On the chiral anomaly in superfluid ^{3}He-A

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The anomalous texture current in ^{3}He-A, which does not fit into the framework of superfluid hydrodynamics, has the same nature as the chiral anomaly in (2 + 1)-dimensional QED, where the magnetic field gives rise to a vacuum charge density. In ^{3}He-A the role of the magnetic field is played by the texture of the orbital angular momentum vector of Cooper pairs. The current has its origin in the existence of an asymmetric branch of the spectrum of fermions which intersect the Fermi surface. The fermions occupying this branch have an uncompensated momentum which gives rise to a finite fermion vacuum current in the texture. The intersection of this branch with the Fermi surface leads to a finite density of states on the Fermi surface, and to the presence of a density of the normal component at $T = 0$, the magnitudes of which have been calculated for weak and strong twist textures.

1. INTRODUCTION

The low-temperature behavior of ^{3}He-A exhibits a series of peculiarities which strongly distinguish this fluid from such "classic" superfluid systems as He-II, ^{3}He-B, and the electron pair fluid in ordinary superconductors (see the review article Ref. 1). These peculiarities are tied to the existence of zeros in the spectrum of fermionic excitations in the A-phase, which begin to manifest themselves for temperatures $T \ll T_c$. One of such consequences is the anomalous current in ^{3}He-A.

For $T = 0$ the expression of the current in ^{3}He-A,

$$\mathbf{j}=\rho\mathbf{v}_s+\frac{1}{2}\operatorname{rot}\left(\frac{\hbar}{2m_3}\rho\mathbf{l}\right)-\frac{\hbar}{2m_3}C_0\mathbf{l}(\mathbf{l}\operatorname{rot}\mathbf{l}), \qquad (1)$$

contains, in addition to the superfluid current $\rho\mathbf{v}_s$ usual for a superfluid ($\mathbf{v}_s$ is the superfluid velocity, ρ the density of the fluid), also orbital currents. The latter are related to the fact that the Cooper pairs have an orbital angular momentum equal to $\hbar$ and are consequently subject to a rotational motion around the vector of the direction of the angular momentum $\mathbf{l}$. In the presence of a texture, i.e., a spatial variation of the vector $\mathbf{l}$, the rotational motions of the Cooper pairs stop compensating each other and a macroscopic orbital current arises. The second term in Eq. (1) is characteristic for any system with an angular momentum, whereas the third term in the current:

$$\mathbf{j}_{an}=-(\hbar/2m_3)C_0\mathbf{l}(\mathbf{l}\operatorname{rot}\mathbf{l}), \qquad (2)$$

where C_0 is close to ρ, turns out to be anomalous: it does not fit into the framework of superfluid dynamics of a fluid at $T = 0$, since it violates both Galilean invariance and the momentum-conservation law.[2] In order to reestablish these conservation laws one would have to admit the existence at $T = 0$ of a normal component ρ_n which could consist of excitations which accumulate in the texture because of the gapless character of the spectrum.[2] The existence of a nonvanishing density of states in the texture and with it of a ρ_n (at $T = 0$) was proved in the papers of Combescot and Dombre.[3,4]

Combescot and Dombre[3,4] have demonstrated that the spectrum of Fermi excitations near zero energy, in the presence of a texture, is analogous to the spectrum of a charged Dirac particle in a magnetic field $\mathbf{B}$, the role of which in the texture is played by curl $\mathbf{l}$ for more details see Sections 1 and 2 of the present paper); the zeroth Landau level is located exactly on the Fermi surface, therefore the density of states is nonzero, since it coincides with the density of states on the zeroth Landau level. In addition, investigating the dynamics of excitations with zero energy, they have shown that in the low-frequency dynamics the normal component does indeed reestablish the momentum conservation law and Galilean invariance.[4]

While the appearance of the normal component ρ_n ($T = 0$) has been cleared up, the physical reason for the origins of the anomalous current has remained obscure. In Ref. 5, where the so-called gradient expansion approximation was used, the anomalous current was related to a nonremovable topologically nontrivial singularity in the phase of the order parameter which is always present in ^{3}He-A, and which leads to zeros in the spectrum of the Fermi excitations. However, the approximation itself loses its validity near the singularity. A similar approximation was used by Stone *et al.*,[6] who related the anomalous current, in the presence of a texture playing the role of a topologically nontrivial configuration of the boson field $\mathbf{l}$, to the appearance of a fractional fermionic charge in the presence of this field.[7] A major step forward was the paper by Ho *et al.*[8] They determined exactly the spectrum of the Fermi excitations in a topologically nontrivial texture (soliton). And observed the spectrum asymmetry that led to a nonzero excitation current at $T \neq 0$.

Later Combescot and Dombre,[4] by calculating the current variation, pointed out the possibility that the spectrum asymmetry leads not only to excitation current, but also to an anomalous current in the ground state. This hypothesis was proved in Ref. 9, where it was shown for the simplest structure curl $\mathbf{l} \parallel \mathbf{l}$ that the anomalous current is the consequence of the chiral anomaly well known in (2 + 1) electrodynamics.[10,11] Namely, parity violation in the spectrum of

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charged Dirac fermions, which is produced in a magnetic field, leads in the case of ^3He-A to a vacuum current directed along the "magnetic" field $\mathbf{B} = \text{curl}\,\mathbf{l}$. This "chiral" current, made up by uncompensated momenta of particles located on a zero Landau level, is in fact the anomalous current (2).

In this paper the chiral anomaly is considered in detail (see Sec. 3). In Sec. 4, the anomalous current is investigated for a larger class of textures, when the Dirac equation is no longer applicable. From the analysis of the Bogolyubov equation (using some properties of supersymmetry[12,13]) it follows that the chiral anomaly is related to a definite topological characteristic of the spectrum which is preserved for a more general texture, namely to the presence of a branch of the spectrum which intersects the Fermi surface. The vacuum fermions on this branch of the spectrum at $T=0$ form a normal subsystem in the superfluid ^3He-A (in distinction from the superfluid subsystem of Cooper pairs), since this branch does not exhibit a gap. This subsystem has a finite density of states which is responsible for the nonzero normal component at $T=0$. The quantity $\rho_n(T=0)$ depends, however, on the details of the spectrum near zero. Its dependence on the "magnetic field" $\mathbf{B}$ is considered in Section 5. In Appendix A we discuss the symmetry of the Bogolyubov equation and the corresponding Dirac equation for fermions to it. The violation of this symmetry leads to the anomalous current in the vacuum. In Appendix B we prove, for the case of an arbitrary twist structure, and making use of qualitative theory of differential equations, that there exists a branch of the spectrum which intersects the Fermi surface and has no gap.

