. RELAXATION MECHANISMS OF NONEQUILIBRIUM MAGNONS AND PHONONS

1. Kinetic equations

Parametric excitation above the threshold creates a narrow packet of parametric spin waves of frequency $\varepsilon_p = \omega_p/2$ (ω_p is the pump frequency). The presence of parametric spin waves alters the distribution function of quasiparticles which do not participate in a parametric resonance. It is shown in Refs. 11 and 12 that the distribution function of such nonresonant quasiparticles obeys a kinetic equation because of the multimode nature of a system of parametric spin waves. The kinetic equations describing the distributions of magnons $n_\mathbf{k}$ and phonons $N_\mathbf{q}$ are as follows for an easy-plane antiferromagnet with a high Néel temperature:

$$\partial n_\mathbf{k}/\partial t = I_\mathbf{k}^{(1)} + I_\mathbf{k}^{(2)} + I_\mathbf{k}^{(3)}, \qquad (1.1)$$

$$\partial N_\mathbf{q}/\partial t = I_\mathbf{q}^{(4)} + I_\mathbf{q}^{(5)} + I_\mathbf{q}^{(6)}, \qquad (1.2)$$

where in the collision integrals,

$$I_\mathbf{k}^{(1)} = \frac{v_0}{(2\pi)^2}\int |V_{\mathbf{kk}'\mathbf{q}}|^2[(n_\mathbf{k}+1)N_\mathbf{q}n_{\mathbf{k}'} - n_\mathbf{k}(N_\mathbf{q}+1)(n_{\mathbf{k}'}+1)] \times\delta(\varepsilon_\mathbf{k}-\varepsilon_{\mathbf{k}'}-\Omega_\mathbf{q})\,\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})\,d\mathbf{k}'\,d\mathbf{q} \qquad (1.3)$$

describes the processes of decay of magnons into a magnon and a phonon,

$$I_k^{(2)} = \frac{v_0}{(2\pi)^2}\int |V_{\mathbf{k}'\mathbf{kq}}|^2[n_{\mathbf{k}'}(n_\mathbf{k}+1)(N_\mathbf{q}+1)-(n_{\mathbf{k}'}+1)n_\mathbf{k}N_\mathbf{q}] \times\delta(\varepsilon_{\mathbf{k}'}-\varepsilon_\mathbf{k}-\Omega_\mathbf{q})\,\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q})\,d\mathbf{k}'\,d\mathbf{q} \qquad (1.4)$$

describes the coalescence of a magnon and a phonon to form a magnon,

$$I_k^{(3)} = \frac{v_0}{2(2\pi)^2}\int |U_{\mathbf{kqq}'}|^2[(n_\mathbf{k}+1)N_\mathbf{q}N_{\mathbf{q}'}-n_\mathbf{k}(N_\mathbf{q}+1) \times(N_{\mathbf{q}'}+1)]\,\delta(\varepsilon_\mathbf{k}-\Omega_\mathbf{q}-\Omega_{\mathbf{q}'})\,\delta(\mathbf{k}-\mathbf{q}-\mathbf{q}')\,d\mathbf{q}\,d\mathbf{q}' \qquad (1.5)$$

describes the decay of a magnon into two phonons,

$$I_\mathbf{q}^{(4)} = I^{(2)}\{N_\mathbf{q}\leftrightarrow n_\mathbf{k}, \mathbf{q}\leftrightarrow\mathbf{k}\} \qquad (1.6)$$

describes the coalescence of a magnon with a phonon to form a magnon,

$$I_\mathbf{q}^{(5)} = \frac{v_0}{(2\pi)^2}\int |U_{\mathbf{kqq}'}|^2[n_\mathbf{k}(N_\mathbf{q}+1)(N_{\mathbf{q}'}+1)-(n_\mathbf{k}+1) \times N_\mathbf{q}N_{\mathbf{q}'}]\,\delta(\varepsilon_\mathbf{k}-\Omega_\mathbf{q}-\Omega_{\mathbf{q}'})\,\delta(\mathbf{k}-\mathbf{q}-\mathbf{q}')\,d\mathbf{q}'\,d\mathbf{k} \qquad (1.7)$$

corresponds to the coalescence of two phonons into a magnon,

$$I_\mathbf{q}^{(6)} = 2\frac{v_0}{(2\pi)^2}\int \{|W_{\mathbf{qq}'\mathbf{q}''}|^2[(N_\mathbf{q}+1)N_{\mathbf{q}'}N_{\mathbf{q}''}-N_\mathbf{q}(N_{\mathbf{q}'}+1) \times(N_{\mathbf{q}''}+1)]\,\delta(\Omega_\mathbf{q}-\Omega_{\mathbf{q}'}-\Omega_{\mathbf{q}''})\,\delta(\mathbf{q}-\mathbf{q}'-\mathbf{q}'')+2|W_{\mathbf{q}'\mathbf{qq}''}|^2 \times[(N_\mathbf{q}+1)(N_{\mathbf{q}''}+1)N_{\mathbf{q}'}-N_\mathbf{q}N_{\mathbf{q}''}(N_{\mathbf{q}'}+1)]\,\delta(\Omega_{\mathbf{q}'}-\Omega_\mathbf{q}-\Omega_{\mathbf{q}''}) \times\delta(\mathbf{q}'-\mathbf{q}-\mathbf{q}'')\}\,d\mathbf{q}'\,d\mathbf{q}'' \qquad (1.8)$$

represents a three-phonon interaction (decay and coalescence). Here,

$$V_{\mathbf{kk}'\mathbf{q}} = 2\Theta\left(\frac{\Omega_\mathbf{q}}{2Mc^2}\right)^{1/2}\frac{J_0}{(\varepsilon_\mathbf{k}\varepsilon_{\mathbf{k}'})^{1/2}}\varphi(\mathbf{e},\mathbf{n}) \qquad (1.9)$$

is the amplitude of the interaction of two magnons with a phonon,[5,6]

$$U_{\mathbf{kqq}'} = i\sqrt{2}J_0\left(\frac{\Omega_\mathbf{q}\Omega_{\mathbf{q}'}J_0}{M^2c_1^2c_2^2\varepsilon_\mathbf{k}}\right)^{1/2} \times\left[\frac{\Theta^2\varphi(\mathbf{q})\varphi'(\mathbf{q}')}{\varepsilon_\mathbf{q}^2-\Omega_\mathbf{q}^2}+\frac{\Theta^2\varphi'(\mathbf{q})\varphi(\mathbf{q}')}{\varepsilon_{\mathbf{q}'}^2-\Omega_{\mathbf{q}'}^2}\right] \qquad (1.10)$$

is the amplitude of the interaction of a magnon and two phonons,[13]

$$W_{\mathbf{qq}'\mathbf{q}''} = -2\sqrt{2}\,\Theta\left(\frac{\Omega_1\Omega_2\Omega_3}{M^3c_1^2c_2^2c_3^2}\right)^{1/2}\Theta^2J_0^2 \times\left[\frac{\varphi(\mathbf{q}_1)\varphi'(\mathbf{q}_2)\varphi'(\mathbf{q}_3)}{(\varepsilon_2^2-\Omega_2^2)(\varepsilon_3^2-\Omega_3^2)}+(2\leftrightarrow1)+(3\leftrightarrow1)\right] \qquad (1.11)$$

is the amplitude of an effective three-phonon interaction,[4,6]

$$\varphi(\mathbf{e},\mathbf{n}) = e_xn_x-e_yn_y+\xi(e_yn_z+e_zn_y), \qquad (1.12a)$$

$$\varphi'(\mathbf{e},\mathbf{n}) = e_xn_y+e_yn_x+\xi(e_xn_z+e_zn_x) \qquad (1.12b)$$

are functions dependent on the direction of the phonon propagation $\mathbf{n} = \mathbf{q}/q$, on its polarization vector $\mathbf{e}$, and on the ratio of the magnetoelastic constants $\tilde{\xi}$,

$$\varepsilon_k=(\varepsilon_0^2+s^2k^2)^{1/2}, \quad \Omega_q=cq \tag{1.13}$$

are the magnon and phonon frequencies, ε_0 is the antiferromagnetic resonance (AFMR) frequency, $J_0 = \omega_E/S$, ω_E is the exchange frequency, S is the spin of an ion, Θ is the magnetoelastic energy, and v_0 and $M = \rho v_0$ are the volume and mass of a unit cell.