1. TEXTURE AS A MAGNETIC FIELD FOR FERMIONS IN ^{3}He-A

The majority of the exotic properties of the A-phase of quperfluid ^3He which distinguish this liquid from all other superfluid and superconducting Fermi systems is a consequence of the fact that the energy of the Fermi excitations vanishes at two points of the Fermi surface. The energy spectrum of the quasiparticles in ^3He-A is the same for fermions which spin "up" and "down" (see the review paper, Ref. 1):

$$E=(\varepsilon^2+\Delta_0^2[\mathbf{kl}]^2/k_F^2)^{1/2}, \tag{1.1}$$

where ε is the energy of the fermions in their normal state, reckoned from the chemical potential μ; k_F is the Fermi momentum; $\mathbf{l}$ is the orbital angular momentum vector of the Cooper pairs; Δ_0 is the amplitude of the gap in the spectrum in the sequel we shall, for simplicity, use for ε the ideal-gas approximation: $\varepsilon = k^2/2m_3 - \mu$. The spectrum (1.1) has no gap for $\mathbf{k}_+ = k_F\mathbf{l}$ and $\mathbf{k}_- = -k_F\mathbf{l}$ on the Fermi surface.

The zeros in the spectrum of the A-phase are topologically stable and do not disappear for small deformations of the order parameter which take the fluid out of the A-phase state[5]: for generic deformations which lift the spin degeneracy, each of the zeros only splits into two $(\mathbf{k}_{+\uparrow},\mathbf{k}_{-\uparrow},\mathbf{k}_{-\downarrow},\mathbf{k}_{-\downarrow})$, corresponding to the two spin projections. Since we will only be interested in orbital currents, we fix the spin state of the A-phase, and therefore the quasiparticle spins remain outside our considerations.

The zeros are also conserved for a spatially nonhomogeneous order parameter, i.e., in the presence of textures. Therefore in the A phase there are always states with arbitrarily small energy E and they determine the properties of the fluid for $T \ll \Delta_0$.

For quasiparticles with small energies E the texture of the vector $\mathbf{l}$ plays the role of an effective "magnetic" field.[3,4] For the case of a classical motion this is easily seen if one makes use of the Hamiltonian equations of motion of the quasiparticles[14]

$$\dot{\mathbf{k}}=-\nabla E, \quad \dot{\mathbf{r}}=\partial E/\partial \mathbf{k}. \tag{1.2}$$

For low-energy states we introduce the small parameter $\mathbf{p} = \mathbf{k} - ek_F\mathbf{l}$, where $e = \pm 1$, depending on the zero near which the excitations are considered. In terms of these momenta the equations (1.2) take the form

$$\dot{\mathbf{p}}=-\nabla E(\mathbf{p},\mathbf{r})+e[\mathbf{vB}]+e\mathbf{E}, \quad \dot{\mathbf{r}}=\mathbf{v}=\partial E/\partial \mathbf{p}, \tag{1.3a}$$

$$\mathbf{E}=-\dot{\mathbf{A}}-\nabla A_0, \quad \mathbf{B}=\text{rot}\,\mathbf{A}, \tag{1.3b}$$

$$\mathbf{A}=k_F\mathbf{l}, \quad A_0=k_F\mathbf{l}\mathbf{v}_s, \tag{1.3c}$$

$$E(\mathbf{p},\mathbf{r})=(c_\parallel^2(\mathbf{pl})^2+c_\perp^2[\mathbf{pl}]^2)^{1/2}, \quad c_\parallel=v_F=k_F/m_3, \quad c_\perp=\Delta_0/k_F. \tag{1.3d}$$

The equations (1.3) describe the classical dynamics of relativistic massless particles with charge $e = \pm 1$ and an anisotropic velocity tensor

$$E^2=(c^2)_{ij}p_ip_j, \quad (c^2)_{ij}=c_\parallel^2 l_il_j+c_\perp^2(\delta_{ij}-l_il_j)$$

(we have neglected the higher powers of p). The vector potential $\mathbf{A}$ appears on account of the inhomogeneous shift of the momentum of the quasiparticle in the texture; we have also introduced the scalar potential A_0, in order to take into account the superfluid velocity field $\mathbf{v}_s$: this leads to $E \to E + \mathbf{k}\cdot\mathbf{v}_s$. The classical approximation is valid for values of p which are large compared to the reciprocal cyclotron radius $(eB/\hbar)^{1/2}$. For small momenta the quasiparticle dynamics is governed by quantum mechanics. The wave function χ of the fermionic excitations in superfluid systems is a Bogolyubov spinor and is described by the Bogolyubov equation.

2. THE BOGOLYUBOV EQUATIONS FOR PARTICLES IN A TEXTURE

In ^3He-A the particles with spin "up" and those with spin "down" are paired independently and therefore the model Hamiltonian can be written down separately for each of the spin projections. If $\Psi_\uparrow^+$ is the creation operator for a particle with spin "up", then the BCS Hamiltonian for triplet pairing has the following form:

$$H=H_\uparrow+H_\downarrow, \quad H_\uparrow=\int d^3x\left\{\frac{1}{2m_3}\nabla\Psi_\uparrow^+\nabla\Psi_\uparrow-\mu\Psi_\uparrow^+\Psi_\uparrow + g\left(\Psi_\uparrow^+\frac{\nabla}{2i}\Psi_\uparrow^+\right)\left(\Psi_\uparrow\frac{\nabla}{2i}\Psi_\uparrow\right)\right\}. \tag{2.1}$$

After the introduction of the quasi-average for the operators with spin "up"

$$\Delta_\uparrow = g\left\langle \Psi_\uparrow \frac{\nabla}{2i} \Psi_\uparrow \right\rangle, \quad \Delta_\uparrow^* = g\left\langle \Psi_\uparrow^+ \frac{\nabla}{2i} \Psi_\uparrow^+ \right\rangle, \tag{2.2}$$

and similarly for spin "down," we obtain an effective Hamiltonian, which is quadratic in Ψ, Ψ^+, in which the spin index has been omitted:

$$H_\uparrow = H_\downarrow = \int d^3x \left\{ \frac{1}{2m_3} \nabla\Psi^+ \nabla\Psi - \mu\Psi^+\Psi + \Delta^* \Psi \frac{\nabla}{2i} \Psi + \Delta\Psi^+ \frac{\nabla}{2i} \Psi^+ \right\}. \tag{2.3}$$

The diagonalization of (2.3) is achieved by means of the unitary transformation

$$\Psi = \sum_s [u_s(\mathbf{r},t) a_s + v_s^*(\mathbf{r},t) a_s^+], \tag{2.4}$$

where a, a^+ are the annihilation and creation operators of the Bogolyubov quasiparticles, and u_S, v_S form the spinor

$$\chi_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix},$$

which is an eigenfunction of the Bogolyubov equation

$$i\frac{\partial}{\partial t}\chi = \begin{pmatrix} \hat{\varepsilon} & -{}^1/_2 i(\Delta\nabla + \nabla\Delta) \\ -{}^1/_2 i(\Delta^*\nabla + \nabla\Delta^*) & -\hat{\varepsilon} \end{pmatrix} \chi,$$

$$\chi_s(\mathbf{r},t) = \exp(-iE_s t)\chi_s(\mathbf{r}), \tag{2.5}$$

where we have introduced the operator

$$\hat{\varepsilon} = -(\nabla^2 + k_F^2)/2m_3. \tag{2.6}$$

The order parameter in ^{3}He-A (more precisely, its orbital part) Δ, is a complex vector[1]:

$$\Delta = (\Delta_0/k_F)(\mathbf{e}_1 + i\mathbf{e}_2), \tag{2.7}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2$ are the basis vector of the orbital coordinate system.

In the texture the position of the basis vectors changes in space. The orientation of the basis can be defined by three functions corresponding to the three Euler angles, functions which we denote by A_x/k_F, A_y/k_F, Φ. Assuming that these functions are small, we have

$$\mathbf{e}_1 + i\mathbf{e}_2 = \{\hat{\mathbf{x}} + i\hat{\mathbf{y}} - \hat{\mathbf{z}}(A_x + iA_y)/k_F\} e^{i\Phi},$$
$$\mathbf{l} = \hat{\mathbf{z}} + (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y)/k_F, \quad \mathbf{v}_s = \hbar\nabla\Phi/2m_3. \tag{2.8}$$

If momenta close to $ek_F\mathbf{l}$ are important, then the operator ∇^2 in (2.6) may be replaced $-iek_F\partial_z$, and in the small terms in the antidiagonal one may set $-i\partial_z = ek_F$. In this case the substitution of (2.8) into Eq. (2.5) leads to an equation of the Dirac type for massless anisotropic charged fermions:

$$\{(i\partial/\partial t - eA_0) - ec_\parallel(-i\partial_z - eA_z)\tau_3 - c_\perp(-i\partial_x - eA_x)\tau_1 + c_\perp(-i\partial_y - eA_y)\tau_2\}\chi = 0. \tag{2.9}$$

Here the Pauli matrices have been denoted by τ in order to distinguish them from the σ matrices corresponding to the usual spins of the quasiparticles, which are not considered here. The scalar potential in Eq. (2.9) equals $A_0 = \frac{1}{2}k_F\partial_z\Phi$, and the vector potential is $\mathbf{A} = (k_F\delta l_x(\mathbf{r}),\ k_F\delta l_y(\mathbf{r}),\ k_F(\mathbf{r}))$, corresponding to the classical limit (1.3c). Thus, the equation (2.9) is a natural generalization of the classical equations (1.3) to the quantum case (the distinction from the usual Dirac equation lies in the coefficient e in front of the second term).

3. THE ANOMALOUS CURRENT IN THE WEAK TWIST TEXTURE

The similarity between the equations for the fermions in ^{3}He-A and those of quantum electrodynamics suggests the existence of ^{3}He-A of an effect similar to the chiral anomaly, which leads to the appearance of a current directed along the magnetic field in the fermion vacuum and thus to a violation of parity. We shall show that the anomalous current in ^{3}He-A is related to just such a chiral anomaly. Since the sought-for anomalous current (2) is proportional to $\mathbf{l}(\mathbf{l}\cdot\text{curl}\,\mathbf{l})$, we choose for simplicity a twist texture where curl $\mathbf{l}\|\mathbf{l}$. In this case the magnetic field is directed along the z axis and the quantization of motion takes place in the transverse plane.

We note that the equation (2.9) is valid only in the case when the reciprocal cyclotron radius $(eB/\hbar)^{1/2}$ is sufficiently small, so that for the classical energy E in Eq. (1.3d) and for the operator $\hat{\varepsilon}$ in Eq. (2.6) one may restrict oneself to the terms linear in p, i.e., when

$$c_\parallel^2 p_\perp^4/k_F^2 \ll c_\perp^2 p_\perp^2. \tag{3.1}$$

Substituting this equation into the square of the transverse momentum in the ground state, $\langle p_\perp^2\rangle \sim \hbar eB$, we obtain the condition of validity of Eq. (2.9):

$$eB \ll \Delta_0^2/\hbar v_F^2. \tag{3.2}$$

We shall designate a texture for which this condition is satisfied as a weak texture. The opposite case of a strong texture, where

$$c_\parallel^2 p_\perp^4/k_F^2 \gg c_\perp^2 p_\perp^2, \quad \Delta_0^2/\hbar v_F^2 \ll eB \ll \Delta_0 m_3/\hbar, \tag{3.3}$$

will be considered in Section 4. The upper bound on B in (3.3) ensures the condition that the characteristic scale of variation of the texture exceeds the coherence length $\xi = \hbar v_F/\Delta_0$.

After these remarks we find the particle current in the vacuum state which appears in a weak texture. Locally the "magnetic field" $\mathbf{B}$ created by the texture is constant, therefore the problem reduces to a determination of the vacuum fermion current which appears as a result of a constant magnetic field. We choose the vector potential in the form

$$A_x = 0, \quad A_y = Bx. \tag{3.4}$$

Since $\mathbf{B}\|\hat{\mathbf{z}}$, the projection k_z on the momentum onto the z axis is a good quantum number, and after the substitution $\chi = \tilde{\chi}\exp(ik_z z)$ the Bogolyubov equation (2.5) coincides with a (2 + 1)-dimensional Dirac equation for isotropic charged fermions placed in a magnetic field:

$$i\frac{\partial}{\partial t}\tilde{\chi} = H\tilde{\chi}, \quad H = Mc_\perp^2\tau_3 + c_\perp\left[\tau_1\frac{1}{i}\partial_x - \tau_2\left(\frac{1}{i}\partial_y - eBx\right)\right]. \tag{3.5}$$

The charge of the fermions depends on k_z and equals

$$e(k_z)=k_z/k_F=\mathbf{kl}/k_F, \tag{3.6}$$

which reduces near the Fermi surface to the old definition $e=\pm 1$. The fermion mass also depends on k_z:

$$Mc_\perp^2=\varepsilon=(k_z^2-k_F^2)/2m_3. \tag{3.7}$$

The current in the direction of the magnetic field can be expressed in terms of the eigenfunctions of the Dirac operator (3.5) as follows:

$$j_z=\frac{1}{2i}(\Psi_\uparrow^+\partial_z\Psi_\uparrow-\partial_z\Psi_\uparrow^+\Psi_\uparrow)+(\uparrow\to\downarrow)$$
$$=2\sum_{k_z}k_z\sum_{E_S<0}(|u_S|^2a_S^+a_S-|v_S|^2a_Sa_S^+)$$
$$=k_F\sum_{k_z}e(k_z)\sum_S\left(\tilde\chi_S^*\frac{1+\tau_3}{2}\tilde\chi_Sa_S^+a_S-a_Sa_S^+\tilde\chi_S\frac{1-\tau_3}{2}\tilde\chi_S^*\right). \tag{3.8}$$