2. Influence of parametric spin waves on thermal magnons and phonons

The occupation numbers of magnons and phonons can be represented in the form

$$n_\mathbf{k}=n_\mathbf{k}^0+n_\mathbf{k}^p+\delta n_\mathbf{k}, \quad N_\mathbf{q}=N_\mathbf{q}^0+\delta N_\mathbf{q}, \tag{1.14}$$

where $n_\mathbf{k}^0$ and $N_\mathbf{q}^0$ are the equilibrium distributions of quasiparticles:

$$n_\mathbf{k}^0=[\exp(\varepsilon_k/T)-1]^{-1}, \quad N_\mathbf{q}^0=[\exp(\Omega_q/T)-1]^{-1}, \tag{1.15}$$

$n_\mathbf{k}^p$ is the distribution function of parametric magnons in $\mathbf{k}$ space, which to first order can be regarded as singular

$$n_\mathbf{k}^p=2\pi^2\frac{\Theta_N^3}{\varepsilon_p sk_p}n^p\delta(\varepsilon_k-\varepsilon_p),$$
$$n^p=\frac{v_0}{(2\pi)^3}\int n_\mathbf{k}^p d\mathbf{k}, \quad \Theta_N^3=\frac{s^3}{v_0}, \tag{1.16}$$

n^p is the total number of parametric spin waves per unit cell, and $\delta n_\mathbf{k}$ and $\delta N_\mathbf{q}$ are perturbations of the distribution functions caused by the interaction of waves with parametric waves.

If the number of parametric spin waves is sufficiently small, then the quantities $\delta n_\mathbf{k}$ and $\delta N_\mathbf{q}$ are also small and they can be found from the linearized kinetic equations

$$2\gamma_\mathbf{k}^0\delta n_\mathbf{k}+J_\mathbf{k}^{(m)}\{\delta N_\mathbf{q}\}=r_\mathbf{k}\{n_\mathbf{k}^p\},$$
$$2\Gamma_\mathbf{q}^0\delta N_\mathbf{q}+J_\mathbf{q}^{(ph)}\{\delta n_\mathbf{k}\}=R_\mathbf{q}\{n_\mathbf{k}^p\}. \tag{1.17}$$

Here, $\gamma_\mathbf{k}^0$ and $\Gamma_\mathbf{q}^0$ are the equilibrium damping factors of magnons and phonons, respectively, $J_\mathbf{k}^{(m)}$ and $J_\mathbf{q}^{(ph)}$ are the integral operators of the linearized kinetic equation, and $r_\mathbf{k}$ and $R_\mathbf{q}$ are the terms due to the presence of a packet of parametric spin waves.

We can seek the solution of the system (1.17) in the form of a series

$$\delta n_\mathbf{k}=\frac{r_\mathbf{k}}{2\gamma_\mathbf{k}^0}-\frac{1}{2\gamma_\mathbf{k}^0}J_\mathbf{k}^{(m)}\left\{\frac{R_\mathbf{q}}{2\Gamma_\mathbf{q}^0}\right\}+\ldots,$$
$$\delta N_\mathbf{q}=\frac{R_\mathbf{q}}{2\Gamma_\mathbf{q}^0}-\frac{1}{2\Gamma_\mathbf{q}^0}J_\mathbf{q}^{(ph)}\left\{\frac{r_\mathbf{k}}{2\gamma_\mathbf{k}^0}\right\}+\ldots. \tag{1.18}$$

If the region of the phase space (defined by the laws of conservation) containing particles responsible for the relaxation of the perturbations $\delta n_\mathbf{k}$ and $\delta N_\mathbf{q}$ is greater than the region where the right-hand sides of $r_\mathbf{k}$ and $R_\mathbf{q}$ in the system (1.17) are large, then the series of Eq. (1.18) converge and they can be terminated at the first term:

$$\delta n_\mathbf{k}=r_\mathbf{k}/2\gamma_\mathbf{k}^0, \quad \delta N_\mathbf{q}=R_\mathbf{q}/2\Gamma_\mathbf{q}^0. \tag{1.19}$$

The strongest of the interactions described by Eqs. (1.9)–(1.11) is the first, whereas the others are weaker by factors

$$\Lambda_\mathbf{q}'=(\zeta_\mathbf{q}'\Omega_\mathbf{q}'/\varepsilon_\mathbf{q}')^{1/2} \text{ and } \Lambda_\mathbf{q}'\Lambda_\mathbf{q}''. \tag{1.20}$$

Here,

$$\zeta_{\mathbf{q}'}=\Theta^2J_0/\varepsilon_{\mathbf{q}'}^2Mc^2 \tag{1.21}$$

is a dimensionless coupling constant which in the limit $\mathbf{q}'\to 0$ ($\zeta_{q'}\to\zeta$) defines the velocity of sound in an antiferromagnet renormalized by a magnetic field: $c(H) = c(\infty)(1-\zeta)^{1/2}$. Therefore, in the expressions for $r_\mathbf{k}$ and $R_\mathbf{q}$ it is sufficient to retain only the principal terms:

$$r_\mathbf{k}=2\pi\alpha^2\zeta_\mathbf{k}J_0T(\varepsilon_p-\varepsilon_k)n^p/sksk_p, \quad \varepsilon_1\leqslant\varepsilon_k\leqslant\varepsilon_p, \tag{1.22}$$
$$R_\mathbf{q}=2\pi\alpha^{-1}\zeta J_0T\varepsilon_0^2n^p/sk_p\Omega_q(\varepsilon_p-\Omega_q), \quad 0\leqslant\Omega_q\leqslant\varepsilon_p-\varepsilon_1, \tag{1.23}$$
$$r_\mathbf{k}=2\pi\alpha^2\zeta_\mathbf{k}J_0T(\varepsilon_k-\varepsilon_p)n^p/sksk_p, \quad \varepsilon_2\leqslant\varepsilon_k\leqslant\varepsilon_3, \tag{1.24}$$
$$R_\mathbf{q}=2\pi\alpha^{-1}\zeta J_0T\varepsilon_0^2n^p/sk_p\Omega_q(\varepsilon_p+\Omega_q), \quad 0\leqslant\Omega_q\leqslant\varepsilon_2-\varepsilon_p, \tag{1.25}$$

where

$$\alpha=s/c, \quad \varepsilon_i=(\varepsilon_0^2+s^2k_i^2)^{1/2} \quad (i=1,2,3),$$
$$sk_1=|sk_p-2(\alpha^2-1)^{-1}(\alpha\varepsilon_p-sk_p)|,$$
$$sk_2=\max\{sk_p, (\alpha^2-1)^{-1}[2\alpha\varepsilon_p-(\alpha^2+1)sk_p]\},$$
$$sk_3=(\alpha^2-1)^{-1}[2\alpha\varepsilon_p+(\alpha^2+1)sk_p]. \tag{1.26}$$

If these inequalities are not obeyed, then $r_\mathbf{k}$ and $R_\mathbf{q}$ given by Eqs. (1.22)–(1.25) vanish.