Here S labels the two-dimensional states, and in the last expression the sum is extended also to positive energies E_S. In a magnetic field with a gauge (3.4) a state S is characterized by the momentum k_y and by the number n of the Landau level. The energy spectrum of the Dirac fermions in a magnetic field has the form

$$E_S=E_n=\pm(M^2c_\perp^4+2nc_\perp^2|eB|)^{1/2},\quad n=0,1,\dots. \tag{3.9}$$

The eigenfunctions of the Hamiltonian (3.5) depend on the sign of eB:

$$\tilde\chi_S=\exp(-iE_nt)\exp(ik_yy)\left[\theta(-eB)\begin{pmatrix}\alpha_nf_n(\tilde x)\\ i\beta_nf_{n-1}(\tilde x)\end{pmatrix}+\theta(eB)\begin{pmatrix}\alpha_nf_{n-1}(\tilde x)\\ i\beta_nf_n(\tilde x)\end{pmatrix}\right], \tag{3.10}$$

where θ is the step function, $\tilde x=x-k_y/eB$, $f_n(\tilde x)$ are the normalized harmonic oscillator eigenfunctions with frequency $\Omega=|eB|$, $f_{-1}=0$, and the parameters α_n and β_n have the form

$$|\alpha_n|^2=(E_n+Mc_\perp^2)/2E_n,\qquad |\beta_n|^2=(E_n-Mc_\perp^2)/2E_n. \tag{3.11}$$

We note that α_0 and β_0 vanish respectively for $E_0/M<0$ and $E_0/M>0$. This leads to a spectral asymmetry: the zeroth Landau level is present only for $eB/ME<0$ (see Fig. 1).

The spectrum of Fig. 1 exhibits the following properties. First, it is symmetric with respect to charge conjugation C (see Appendix A): $CE=-E$, $Ce=-e$, i.e., it is invariant under the simultaneous substitution $E\to -E$, $e\to -e$. Second, there is invariance with respect to a combined time reversal TP_M (Appendix A): $e\to -e$, $M\to -M$. Third, if one excludes the zeroth Landau level, the remaining spectrum has an additional anomalous symmetry with respect to each of the operations $E\to -E$, $eB\to -eB$, $M\to -M$ separately. In $(2+1)$-dimensional electrodynamics this corresponds to the conservation of two-dimensional spatial parity in the vacuum (Appendix A). This parity is violated by the zeroth Landau level, which leads to the anomalous current both in ^{3}He-A and in $(2+1)$-electrodynamics.

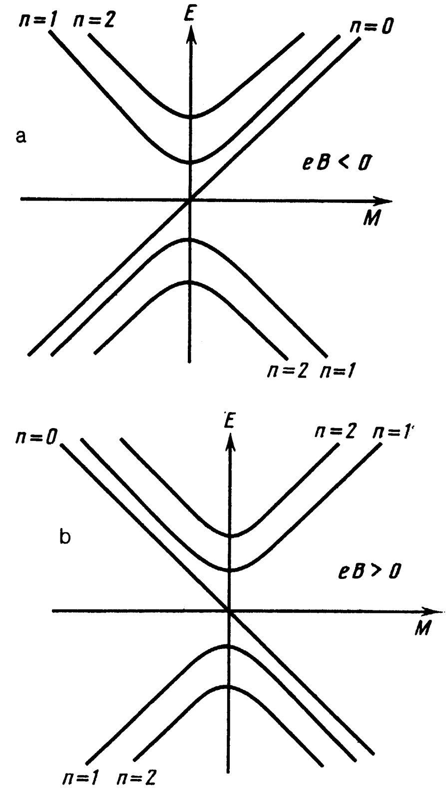

FIG. 1. The spectrum of Fermi excitations in ^{3}He-A in the presence of a twist texture and in $(2+1)$-dimensional electrodynamics in the presence of a magnetic field. With the exception of the branch corresponding to the zeroth Landau level the spectrum is symmetric with respect to each of the transformations: $E\to -E$, $M\to -M$, $eB\to -eB$. The level with $n=0$ violates this symmetry. The violation of the $eB\to -eB$ symmetry leads to the appearance of the anomalous fermion current (2) in ^{3}He-A, and the violation of the $E\to -E$ symmetry by the same level leads to the appearance of a nonzero charge density in the vacuum of $(2+1)$-QED. Figure a corresponds to $eB<0$; figure b corresponds to $eB>0$.

Let us consider the current (3.8). The contributions from the nonzero Landau levels to the current cancel on account of the odd character of the current under the substitution $e\to -e$. Therefore, only the parity-violating contribution from the zero Landau level remains, as a result of which there appears an uncompensated vacuum fermion current. For $E<0$, $|u_0|^2$ contributes to the current, being nonzero for $eB<0$ and $M<0$, whereas for $E>0$, $|v_0|^2$ contributes, being nonzero for $eB>0$ and $M<0$. As a result, we obtain

$$\langle 0|j_z|0\rangle=k_F\int\frac{dk_z}{2\pi}j(k_z), \tag{3.12a}$$

$$j(k_z)=-e\,\mathrm{sign}(eB)\theta(-M)\int\frac{dk_y}{2\pi}|f_0|^2. \tag{3.12b}$$

Taking into account the fact that $\int dk_y|f_n|^2=|eB|$, we obtain

$$j(k_z)=-(e^2B/2\pi)\theta(-M) \tag{3.13a}$$

and substituting into (3.12a), we get the final expression for the chiral vacuum current:

$$\langle 0|j_z|0\rangle=-(k_F^2/6\pi^2)B=-\rho(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})/2m_3, \tag{3.13b}$$

where $\rho = k_F^2/3\pi^2$ is the density of the ^{3}He, in agreement with Eq. (2).

The same spectral asymmetry leads to the appearance of a vacuum charge Q in $(2+1)$-electrodynamics (see, e.g., Refs. 10, 11). The operator Q has the form

$$Q = {}^1/_2 e(\chi^+\chi - \chi\chi^+), \quad (3.14)$$

where χ is expressed in terms of the electron and positron operators:

$$\chi = \sum_{E_S>0} \chi_S a_S + \sum_{E_S<0} \chi_S b_S^+. \quad (3.15)$$

Then the excess vacuum charge

$$\langle 0|Q|0\rangle = {}^1/_2 e\left(\sum_{E_S>0} |\chi_S|^2 - \sum_{E_S<0} |\chi_S|^2\right) \quad (3.16)$$

appears on account of the violation of the same separate symmetry $e \to -e$, $E \to -E$, $M \to -M$ due to the zeroth Landau level. Only in this case it is important that the symmetry under $E \to -E$ is violated. As a result, the difference between the number of positive and negative levels is nonzero:

$$N_> - N_< = -\mathrm{sign}\,(MeB), \quad (3.17)$$

and we obtain for the charge density

$$\langle 0|Q|0\rangle = {}^1/_2 e(N_> - N_<) \int \frac{dk_y}{2\pi} |f_0(\tilde{x})|^2 = -\frac{e^2 B}{4\pi} \mathrm{sign}\, M. \quad (3.18)$$

This expression is completely analogous to the expression (3.18a), and the differences are related to the different physical definitions of the current in ^{3}He-A [Eq. (3.8)] and in $(2+1)$-electrodynamics [Eq. (3.14)], although the fermions are described by the same Dirac equation (3.5), and the anomalous current appears as a consequence of the violation of the same symmetry. The expressions (3.13a) and (3.18) would coincide if the matrix τ_3 were not present in the definition of the ^{3}He-A current, Eq. (3.8).

4. THE ANOMALOUS CURRENT IN AN ARBITRARY TWIST TEXTURE AND THE TOPOLOGY OF THE SPECTRUM

We have determined the anomalous current (3.1b) in the limit when the Bogolyubov equations (2.5) reduce to a Dirac equation and the anomaly coincides with the chiral anomaly in QED. In reality the result holds also in a more general case, since it is topologically stable and does not change under a weak modification of the Dirac equation. The anomalous current (3.1b) is determined by the topology of the spectrum in Fig. 1. Indeed, the vacuum current (3.1b) can be obtained from simple considerations relative to Fermi occupation of levels. The levels with $E<0$ are filled, levels which are below the Fermi surface, i.e., have $M<0$. Each level in a magnetic field has the same density of states $\nu = |2eB|/2\pi$ (taking spin into account) and therefore the total momentum of the Fermi excitation equals

$$\langle 0|j_z|0\rangle = \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} k_z \nu N_<. \quad (4.1)$$

The expression (4.1) is different from zero if the number $N_<$ is different for positive and for negative k_z, which is indeed true for the spectrum of Fig. 1. The branch of the spectrum which intersects the M axis (or the ε axis) contributes to Eq. (4.1) and this leads to Eq. (3.1b). Thus, for the existence of an anomalous vacuum current it is necessary that one or several branches of the spectrum should intersect the abscissa axis, and this property of the spectrum is stable under small deformations of the Dirac equation. The number of intersections (more precisely, their algebraic sum) is thus a topological invariant. The particles which occupy this branch do not have a gap in the spectrum and therefore form a normal Fermi system, which carries a current even in the ground state.

In the limiting case of a weak texture, when the Bogolyubov equation reduces to the Dipac equation (2.9) for an electron in a magnetic field, the passing through zero of one of the branches of the energy spectrum is a consequence of the supersymmetry of the Dirac equation in the case when the magnetic field is directed along one of the axes and depends on the other two coordinates: $B_x = B_y = 0$, $B_z = B(x,y)$, the branch being infinitely degenerate if the magnetic flux is infinite (see Ref. 13).

Moving away from the limiting case we must take into account the fact that the operator (2.6) contains differentiations. We consider again the texture (3.4), (2.8) with $\Phi = 0$, when k_z and k_y are good quantum numbers, and $\hat{\varepsilon}$ has the form

$$\hat{\varepsilon} = \tilde{\varepsilon} - (2m_3)^{-1}\partial_x^2, \quad \tilde{\varepsilon} = (k_z^2 + k_y^2 - k_F^2)/2m_3. \quad (4.2)$$

At the same time we obtain from (2.5) in place of the Dirac Hamiltonian (3.5) the following Hamiltonian

$$H = \tau_3[\tilde{\varepsilon} - (2m_3)^{-1}\partial_x^2] - c_\perp[i\tau_1\partial_x + \tau_2(k_y - eBx)], \quad (4.3)$$

which no longer exhibits supersymmetry. We shall show nevertheless that the topological character of the spectrum of Fig. 1 is preserved in this case also, i.e., the number of intersections of the spectrum with the $\tilde{\varepsilon}$ axis is equal to one, and thus the anomalous current (3.1b) does not change.

For this it is convenient to go over in Eq. (4.3) to the momentum representation

$$i\partial_x \to p, \quad \tilde{x} = (x - k_y/eB) \to -i\partial_p.$$

As a result of this the Hamiltonian (4.3) takes the form

$$H = \tau_3(\tilde{\varepsilon} + p^2/2m_3) - c_\perp eB\tau_2 i\partial_p - c_\perp\tau_1 p. \quad (4.4)$$

We consider first the case of a weak texture (3.2). In this case the term quadratic in p can be considered as a perturbation, so that the correction to the energy of the zeroth Landau level equals

$$\delta E_0 = \langle 0|\tau_3 p^2/2m_3|0\rangle = -eB/4m_3, \quad (4.5)$$

and the energy of that level

$$E_0 = -(\tilde{\varepsilon} + |eB|/4m_3)\,\mathrm{sign}\,(eB) \quad (4.6)$$

vanishes for $\tilde{\varepsilon} = -|eB|/4m_3$ (Fig. 2).

Thus, in distinction from the erroneous assertion in

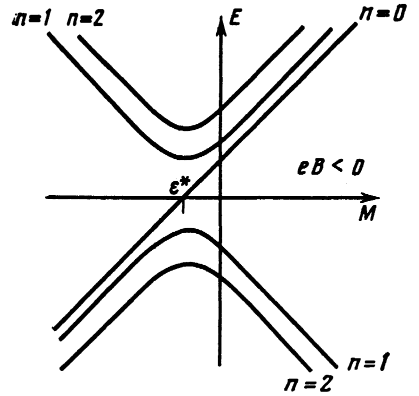

FIG. 2. Deformation of the spectrum in a weak twist texture on account of the deviation of the Bogolyubov equation from the Dirac equation. The spectrum is shown for $eB<0$; the spectrum for $eB>0$ is obtained by the substitution $E\to -E$, on account of charge conjugation (see Appendix A). The branch of the spectrum corresponding to the zeroth Landau level intersects the Fermi surface for $\tilde\varepsilon = \varepsilon^* = -|eB|/4m_3$. The topology of the spectrum does not change compared to Fig. 1, therefore magnitude of the anomalous current (2) does not change.

Ref. 9, no gap appears in the spectrum. Moreover, near zero the spectrum remains infinitely degenerate (there is a set of values of k_y and k_z for which $E_0=0$), just like the spectrum of charged particles in a magnetic field. This leads to a nonvanishing density of states for $E=0$ and to a finite magnitude of the density of the normal component for $T=0$; $\rho_n(T=0)\neq 0$ (see the following section).

We go over to the case of a strong texture (3.3) and verify whether as the field B is increased there does not appear a bifurcation, leading to a change of the topological character of the spectrum. In a strong texture the term linear in p in the Hamiltonian (4.4) is small compared to p^2/m_3, on account of the inequality (3.3). In this case one may use perturbation theory:

$$\begin{gathered} H=H^0+H^1, \\ H^0=\tau_1 W(p)-\tau_2\tilde{B}i\partial_p, \quad H^1=c_\perp\tau_3 p, \\ W(p)=\tilde\varepsilon+p^2/2m_3, \quad \tilde{B}=c_\perp eB. \end{gathered} \tag{4.7}$$

For convenience we have carried out a unitary transformation of the Hamiltonian (4.4)—an isospin rotation by $\pi/2$ around the τ_2 axis. The unperturbed Hamiltonian H^0 is supersymmetric; indeed, there is a superalgebra with generators Q_1 and Q_2:

$$Q_1=H^0, \quad Q_2=i\tau_3H^0, \quad \{Q_i, Q_j\}=\delta_{ij}(H^0)^2, \quad i, j=1, 2, \tag{4.8}$$

where

$$(H^0)^2=W^2(p)-\tilde{B}^2\partial_p^2-\tau_3\tilde{B}\partial W/\partial p. \tag{4.9}$$