Here and below we shall assume that the magnon spectrum and the elastic subsystem of the investigated crystal are isotropic. Summation over the polarizations of phonons and integration with respect to the angles of the functions φ, φ' give rise to a numerical coefficient of the order of unity, which we shall ignore. It is clear from Eqs. (1.22)–(1.25) that the quantity $R_\mathbf{q}$ is α^3 times less than $r_\mathbf{k}$, so that perturbations of the phonon distribution function are relatively small (in the case of $FeBO_3$, the parameter is $\alpha\approx 3\times10^{-2}$).

3. Nonlinear damping of parametric spin waves with low amplitudes

The presence of nonequilibrium secondary magnons and phonons in accordance with Eq. (1.19) gives rise to, e.g., additional damping of parametrically excited spin waves $\delta\gamma_k^p = \delta\gamma_1 + \delta\gamma_2$, where

$$\delta\gamma_1=\frac{v_0}{8\pi^2}\int|V_{\mathbf{k}\mathbf{k}'\mathbf{q}}|^2(\delta n_{\mathbf{k}'}+\delta N_\mathbf{q})\delta(\varepsilon_k-\varepsilon_{k'}-\Omega_q)\times\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})d\mathbf{k}'\,d\mathbf{q}, \tag{1.27}$$

$$\delta\gamma_2=\frac{v_0}{8\pi^2}\int|V_{\mathbf{k}'\mathbf{k}\mathbf{q}}|^2(\delta N_\mathbf{q}-\delta n_{\mathbf{k}'})\delta(\varepsilon_{k'}-\varepsilon_k-\Omega_q)\times\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q})d\mathbf{k}'\,d\mathbf{q}. \tag{1.28}$$

The substitution of Eqs. (1.22) and (1.23) into the expressions for the damping $\delta\gamma_1$ and $\delta\gamma_2$ gives

$$\delta\gamma_k^p=\alpha^4\zeta^2\frac{J_0^2T\varepsilon_0^5n^p}{2\Theta_N^3\varepsilon_p(sk_p)^2\bar{\gamma}}\left[f\left(\frac{\varepsilon_1}{\varepsilon_p}\right)-f\left(\frac{\varepsilon_3}{\varepsilon_p}\right)\right],$$

$$f(x)=\left(\frac{\varepsilon_p}{\varepsilon_0}\right)^2\left\{\frac{(x^2-1)^{1/2}}{x}-6\,\mathrm{arctg}[x+(x^2-1)^{1/2}]\right\}$$
$$+3\frac{\varepsilon_p}{\varepsilon_0}\ln[x+(x^2-1)^{1/2}]-(x^2-1)^{1/2},\qquad(1.29)$$

where $\bar{\gamma}$ is the damping γ_k^0 averaged over a packet of secondary magnons. It should be pointed out that the contribution to the damping $\delta\gamma_1$ due to decay of parametric spin waves is positive, whereas $\delta\gamma_2$ due to coalescence of secondary magnons with parametric spin waves is negative, and the two contributions have similar absolute values. In particular, if $\varepsilon_3-\varepsilon_1\ll\varepsilon_p$, then

$$\delta\gamma_k^p\approx-[1-(1+4\alpha^{-2})^{1/2}]^3\alpha^4\zeta^2J_0T\varepsilon_0^4n^p/\varepsilon_p sk_p\bar{\gamma}.\qquad(1.30)$$

It therefore follows that in the case of $FeBO_3$ the nonlinear damping of Eq. (1.30) corresponding to small amplitudes of parametric spin waves is negative and is two orders of magnitude less than the result given in Ref. 14, which only allowed for the damping $\delta\gamma_1$.

Consequently, in calculating the nonlinear damping of parametric spin waves we must allow also for the three-wave process of decay of a magnon into two phonons. In spite of the relatively small amplitude of this process, it is important because there is no contribution which would balance it (the process of coalescence of a magnon with a phonon to form a phonon is forbidden by the laws of conservation). Calculations carried out in accordance with Eq. (1.28) give

$$\delta\gamma_3=\frac{\alpha^4}{4}\zeta^4\frac{J_0^2T}{\Theta_N^3}\frac{\varepsilon_p^3}{sk_p\Gamma}n^p\left[1+\left(\frac{\varepsilon_p}{2\varepsilon_0}\right)^2(\alpha^2-1)\right]^{-4}.\qquad(1.31)$$

In the case of an experimentally investigated situation[7-10] the quantities $\delta\gamma_1+\delta\gamma_2$ and $\delta\gamma_3$ are of the same order of magnitude.

§ 2. FIRST KINETIC INSTABILITY IN A MAGNON-PHONON SYSTEM AND MECHANISMS LIMITING IT

1. First kinetic instability

An increase in the number of parametric spin waves n^p enhances their influence on thermal magnons and phonons, and these effects cannot be allowed for in the approximation which is linear in the number of parametric spin waves. If n^p is not small compared with the number of thermal waves interacting effectively with parametric spin waves, the influence of spin waves must be allowed for exactly. If parametric spin waves are in the decay part of the spectrum, perturbations of the distribution function

$$\delta n_k=2\pi^2\frac{\Theta_N^3\delta n(\varepsilon_k)}{\varepsilon_k(\varepsilon_k^2-\varepsilon_0^2)^{1/2}},\quad \delta N_q=2\pi^2\frac{\Theta_N^3}{\alpha^3\Omega_q^2}\delta N(\Omega_q)\quad(2.1)$$

are described by the following system of equations obtained from Eqs. (1.1) and (1.2):

$$(\gamma_k^0-\Delta\gamma_k)\delta n(\varepsilon_k)=\alpha^{-3}D\delta N(\varepsilon_p-\varepsilon_k)+r(\varepsilon_k),$$
$$(\Gamma_q^0-\Delta\Gamma_q)\delta N(\varepsilon_p-\varepsilon_k)=D(\varepsilon_p-\varepsilon_k)^2\delta n(\varepsilon_k)/\varepsilon_k sk+R(\varepsilon_p-\varepsilon_k),\qquad(2.2)$$

where

$$\Delta\gamma_k=\pi\zeta\alpha^2J_0\varepsilon_0^2(\varepsilon_p-\varepsilon_k)^2n^p/\varepsilon_p sk_p\varepsilon_k sk,$$
$$\Delta\Gamma_q=\Delta\gamma_k\varepsilon_k sk/\alpha^3(\varepsilon_p-\varepsilon_k)^2,\quad D=\pi\zeta\alpha^2J_0\varepsilon_0^2n^p/\varepsilon_p sk_p,\qquad(2.3)$$