On account of the supersymmetry the spectrum of the Hamiltonian is symmetric under the transformation $E\to -E$ for all $\tilde\varepsilon$. Indeed, if there is a state ψ_E with energy E, i.e., $H^0\psi_E=E\psi_E$, then the state $Q_2\psi_E$ has energy $-E$: $H^0Q_2\psi_E=-Q_2H^0\psi_E=-EQ_2\psi_E$. However, the unperturbed Hamiltonian does not have a zero level for any $\tilde\varepsilon$, since the solution

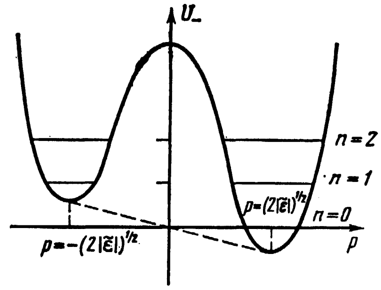

FIG. 3. The spectrum of E^2 in the supersymmetric potential U_- (Eq. (4.11)), formed in a strong twist texture. Each of the wells is harmonic and leads to an oscillator spectrum. The level with $n=0$ in the right well has zero energy. Taking into account tunneling between the wells has the effect that the energy of this level becomes nonzero, albeit exponentially small. The spectrum is shown $eB>0$; for $eB<0$ one must replace p by $-p$.

$$\psi_0\propto\exp\left[-\frac{\tau_3}{\tilde{B}}\int^p dp\, W(p)\right] \tag{4.10}$$

is not normalizable, since it grows without bound either as $p\to\infty$ or as $p\to-\infty$, depending on the sign of $\tilde{B}$. Therefore the Witten condition[12,13] is violated, according to which a zero mode exists if the superpotential $W(p)$ has different signs for $p\to\infty$. In other words, supersymmetry is spontaneously broken. Therefore, at a first glance, it might seem that the topological characteristic of the spectrum has changed in an extremely strong texture. We shall see, however, that this is only a property of the limit Hamiltonian H^0. Switching on an interaction H^1, no matter how weak, restores the character of the spectrum, i.e., the bifurcation occurs in the limit $B\to\infty$, which is never realized, on account of the upper bound (3.3) on B (See Appendix B).

Let us find the value of $\tilde\varepsilon$ for which the energy vanishes. For this we consider the spectrum of the Hamiltonian H^0 for large negative $\tilde\varepsilon$. For this purpose it is more convenient to look for the spectrum E^2 of the Hamiltonian $(H^0)^2$, Eq. (4.9), which reduces to the spectrum of the one-dimensional Schrödinger equation

$$E^2\psi=(-\tilde{B}^2\partial_p^2+U_-)\psi, \quad U_-=W^2-\tilde{B}W' \tag{4.11a}$$

or

$$E^2\psi=(-\tilde{B}\partial_p+W)(\tilde{B}\partial_p+W)\psi. \tag{4.11b}$$

For large negative $\tilde\varepsilon$ the potential U_- represents a double-well potential with almost independent wells (Fig. 3), slanted on account of the weak "electric" field $\tilde{B}$: $U_-=W^2-p\tilde{B}$ ($\tilde{B}>0$ in Fig. 3).

Neglecting tunneling, the spectrum in each of the wells is given by

$$E_n^2=2n|\tilde{B}|(2|\tilde\varepsilon|)^{1/2}, \tag{4.12}$$

and $E_0^2=0$ is obtained to all orders of perturbation theory in $|\tilde\varepsilon|^{-1}$, when the potential U_- is expanded near the bottom of the right well ($\tilde{B}>0$). Since the energy of the zeroth level cannot be exactly equal to zero on account of the violation of the Witten condition, this means that the energy E_0^2 of the zeroth level is not analytic in reciprocal powers of $|\tilde{\varepsilon}|$ and consequently is exponentially small. On account of this the wave function ψ_0 near the right well can be taken as a solution of the equation (4.11) for $E=0$, i.e., $(\tilde{B}\partial_p + W)\psi_0 = 0$, and the function can be continued to values of p for which ψ_0 still falls of on the left, i.e., to $p \sim -(2|\tilde{\varepsilon}|)^{1/2}$ $(m_3 = 1)$:

$$\psi_0 = C\exp\left[-(\tilde{\varepsilon}p + {}^1/_6 p^3)/\tilde{B}\right], \quad p > -(2|\tilde{\varepsilon}|)^{1/2}. \tag{4.13a}$$

On the other hand, far from the well, one may use for $p < (2|\tilde{\varepsilon}|)^{1/2}$ the WKB approximation:

$$\psi_0 \approx \frac{C}{U_-^{1/4}}\exp\left(\frac{1}{\tilde{B}}\int_0^p dp\, U_-^{1/2}\right), \quad p < (2|\tilde{\varepsilon}|)^{1/2}. \tag{4.13b}$$

Since far from the well

$$U_-^{1/2} \approx |W| + {}^1/_2 \tilde{B}W'/|W|, \tag{4.14}$$

the expression (4.13a) and (4.13b) coincide everywhere where $W > 0$ and $W^2 \gg BW'$, i.e., the matching of the solutions (4.13a) and (4.13b) takes place practically in the whole region $(2|\tilde{\varepsilon}|)^{1/2} < p < (2|\tilde{\varepsilon}|)^{1/2}$.

Substituting the wave function (4.13) into the energy functional

$$E_0^2 = \int dp\,|(\tilde{B}\partial_p + W)\psi_0|^2 \Big/ \int dp\,|\psi_0|^2 \tag{4.15}$$

and recognizing that the contribution to the energy from the region $p > -(2|\tilde{\varepsilon}|)^{1/2}$ vanishes on account of $(\tilde{B}\partial_p + W)\psi_0 = 0$ for $p \geqslant -(2|\tilde{\varepsilon}|)^{1/2}$, we obtain that the contribution to the energy comes from the region $p < -(2|\tilde{\varepsilon}|)^{1/2}$ where the wave function (4.13b) is exponentially small compared to $\psi_0(2|\tilde{\varepsilon}|)^{1/2})$:

$$E_0^2 \sim \int_{-\infty}^{-(2|\tilde{\varepsilon}|)^{1/2}} dp\,4W^2|\psi_0|^2 \Big/ \int dp\,|\psi_0|^2$$

$$\sim \psi_0^2(-(2|\tilde{\varepsilon}|)^{1/2})/\psi_0^2((2|\tilde{\varepsilon}|)^{1/2})$$

$$\sim \exp\left[-\frac{2}{\tilde{B}}\int_{-(2|\tilde{\varepsilon}|)^{1/2}}^{(2|\tilde{\varepsilon}|)^{1/2}} dp\,U_-^{1/2}\right] \sim \exp\left(-\frac{2^{7/2}}{3}\frac{|\tilde{\varepsilon}|^{3/2}}{\tilde{B}}\right). \tag{4.16}$$

(The case $\tilde{B} < 0$ is discussed similarly.) This is nothing but the "instanton" exponential which describes the tunneling between the wells. The generic form of the spectrum of the Hamiltonian H^0 is shown in Fig. 4, *a*.