γ_k^0 and Γ_q^0 are the equilibrium damping factors of magnons and phonons. We then have

$$n^p=n_{cr}^p=\frac{\varepsilon_p sk_p}{\pi\alpha^2\zeta J_0\varepsilon_0^2}\frac{\gamma_k^0\Gamma_q^0}{\gamma_k^0\alpha^{-3}+\Gamma_q^0(\varepsilon_p-\varepsilon_k)^2/\varepsilon_k sk}.\qquad(2.4)$$

The determinant of the system (2.2) vanishes and a kinetic instability of parametric spin waves develops[15-17] (by analogy with the Suhl parametric processes, this is a first-order kinetic instability). In the case of $FeBO_3$ the relaxation frequencies of magnons and phonons are of the same order of magnitude, $\alpha^{-3}\ll1$, so that

$$n_{cr}^p\approx\varepsilon_p\varepsilon_k sk_p sk\gamma_k^0/\pi\alpha^2\zeta J_0\varepsilon_0^2(\varepsilon_p-\varepsilon_k)^2,\qquad(2.5)$$

which corresponds to vanishing of the combined damping of magnons $\gamma_k^0-\Delta\gamma_k$. It follows from Eq. (2.5) that the kinetic instability threshold is lowest for the smallest k of secondary magnons allowed by the laws of conservation. We can easily show that in the case of decay of a parametric spin wave into a magnon and a phonon, we have

$$sk_{min}=|sk_p-2(\alpha^2-1)^{-1}(\alpha\varepsilon_p-sk_p)|.\qquad(2.6)$$

Under the experimental conditions of Ref. 10, we obtain

$$n_{cr}^p\approx2\cdot10^{-8}sk_p sk_{min}/(\varepsilon_p-\varepsilon_{kmin})^2.\qquad(2.7)$$

The factor $k_pk_{min}(\varepsilon_p-\varepsilon_{kmin})^{-2}$ depends on the applied magnetic field and ranges from $(\alpha^2-1)/4\approx2$ in a field $H=H_c$ ($k_p=0$) to zero when $H=240$ Oe. If in our estimates we use the experimental value of the nonlinear damping factor,[9] we find that the kinetic instability threshold $n_{cr}\approx2\times10^{-8}$ is attained for $(h/h_c)-1\approx0.1$.

2. Mechanisms limiting the first kinetic instability

We shall assume that secondary magnons formed as a result of decay of parametric spin waves have a frequency which is not too far from ζ and, therefore, that they cannot decay into a magnon and a phonon. In this case the main mechanism for relaxation of secondary spin waves with respect to the magnetostriction parameter ζ is the process in which they coalesce with thermal and secondary phonons.

2.1. We shall now determine the contribution made to the damping of secondary spin waves by the processes of coalescence with thermal phonons. The kinetic equation for phonons disturbed from equilibrium by a packet of secondary spin waves is

$$\frac{\partial}{\partial t}\delta N_q+2(\Gamma_q^0+\mu n^s)\delta N_q$$
$$=-2\frac{T\varepsilon_s}{\Omega_q(\varepsilon_s+\Omega_q)}\mu n^s,\quad \mu=\frac{\pi\zeta J_0\varepsilon_0^2}{\alpha\varepsilon_s sk_s},\qquad(2.8)$$

where ε_s, k_s, and n^s are the frequency, wave vector, and number of secondary spin waves. Similarly, the deviation from the magnon occupation number δn_k is described by

$$\frac{\partial}{\partial t}\delta n_k+2(\gamma_k^0+\mu_1(\mathbf{k})n^s]\delta n_k=2\frac{T\varepsilon_s}{\varepsilon_k(\varepsilon_k-\varepsilon_s)}\mu_1(\mathbf{k})n^s,$$
$$\mu_1(\mathbf{k})=\pi\alpha^2\zeta J_0\varepsilon_0^2(\varepsilon_k-\varepsilon_s)^2/\varepsilon_k sk\varepsilon_s sk_s.\qquad(2.9)$$

Consequently, the contribution to the damping of secondary spin waves made by the perturbations $\delta N_{\mathbf{q}}$ and $\delta n_{\mathbf{k}}$ is

$$\delta\gamma_s=\delta\gamma_s(\mathbf{k}_s)=-\frac{\alpha^2\zeta J_0T\varepsilon_0^2}{2\pi\Theta_N^3 sk_s}\int_{\Omega_1}^{\Omega_2}\frac{\Omega}{\varepsilon'}\left[\frac{\mu n^s}{\Gamma(\Omega)+\mu n^s}+\frac{\mu_1(\varepsilon')n^s}{\gamma(\varepsilon')+\mu_1(\varepsilon')n^s}\right]\delta(\varepsilon'-\varepsilon_s-\Omega)\,d\varepsilon'\,d\Omega, \quad (2.10)$$

$$\Omega_{1,2}=2(\varepsilon_s\pm\alpha sk_s)/(\alpha^2-1).$$

In the case of small wave vectors of secondary spin waves ($\alpha sk_s\ll\varepsilon_k$), we have

$$\delta\gamma_s=-\frac{4\alpha^3\zeta J_0T\varepsilon_0^2}{\pi(\alpha^4-1)\Theta_N^3}\left[\frac{\mu n^s}{\Gamma(\bar\Omega)+\mu n^s}+\frac{\mu_1(\bar\varepsilon)n^s}{\gamma(\bar\varepsilon)+\mu_1(\bar\varepsilon)n^s}\right],$$

$$\bar\Omega=2\varepsilon_s/(\alpha^2-1),\quad \bar\varepsilon=\varepsilon_s+\bar\Omega. \quad (2.11)$$

This contribution to the damping is negative and it saturates at n^s of the order of the threshold intensity of parametric spin waves of Eq. (2.5), necessary for the excitation of the kinetic instability. Therefore, this mechanism does not limit the instability.

2.2. We shall now consider the processes of interaction of secondary spin waves with nonequilibrium secondary phonons, which appear because of the development of a kinetic instability. We shall assume that a packet of secondary spin waves is sufficiently narrow so that $\Delta\varepsilon_s/\varepsilon_s\ll1$, but wide compared with a packet of parametric spin waves ($\Delta\varepsilon_s\gg v\Delta k_p$). We shall demonstrate the validity of these assumptions below (§ 4). It is clear from Eq. (2.2) that the width of a secondary phonon packet is equal to the width of a packet of secondary spin waves. Secondary spin waves merge with secondary phonons to form a packet of nonequilibrium magnons $\tilde n_{\mathbf{k}}^s$ with a frequency close to ε_p and a width $\Delta\varepsilon_s$ (we shall call it SSW2). Superposed on this packet is a narrower packet of parametric spin waves. Therefore, above the threshold of the first kinetic instability there are four coupled magnon and phonon packets: parametrically excited spin waves, secondary spin waves, secondary phonons, and SSW2, representing the product of coalescence of secondary spin waves with secondary phonons. The narrowest is the packet of parametric spin waves: its width on the frequency scale less than the damping factor $\gamma_{\mathbf{k}}$. The distribution function of parametric spin waves

$$n_{\mathbf{k}}^p=\int n_{\mathbf{k},\omega}^p\,d\omega \quad (2.12)$$

is described by

$$n_{\mathbf{k}}^p=\frac{\pi|\Pi_{\mathbf{k}}|^2\Phi_{\mathbf{k}}}{\gamma_{\mathbf{k}}[\nu_{\mathbf{k}}^2+(\varepsilon_{\mathbf{k}}-\varepsilon_p)^2]},$$

$$n_p=\frac{v_0}{(2\pi)^3}\int n_{\mathbf{k}}^p\,d\mathbf{k}=\frac{|\Pi|^2\varepsilon_p sk_p\Phi}{\Theta_N^3\gamma_p\nu}, \quad (2.13)$$

where $\Pi_{\mathbf{k}}$ is the total pumping (in our case $\Pi_{\mathbf{k}}=hV_{\mathbf{k}}$),

$$\nu_{\mathbf{k}}^2=\gamma_{\mathbf{k}}^2-|\Pi_{\mathbf{k}}|^2,\quad \Phi_{\mathbf{k}}=\Phi_{\mathbf{k}}^0+\tilde\Phi_{\mathbf{k}}, \quad (2.14)$$

$$\pi\Phi_{\mathbf{k}}^0=\gamma_{\mathbf{k}}^0n_{\mathbf{k}}^0,\quad \tilde\Phi_{\mathbf{k}}=\frac{v_0}{(2\pi)^3}\int|V_{\mathbf{kk}'\mathbf{q}}|^2n_{\mathbf{k}'}^sN_{\mathbf{q}}^s\delta(\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{k}'}-\Omega_{\mathbf{q}})\,\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})\,d\mathbf{k}\,d\mathbf{q}. \quad (2.15)$$

The incoming term $\Phi_{\mathbf{k}}^0$ is due to thermal noise, whereas $\tilde\Phi_{\mathbf{k}}$ is due to the interaction with magnons and phonons of the second generation. In the case of integral numbers of secondary spin waves, secondary phonons, and SSW2, it follows from the kinetic equations (1.1)–(1.4) and (1.6) that

$$\gamma_s^0n^s=\pi\alpha^2\zeta J_0\varepsilon_0^2\Omega_s^2\times\left[\frac{n^s(n^p+\tilde n^s)}{\varepsilon_ssk_s\varepsilon_psk_p}-\frac{n^sN^s}{\alpha^3\Omega_s^2\varepsilon_ssk_s}+\frac{N^s(n^p+\tilde n^s)}{\alpha^3\Omega_s^2\varepsilon_psk_p}\right], \quad (2.16)$$

$$\Gamma_s^0N^s=\pi\alpha^2\zeta J_0\varepsilon_0^2\Omega_s^2\times\left[\frac{n^s(n^p+\tilde n^s)}{\varepsilon_ssk_s\varepsilon_psk_p}-\frac{n^sN^s}{\alpha^3\Omega_s^2\varepsilon_ssk_s}+\frac{N^s(n^p+\tilde n^s)}{\alpha^3\Omega_s^2\varepsilon_psk_p}\right], \quad (2.17)$$

$$\tilde\gamma\tilde n^s=\pi\alpha^2\zeta J_0\varepsilon_0^2\Omega_s^2\times\left[\frac{n^sN^s}{\alpha^3\Omega_s^2\varepsilon_ssk_s}-\frac{n^s\tilde n^s}{\varepsilon_ssk_s\varepsilon_psk_p}-\frac{N^s\tilde n^s}{\alpha^3\Omega_s^2\varepsilon_psk_p}\right]. \quad (2.18)$$

Here, γ_s^0, Γ_s^0, and $\tilde\gamma$ are the equilibrium damping factors of secondary spin waves, secondary phonons, and SSW2. The damping factor of parametric spin waves is the sum of the equilibrium factor γ_p^0 and the contributions due to secondary spin waves and secondary phonons:

$$\gamma_p=\gamma_p^0+\pi\alpha^2\zeta J_0\frac{\varepsilon_0^2\Omega_s^2}{\varepsilon_psk_p}\left(\frac{n^s}{\varepsilon_ssk_s}+\frac{N^s}{\alpha^3\Omega_s^2}\right). \quad (2.19)$$

Equations (2.16) and (2.17) yield the relationship

$$\gamma_s^0n^s=\Gamma_s^0N^s, \quad (2.20)$$

which demonstrates that one secondary magnon and one secondary phonon participate in the coalescence processes.

It should be pointed out that we are allowing here for just one mechanism for limiting the amplitude of parametric spin waves, namely, an increase in their damping because of the scattering by secondary spin waves and secondary phonons. Consequently, within the framework of the system of equations (2.16)–(2.19) the kinetic instability threshold is identical with the parametric excitation threshold.

It follows from Eqs. (2.13) and (2.14) that if

$$\nu=\gamma_p\varepsilon_psk_p\Phi/2\Theta_N^3n^p \quad (2.21)$$

is small compared with γ_p, then the condition for the balance of energy in the case of parametric spin waves can be represented in the form

$$|hV|=\gamma_p, \quad (2.22)$$

and this equation closes the system (2.16)–(2.19).

The solution of this system of equations is

$$n^s=\frac{\gamma_p(\xi-1)}{\lambda(a+1)},\quad \tilde n^s=\frac{\lambda(n^s)^2}{\gamma_p^0\xi},\quad N^s=\frac{n^s\gamma_s^0}{\Gamma_s^0},$$

$$n^p=\frac{1}{\lambda(1+a)}\left[\gamma_s^0+\frac{b}{1+a}\gamma_p^0\frac{\xi-1}{\xi}\right], \quad (2.23)$$

where

$$a=\frac{\gamma_s^0}{\Gamma_s^0}\frac{\varepsilon_ssk_s}{\alpha^3(\varepsilon_p-\varepsilon_s)^2},\quad b=\frac{\gamma_s^0}{\Gamma_s^0}\frac{\varepsilon_psk_p}{\alpha^3(\varepsilon_p-\varepsilon_s)^2},$$

$$\xi=\frac{|hV|}{\gamma_p^0},\quad \lambda=\frac{\pi\alpha^2\zeta J_0\varepsilon_0^2(\varepsilon_p-\varepsilon_s)^2}{\varepsilon_psk_p\varepsilon_ssk_s}. \quad (2.24)$$

It is clear from Eq. (2.23) that just above the kinetic instability threshold the number of parametric spin waves rises linearly with the excess above the threshold. An increase in the pump amplitude saturates the dependence $n^p(\xi)$. The coefficient of proportionality in the dependence of n^p on $(\xi-1)/\xi$ under the experimental conditions of Refs. 7–10 is 10^{-7}–10^{-6}, which is close to the results of measurements reported in Ref. 9. The dependence of this coefficient on the temperature of the crystal and on the magnetic field H are also in good agreement with the experimental results.