We now consider the influence on the energy of the level with $n = 0$ of a perturbation H^1 to first order of perturbation theory:

$$\delta E_0^1 = \langle 0|H^1|0\rangle = c_\perp (2|\tilde{\varepsilon}|)^{1/2}\,\mathrm{sign}\,(eB). \tag{4.17}$$

This supersymmetry-violating correction simultaneously moves both levels $E_0 = \pm(E_0^2)^{1/2}$ down by the same amount for $eB < 0$ (or up for $eB > 0$), levels which before the perturbation were symmetric, and therefore the positive branch of the spectrum intersects the horizontal axis (Fig. 4b), when $E_0 + \delta E_0^1 = 0$, i.e., for

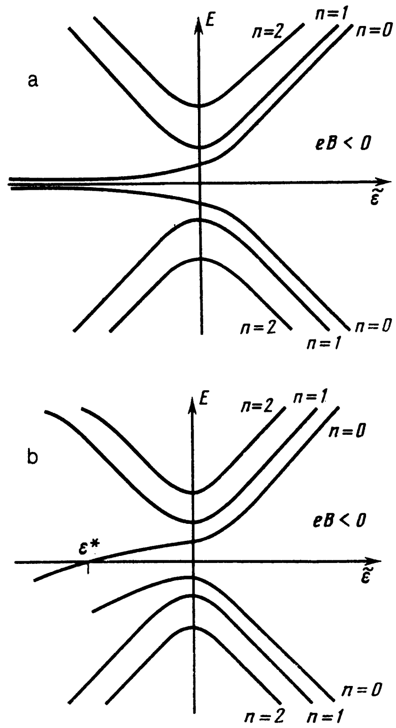

FIG. 4. The spectrum of Fermi excitations in a strong twist texture for $eB < 0$. a) The spectrum of Eq. (4.4) neglecting the term linear in p corresponds to the spectrum in the supersymmetric potential on Fig. 3. The supersymmetry ensures a twofold degeneracy of the spectrum of E^2, i.e., symmetry under the interchange $E \to -E$. However, the spectrum vanishes nowhere on account of the violation of the Witten condition, although for $\tilde{\varepsilon} \to -\infty$ the energy tends to zero according to the law $E \propto \exp(-|\varepsilon|^{3/2}/\tilde{B})$. b) Taking into account the small term linear in p, $c_\perp \tau_1 p$ in Eq. (4) by means of perturbation theory has the effect that one of the branches with $n = 0$ intersects the abscissa axis. Thus, the topology of the spectrum in a strong texture is the same as in a weak twist texture (Figs. 1 and 2). Therefore the magnitude of the anomalous current is conserved. The density of states is, however, different for the weak and the strong textures (see Eq. (5.3)).

$$\tilde{\varepsilon} = \varepsilon^* \sim -|eB|\left(\frac{c_\perp^2}{|eB|}\ln^2\frac{|eB|}{c_\perp^2}\right)^{1/3}. \tag{4.18}$$

Thus the character of the spectrum does not change in a strong texture. It is shown in Appendix B that the topology of the spectrum is preserved also in intermediate textures. Consequently, the anomalous current (3.13b) is conserved in any twist texture.

5. THE DENSITY OF STATES FOR $E=0$

The infinite degeneracy of the levels near $E = 0$ should lead to a finite density of states for $E = 0$ in any twist texture and, consequently, to a finite density of the normal component for $T = 0$. However, whereas the magnitude of the anomalous current does not depend on the details of the spectrum and is determined by its topological characteristics, the magnitude of the density of states depends on the details of the spectrum near $E = 0$ and therefore has a different form in strong and weak textures.

Only the branch of the spectrum which intersects the horizontal axis contributes to the density of states at $E = 0$:

$$N(0)=2\int\frac{dk_z}{2\pi}\int\frac{dk_y}{2\pi}|\chi_0(x=0)|^2\delta(E_0). \quad (5.1)$$

Since χ_0 depends on x only via the combination $\tilde{x} = x - k_y/eB$ the integral with respect to k_y of $|\chi_0|^2$ yields $|eB|$, hence

$$N(0)=\frac{|eB|}{2\pi^2}\int\frac{d\xi}{k_F}\delta(E_0). \quad (5.2)$$

Substituting E_0 from Eqs. (4.6) and (4.16) + (4.17), for weak and strong textures, respectively, we obtain

$$N(0)=|\mathbf{l}\,\mathrm{rot}\,\mathbf{l}|f(\eta)/2\pi^2, \quad (5.3a)$$

where

$$f(\eta)=\begin{cases}1, & \eta\ll1,\\ \sim\eta^{1/3}\ln^{-2/3}\eta, & \eta\gg1,\end{cases}$$

$$\eta=|\mathbf{l}\,\mathrm{rot}\,\mathbf{l}|v_Fk_F^2/\Delta_0^2. \quad (5.3b)$$

For the case of a weak texture ($\eta \ll 1$) this result was derived in Ref. 4. Although the density of states grows rapidly in a strong texture ($\eta \gg 1$), it still remains small compared to that which appear in a texture where curl **l** is parallel to **l**, when, according to Ref. 3,

$$N(0)=\frac{1}{4\pi^2}\frac{k_Fv_F}{\Delta_0}|[\mathbf{l},\mathrm{rot}\,\mathbf{l}]|. \quad (5.4)$$

The contributions of the textures to the density of states become equal for

$$|[\mathbf{l},\mathrm{rot}\,\mathbf{l}]|\sim|\mathbf{l}\,\mathrm{rot}\,\mathbf{l}|(|\mathbf{l}\,\mathrm{rot}\,\mathbf{l}|\Delta_0/k_F\varepsilon_F)^{1/3},$$

i.e., in order to observe the strong non-analyticity created by the twist texture, the latter must be very pure, i.e., contain a small admixture of a texture of a different kind with $\mathbf{l}\times\mathrm{curl}\,\mathbf{l}\neq0$.

A finite density of states $N(0)$ must lead to a nonvanishing density of the normal component $\rho_n\ (T=0) = k_F^2N(0)$ and a linear specific heat $C(T) = N(0)T$.