2.3. Decay of secondary spin waves into two phonons. This decay can also contribute to the relaxation of secondary spin waves. If the frequency of the secondary spin waves is close to ε_0, then the phonons which appear as a result of decay of these waves have frequencies close to $\varepsilon_0/2$. The change in the distribution function of phonons in these processes is

$$\delta N_{\mathbf{q}}=\frac{64\zeta^2}{(\alpha^2+3)^2}\frac{\varepsilon_0{}^2\Omega}{\varepsilon_s sk_s\Gamma(\varepsilon_0/2)}\frac{TJ_0(\varepsilon_s-\varepsilon_0)}{\Theta_N{}^3}n^s. \qquad (2.25)$$

Consequently, the contribution to the damping because of decay of secondary spin waves into two phonons is

$$\delta\gamma_s{}^{(2)}=64\left(\frac{\alpha}{\alpha^2+3}\right)^4\zeta^4\frac{TJ_0{}^2}{\Theta_N{}^3}\frac{\varepsilon_s{}^2}{sk_s\Gamma(\varepsilon_0/2)}n^s. \qquad (2.26)$$

It follows that the processes of decay of secondary spin waves into two phonons give rise to nonlinear damping of these waves in addition to the processes SSW + SP→SSW2, where SSW represents second spin waves and SP represents secondary phonons which set the limit on the kinetic instability. An allowance for just these processes yields the following dependence of the number of parametric spin waves on the pump amplitude:

$$n^p=\frac{64}{\pi^2}\frac{\zeta^2}{(\alpha^2+3)^4}\frac{\gamma_p{}^0}{\Gamma^0(\varepsilon_0/2)}\frac{(\varepsilon_p sk_p)^2(\varepsilon_s sk_s)T}{(\varepsilon_p-\varepsilon_s)^2\Theta_N{}^3\varepsilon_0{}^2}\left(\frac{\varepsilon_s}{\varepsilon_0}\right)^4(\xi-1). \qquad (2.27)$$

Comparing Eq. (2.27) with the expression for n^p given by Eq. (2.23), which is obtained allowing for the interaction of packets of secondary spin waves, secondary phonons, and parametric spin waves, we can see that the processes of decay of secondary spin waves into two phonons are two or three orders of magnitude less effective. Therefore, the main mechanism which limits the kinetic instability is the interaction of four packets: parametric spin waves, secondary spin waves, secondary phonons, and SSW2.

3. Position of a packet of secondary spin waves in k space

In the derivation of Eqs. (2.16)–(2.19) we have assumed that a packet of secondary spin waves is narrow in **k** space, i.e., that $\Delta k_s \ll k_s$. We shall show that the relaxation mechanisms discussed above do indeed give rise to a narrow packet of secondary spin waves. We shall do this by considering the behavior in the **k** space of the total magnon damping:

$$\gamma_{\mathbf{k}}=\gamma_{\mathbf{k}}{}^0-\Delta\gamma_{\mathbf{k}}+\delta\gamma_{\mathbf{k}}. \qquad (2.28)$$

A steady equilibrium state of magnons is stable when

$$\gamma_{\mathbf{k}}\geqslant 0, \qquad (2.29)$$

where $n_{\mathbf{k}}\neq 0$ in those parts of **k** space where[18]

$$\gamma_{\mathbf{k}}=0. \qquad (2.30)$$

We can easily show that the quantity $\gamma_{\mathbf{k}}$ of Eq. (2.28) calculated by us has its minimum value at $k=k_{\min}$ of Eq. (2.6), and at this point we have

$$\partial\gamma_k/\partial k\approx\gamma_k{}^0/k>0. \qquad (2.31)$$

Consequently, a packet of secondary spin waves is concentrated near the $k=k_s=k_{\min}$ surface in the **k** space.

§ 3. SECOND AND THIRD KINETIC INSTABILITIES

1. Second kinetic instability

An increase in the number of secondary spin waves reduces the damping of phonons of frequency close to $\varepsilon_s/2$. Perturbations of the phonon distribution function at frequencies $\Omega_{\mathbf{q}}$ and $\Omega_{\mathbf{q}'}=\varepsilon_s-\Omega_{\mathbf{q}}$ are described by the following coupled systems of linearized kinetic equations:

$$\begin{aligned}&{}^1/_2\partial\delta N_{\mathbf{q}}/\partial t+(\Gamma_{\mathbf{q}}{}^0-\Delta\Gamma_{\mathbf{q}})\delta N_{\mathbf{q}}-\Delta\Gamma_{\mathbf{q}}\delta N_{\mathbf{q}'}=0,\\&{}^1/_2\partial\delta N_{\mathbf{q}'}/\partial t+(\Gamma_{\mathbf{q}'}{}^0-\Delta\Gamma_{\mathbf{q}'})\delta N_{\mathbf{q}'}-\Delta\Gamma_{\mathbf{q}'}\delta N_{\mathbf{q}}=0,\end{aligned} \qquad (3.1)$$

where

$$\Delta\Gamma_{\mathbf{q}}=\frac{v_0}{(2\pi)^2}\int|U_{\mathbf{kqq}'}|^2 n_{\mathbf{k}}{}^s\delta(\varepsilon_{\mathbf{k}}-\Omega_{\mathbf{q}}-\Omega_{\mathbf{q}'})\delta(\mathbf{k}-\mathbf{q}-\mathbf{q}')\,d\mathbf{q}\,d\mathbf{q}'$$
$$\approx 2\pi\alpha\zeta^2J_0\Omega_{\mathbf{q}'}{}^2(\varepsilon_s sk_s)^{-1}g(\Omega_{\mathbf{q}})n^s, \qquad (3.2)$$
$$g(\Omega)=\{1+[1+\alpha^2(\varepsilon_s-\Omega^2)/\varepsilon_s{}^2]\}^2(1+\alpha^2\Omega^2/\varepsilon_0{}^2)^{-2}.$$

If $sk_s\ll\varepsilon_s$ and $\alpha\approx 3$ ($FeBO_3$), we have $g(\Omega)\approx 1/5$. In this case we obtain $\Omega_{\mathbf{q}}\approx\Omega_{\mathbf{q}'}\approx\varepsilon_s/2$. It follows from Eq. (3.1) that the threshold of the second kinetic instability is reached at

$$2\Delta\Gamma_{\mathbf{q}}=\Gamma_{\mathbf{q}}{}^0, \qquad (3.3)$$

where

$$n_{cr}{}^s=5sk_s\Gamma^0(\varepsilon_0/2)/\pi\alpha\zeta^2\varepsilon_0J_0. \qquad (3.4)$$

In the case of iron borate under the experimental conditions of Ref. 10 [$\zeta=0.2$, $\Gamma_{\mathbf{q}}=(1\text{–}3)\times10^{-5}\Omega$], this gives

$$n_{cr}{}^s\approx(2\div 6)\cdot 10^{-8}sk_s/\varepsilon_0. \qquad (3.5)$$

It follows from Eqs. (2.19)–(2.21) that the critical number of secondary spin waves is reached at a pump amplitude ξ_{c2} given by

$$\xi_{c2}-1=5(1+\alpha)\alpha\zeta^{-4}\frac{(\varepsilon_p-\varepsilon_s)^2}{\varepsilon_p sk_p}\frac{\Gamma^0(\varepsilon_0/2)}{\gamma_p{}^0}. \qquad (3.6)$$

For example, in the case of iron borate in a field $H=200$ Oe, we have

$$\xi_{c2}\approx 1+8\Gamma^0(\varepsilon_0/2)/\gamma_p{}^0. \qquad (3.7)$$

The phonon damping decrement $\Gamma^0(\varepsilon_0/2)$ of iron borate is not known reliably, but it is probable that the damping is governed by the same microscopic processes as the magnon damping γ_p^0. Therefore, we can expect $0.3\lesssim\Gamma^0(\varepsilon_0/2)/\gamma_p^0\lesssim 3$ and, consequently, $3.5\lesssim\xi_{c2}\lesssim 25$.

2. Third kinetic instability

Third-generation phonons may not only coalesce with parametric spin waves, but also decay into two phonons, which we shall call fourth-generation phonons. The kinetic equations for the perturbations of the distribution function of fourth-generation phonons δN^f of frequency Ω_f and $\delta N^{f'}$ of frequency $\Omega_f' = \Omega_t - \Omega_f$ are as follows:

$$(\partial/\partial t+2\Gamma_f{}^0-2\Delta\Gamma_f)\delta N^f=2\Delta\Gamma_f\delta N^{f'},$$
$$(\partial/\partial t+2\Gamma_f{}^{0'}-2\Delta\Gamma_f')\delta N^{f'}=2\Delta\Gamma_f'\delta N^f, \quad (3.8)$$

where

$$\Delta\Gamma_f=72\pi\zeta^2J_0\Omega_t(\Omega_t-\Omega_f)\varepsilon_0^{-2}F(\Omega_t,\Omega_f,\varepsilon_0)N^t,$$
$$F=\frac{[1+\alpha^2(\Omega_f^2+\Omega_f'^2+\Omega_t^2)/3\varepsilon_0^2]^2}{(1+\alpha^2\Omega_t^2/\varepsilon_0^2)(1+\alpha^2\Omega_f^2/\varepsilon_0^2)(1+\alpha^2\Omega_f'^2/\varepsilon_0^2)}. \quad (3.9)$$

The threshold number of third-generation phonons attains a minimum in the limit $\Omega_f\to 0$. It is equal to

$$N_{cr}{}^t=\frac{\Gamma_f(1+\alpha^2/4)}{18\pi\zeta^2J_0(1+\alpha^2/6)}. \quad (3.10)$$

The damping rate is $\Gamma_f\approx 10^4$–10^5 sec^{-1} (Ref. 8), which gives $N^t_{cr}\approx 10^{-8}$; therefore, the third kinetic instability threshold may be crossed when the threshold of the second instability is exceeded only slightly. It is probable that for $\xi\gtrsim 10$ the generation of low-frequency phonons[7,10] is due to the second kinetic instability.

§ 4. WIDTHS OF SECONDARY SPIN WAVE AND PARAMETRIC SPIN WAVE DISTRIBUTIONS

1. Form of a packet of secondary spin waves

In § 2 we have shown that a packet of secondary spin waves is located on a surface in **k** space on which the relationship $\gamma^s_{\mathbf{k}}=0$ is satisfied. In fact, a packet of secondary spin waves is always broadened by four-magnon scattering by secondary phonons and thermal noise.[19] The kinetic equation for the distribution function of secondary magnons in the eigenfrequencies $n(\varepsilon)$ has the following form when these factors are allowed for:

$$\gamma_s(\varepsilon)n(\varepsilon)=\frac{|A|^2\varepsilon_s}{2(sk_s)^2}\int n(\varepsilon_1)n(\varepsilon_2)n(\varepsilon_3)\times\delta(\varepsilon+\varepsilon_1-\varepsilon_2-\varepsilon_3)d\varepsilon_1\,d\varepsilon_2\,d\varepsilon_3$$
$$+\pi\frac{\zeta J_0\varepsilon_0^2}{\alpha\varepsilon_s sk_s}\int[n^p(\varepsilon_1)+\bar{n}^s(\varepsilon_1)]N^s(\varepsilon_2)\delta(\varepsilon_1-\varepsilon-\varepsilon_2)d\varepsilon_1\,d\varepsilon_2$$
$$+\frac{sk_s\varepsilon_s}{2\pi^2\Theta_N^3}\gamma_s{}^0n^0(\varepsilon_s), \quad (4.1)$$

where $A=J_0(\frac{1}{8}-8\zeta\varepsilon_0^2/\varepsilon_s^2)$ is the amplitude of the four-magnon interaction calculated allowing for the processes of virtual phonon exchange.[6] It follows from Eq. (2.17) that

$$\gamma_s(\varepsilon)=\begin{cases}\gamma_s{}^0(\varepsilon), & \varepsilon<\varepsilon_s,\\ -\Delta+\gamma'(\varepsilon-\varepsilon_s), & \varepsilon>\varepsilon_s,\end{cases} \quad (4.2)$$

where $\gamma'=\gamma_s^0\varepsilon_s/(sk_s)^2$, and $\gamma_s^0\gg\Delta$. The influence of the four-magnon scattering reduces to the transfer of magnons from the negative damping region to the positive one. If $n^s>Tsk_s\gamma_s^0/\Theta_N^3\approx 10^{-12}$, i.e., practically always, we can ignore the thermal noise in Eq. (4.1). The noise associated with the scattering by secondary phonons can be ignored if $n^s>10^{-5}(\xi-1)^{1/2}b/(1+b)\approx 3\cdot 10^{-7}(\xi-1)^{1/2}$. Then, integrating Eq. (4.1) with respect to the frequency ε, we can obtain

$$\langle\gamma\rangle=\frac{1}{n^s}\int\gamma_s(\varepsilon)n(\varepsilon)d\varepsilon=\frac{A^2\varepsilon_s}{2(sk_s)^2}(n^s)^2. \quad (4.3)$$

In our case, if $\varepsilon<\varepsilon_s$ the damping is very strong but $n(\varepsilon)$ is small. In the limit $\gamma_s^0/\Delta\to\infty$ the value of $n(\varepsilon)$ vanishes in this region. Therefore,

$$\langle\gamma\rangle=-\Delta+\gamma'\int_0^\infty\omega n(\omega)d\omega,\quad \omega=\varepsilon-\varepsilon_s. \quad (4.4)$$

It follows that Eqs. (4.3) and (4.4) give the relationship between the zeroth and first moments of the distribution function $n(\varepsilon)$.

In the absence of the thermal noise, Eq. (4.1) is equivalent to a condition for an extremum of the functional

$$F=\frac{1}{2}\int\gamma_s(\varepsilon)n^2(\varepsilon)d\varepsilon-\frac{B}{4}\int n(\varepsilon_1)n(\varepsilon_2)n(\varepsilon_3)n(\varepsilon_4)$$
$$\times\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)d\varepsilon_1\,d\varepsilon_2\,d\varepsilon_3\,d\varepsilon_4,\quad B=A^2\varepsilon_s/2(sk_s)^2. \quad (4.5)$$

It is shown in Ref. 16 that the solution of Eq. (4.1) has an exponential asymptote. Approximate determination of the parameters of a packet of parametric spin waves can be made by taking $n(\varepsilon)$ in the form of the test function

$$n(\varepsilon)=\bar{n}\exp(-\lambda\omega), \quad (4.6)$$

with the arbitrary parameters $\bar{n}$ and λ. The functional for the class of such functions is

$$F={}^1/_4(-\Delta\bar{n}^2\lambda^{-2}+{}^1/_2\gamma'\bar{n}^2\lambda^{-2})-{}^1/_{16}B\bar{n}^4\lambda^{-3}, \quad (4.7)$$

whereas the condition (4.3) gives

$$-\Delta+\gamma'\lambda^{-1}=B(n^s)^2. \quad (4.8)$$

Minimization of Eq. (4.7) subject to the condition (4.8) gives

$$B(n^s)^2=(\sqrt{3}-1)\Delta,\quad \Delta\varepsilon_s=\lambda^{-1}=\sqrt{3}\frac{\Delta}{\gamma'}=\frac{\sqrt{3}}{\sqrt{3}-1}\frac{B(n^s)^2}{\gamma'}. \quad (4.9)$$

Substitution of numerical values for iron borate in the case when $n^s=10^{-6}$ yields

$$\Delta\varepsilon_s\approx 5\cdot 10^8\ \text{sec}^{-1}.$$

As shown in § 2, the strongest among the second-generation waves is a packet of secondary spin waves. Secondary phonons are formed as a result of induced decay of parametric spin waves into secondary spin waves and secondary phonons, and we have

$$N_q{}^s=2\pi^2\alpha^{-1}\zeta(J_0\varepsilon_0^2\Theta_N^3/\varepsilon_s sk_s\varepsilon_p sk_p\Gamma_q)n^pn^s(\varepsilon_p-\Omega_q). \quad (4.10)$$

The width of a packet of secondary phonons is equal to the width of a packet of secondary spin waves and the total number of secondary phonons is

$$N^s=\pi\alpha^2\zeta[J_0\varepsilon_0^2(\varepsilon_p-\varepsilon_s)^2/\varepsilon_s sk_s\varepsilon_p sk_p\Gamma_s]n^sn^p. \quad (4.11)$$

2. Form of a packet of parametric spin waves

The formula (2.13) for $n_{\mathbf{k}}^{p}$, representing the distribution function of parametric spin waves in the **k** space, is given in § 2. The distribution function of the frequencies of parametric spin waves

$$n_{\omega}^{p}=\frac{v_0}{(2\pi)^3}\int n_{\mathbf{k},\omega}\,d\mathbf{k} \tag{4.12}$$

has a characteristic width $\nu^2/2\gamma_p$ [the definition of ν is given by Eq. (2.14)] and in the case of a wide-band incoming term Φ_k for parametric spin waves, this distribution function is

$$n_{\omega}^{p}=\frac{1}{2\pi^2}\frac{\varepsilon_p s k_p}{\Theta_N^3}\Phi_k\frac{\varpi^2+\gamma_p^2+|\Pi|^2+|\mu_\omega|^2}{2\,\mathrm{Re}\,\mu_\omega|\mu_\omega|^2} \tag{4.13}$$

$$\mu_\omega=(\nu^2-2i\gamma_p\varpi-\varpi^2)^{1/2},\quad \varpi=\omega-\varepsilon_p.$$

In particular, the width of the distribution at 10% from the maximum is

$$\delta\omega\approx 3\nu^2/\gamma_p. \tag{4.14}$$

The incoming term Φ_k represents the sum of the thermal noise Φ_k^0 and of the noise created by the waves of the second, third, and later generations in the region of a parametric resonance:

$$\Phi_k^0=\gamma_k^0T/\pi\varepsilon_k, \tag{4.15}$$

$$\begin{aligned}\Phi_{\mathbf{k}}^{s}&=\frac{v_0}{(2\pi)^3}\int|V_{\mathbf{k}\mathbf{k}'\mathbf{q}}|^2n_{\mathbf{k}'}^{s}N_{\mathbf{q}}^{s}\delta(\varepsilon_p-\varepsilon_{\mathbf{k}'}-\Omega_{\mathbf{q}})\\&\quad\times\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})\,d\mathbf{k}'\,d\mathbf{q}\\&=2\pi^2\alpha^{-1}\zeta J_0\varepsilon_0^2\Theta_N^3(\varepsilon_s sk_s\varepsilon_p sk_p\Delta\varepsilon_s)^{-1}n^sN^s,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\Phi_{\mathbf{k}}^{t}&=\frac{v_0}{(2\pi)^3}\int|V_{\mathbf{k}'\mathbf{k}\mathbf{q}}|^2n_{\mathbf{k}'}^{t}N_{\mathbf{q}}^{t}\delta(\varepsilon_{\mathbf{k}'}-\varepsilon_p-\Omega_{\mathbf{q}})\\&\quad\times\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q})\,d\mathbf{k}'\,d\mathbf{q}\\&=2\pi^2\alpha^{-1}\zeta J_0\varepsilon_0^2\Theta_N^3(\varepsilon_s sk_s\varepsilon_p sk_p\Delta\varepsilon_t)\,n^tN^t.\end{aligned} \tag{4.17}$$

If $n^p\gtrsim 10^{-12}$, then—as already pointed out—the thermal noise can be ignored. We can identify the mechanism which dominates the frequency width of parametric spin waves, Φ^s of Eq. (4.16) or Φ^t of Eq. (4.17), by determining the width of a packet $\Delta\varepsilon_t$ of third-generation waves. However, in the range of fields where $k_s\to 0$ it is obvious that the main contribution comes from the scattering by second-generation waves. This is supported by the experimental results of Ref. 10, showing a considerable increase in the width of a packet of parametric spin waves $\delta\omega$ in the range of magnetic fields close to $H=H^*$ where $k_s=0$ (i.e., when $\varepsilon_0+ck_p=\varepsilon_p$). A simple calculation shows that under the experimental conditions of Ref. 10 we have $H^*\approx 240$ Oe. When the scattering by second-generation waves predominates, we have

$$\nu=\frac{\pi^2}{2}\frac{\sqrt{3}-1}{\sqrt{3}}\left(8\zeta-\frac{1}{8}\right)^{-2}\alpha\zeta^2\frac{\varepsilon_0^4(\varepsilon_p-\varepsilon_s)^2\gamma_s^0}{\varepsilon_p sk_p(\varepsilon_s sk_s)^2\Gamma_s^0}\gamma_p. \tag{4.18}$$

Substitution of Eq. (4.18) into the expression for the width of a packet of parametric spin waves given by Eq. (4.14) yields

$$\delta\omega=\pi^4(2-\sqrt{3})2^{-13}\alpha^2\left(\frac{\varepsilon_0^2}{\varepsilon_s sk_s}\right)^4\frac{(\varepsilon_p-\varepsilon_s)^4}{(\varepsilon_p sk_p)^2}\left(\frac{\gamma_s^0}{\Gamma_s^0}\right)^2\gamma_p^0\xi. \tag{4.19}$$

We have expressed here the damping factor of parametric spin waves γ_p in terms of the pump amplitude reduced to the threshold value $\xi=h/h_c$ [Eq. (2.24)]. The expression for $\delta\omega$ given by Eq. (4.19) is in full qualitative and good quantitative agreement with the experimental results. The magnetic field dependence of $\delta\omega$ has a sharp peak for $H\to H^*$ ($k_s\to 0$) and is a linear function of temperature (because γ_s^0, Γ_s^0, and γ_p^0 are proportional to temperature T) and of the excess above the threshold ξ. If $sk_s\approx\varepsilon_0$, we have $\delta\omega/\gamma_p^0\xi\approx 0.4$, which is close to the experimental value of this quantity.

It should be stressed that the dependence of $\delta\omega$ on the experimental parameters which we have found is due to the interaction of parametric spin waves with intense packets of secondary spin waves and secondary phonons generated as a result of a kinetic instability. The width of the spectrum of parametric spin waves in iron borate was also considered in Ref. 14, but the authors reached the conclusion that the kinetic instability threshold cannot be attained and the incoming term for parametric spin waves is entirely due to the thermal noise. Therefore, the width $\delta\omega$ of a packet of parametric spin waves calculated in Ref. 14 for $\xi<\xi_{cr}$ gives a completely different dependence $\delta\omega(\xi)$ from that observed experimentally in Ref. 10.

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