6. CONCLUSION

Until now the C_0-anomaly (2) in the current, related to the existence of zeros in the quasiparticle spectrum, has been studied starting from an approximate description of the type of a gradient expansion (Refs. 5, 6, 15). In that approach a fundamental role is played by the phase $\Phi(\mathbf{k},\mathbf{r})$ of the order parameter, which depends both on coordinates and momenta,

$$\Delta(\mathbf{r})\mathbf{k}=|\Delta(\mathbf{r})\mathbf{k}|\exp[i\Phi(\mathbf{k},\mathbf{r})] \quad (6.1)$$

(in the general case of an order parameter which depends on spin this is the phase of the determinant of the order parameter). In ^{3}He-A this phase has a topologically nonremovable vortex singularity on the Fermi surface; going around this singularity on the Fermi surface changes the phase by 2π, as a result of which in the core of this singularity the order parameter vanishes, and with it the gap in the excitation spectrum. This vortex singularity is not removed from the texture and leads to the anomalous current (3.13b), which is wholly concentrated in the core of the singularity (Refs. 5, 16):

$$\mathbf{j}_{an}=\frac{1}{2}\sum_{\mathbf{k}}\mathbf{k}n(\mathbf{k},\mathbf{r})(\partial_{\mathbf{r}}\partial_{\mathbf{k}}-\partial_{\mathbf{k}}\partial_{\mathbf{r}})\Phi(\mathbf{k},\mathbf{r}) \quad (6.2)$$

(where $n(\mathbf{k},\mathbf{r})$ is the quasiparticle distribution function), although at the singularity proper the gradient expansion does not work anymore. Near the singularity one must solve the exact quantum-mechanical problem of the quasiparticle spectrum near zero energy, a task which was accomplished in the present paper. Our results have shown that the result for the anomalous current was not changed. The topological invariant which had guaranteed the existence of an anomalous current in the gradient expansion (the accumulation of phase near the singularity) has been converted in the quantum-mechanical problem into a topological characteristic of the spectrum in the Bogolyubov-Dirac equation for fermions in a magnetic field, thus guaranteeing the existence (and texture-type independence) of the same current, but now in the form of a chiral anomaly.

Although the correspondence of these two (in principle different) topological characteristics was established in this paper, the reason why one of them (the change of phase by 2π on going around the vortex singularity) implies the existence of the second (the number of intersections of the quasiparticle spectrum with the horizontal axis equals one) is not clear for the moment. Moreover, the gradient expansion leads to a complete expression (1) for the current in the texture, whereas in the quantum case considered here all quasiparticle levels with $n\neq0$ do not contribute to the current. Therefore, either the other terms in the current (1) appear when one takes into account levels with high quantum numbers, i.e., in the quasiclassical approximation, or the expression for the current may be essentially different from Eq. (1), expression obtained by means of the gradient expansion, e.g., it could be nonlocal (see note added in proof).

Further, the same topological characteristic of the spectrum which ensures the intersection of one of the branches with the zero-energy axis also leads to a nonvanishing density of states $N(0)$. Here there is also a qualitative agreement with the result of the gradient expansion, which shows that the energy spectrum has the form

$$E=(\varepsilon^2+\Delta_0^2[\mathbf{k},\mathbf{l}]^2/k_F^2)^{1/2}+{}^1/_2\mathbf{k}\nabla\Phi(\mathbf{k},\mathbf{r}). \quad (6.3)$$

In this case the energy vanishes on a two-dimensional surface, which leads to a nonzero density of states $N(0)$ and, consequently, to a nonvanishing density ρ_n of the normal component for $T=0$.[2]

However, although the nonvanishing of $N(0)$ and $\rho_n\ (T=0)$ is in itself topologically stable, the magnitudes of these quantities are no longer topological characteristics (in distinction from the value of the anomalous current) but depend on details of the fermion spectrum near zero energy. Therefore the results for $\rho_n\ (T=0)$ in the exact quantum-mechanical problem turned out to be different from the result predicted in the framework of the gradient expansion ($N(0)\sim B^2/\Delta_0^2$ for a twist texture).

We note that in those superfluid phases in which the zeros of the spectral gap are unstable with respect to small "stirrings" (deformations) (for instance, a polar or a planar phase), the fermion spectrum may in principle acquire a gap in definite textures, and in these textures the chiral anomaly must be absent from the current (on the magnitude of ρ_n ($T=0$) in the polar phase, see Ref. 17).

The existence of an anomalous current carried by vacuum fermions situated in the gapless part of the spectrum must lead to a reconsideration of the dynamics of ^{3}He-A as $T\to 0$. These vacuum fermions form an additional subsystem in ^{3}He-A, the interaction of which with the remainder of the superfluid and with the excitations which form a normal fluid must still be investigated. In particular, one must determine whether there exists a characteristic time τ which separates the hydrodynamic (low-frequency) regime from the collision-free (high-frequency) regime at $T=0$, when the gapless subsystem either manages, or does not manage, respectively, to follow the superfluid system. In the latter case the anomalous current becomes a dynamical invariant in the equation of superfluid dynamics, as discussed in Ref. 15.

The authors express their gratitude to S. P. Novikov and S. B. Khokhlachev for valuable discussions.

APPENDIX A

Symmetry in ^{3}He-A and in (2+1)-QED

The symmetry of the equations for the fermionic excitations of the vacuum, described by the spinor χ, and for the bosonic variables $\mathbf{A}=k_F\delta\mathbf{l}$, which describe the deviations of the order parameter from the equilibrium vacuum state

$$\Delta^{(0)}=(\Delta_0/k_F)(\hat{\mathbf{x}}+i\hat{\mathbf{y}}), \qquad (A.1)$$

is determined by the symmetry of this vacuum state. The following are the symmetry operations to which the vacuum state (A.1) can be subjected.

1. Combined gauge-rotational symmetry $U^{comb}(1)$. Each element $O_\alpha U_{\alpha/2}$ of this group is a combination of the rotation $O_\alpha U_{\alpha/2}$ by an angle α around the z axis with the simultaneous gauge transformation $U_{\alpha/2}$ of the fermions, which for the two-fermion order parameter (2.2) consists in multiplication by $\exp(i\alpha)$. The transformation $O_\alpha U_{\alpha/2}$ does not change the vacuum state

$$O_\alpha^z U_{\alpha/2}\Delta^{(0)}=\Delta^{(0)}. \qquad (A.2)$$

Therefore, under such a transformation the Bogolyubov equation for χ and the hydrodynamic equation for $\mathbf{A}$ do not change, whereas the variables themselves transform under $U^{comb}(1)$ in the following manner:

$$O_\alpha^z U_{\alpha/2}\chi=\exp(i\alpha\tau_3/2)\chi(O_\alpha^z\mathbf{r}), \qquad (A.3a)$$

$$O_\alpha^z U_{\alpha/2}\mathbf{A}=O_\alpha^z\mathbf{A}(O_\alpha^z\mathbf{r}). \qquad (A.3b)$$

In the case when the Bogolyubov equation reduces to the Dirac equation (3.5) in (2 + 1) dimensions, $U^{comb}(1)$ coincides with the normally defined rotations in the x(y-plane which are part of the Lorentz group. The conservation of the Noether current in ^{3}He-A and in (2 + 1)-QED which follows from invariance with respect to the continuous group $U^{comb}(1)$ corresponds to the conservation of the projection of the angular momentum onto the z axis.

2. The parity $\tilde{P}$. The Cooper pairing in ^{3}He-A occurs in the odd p-state, therefore parity P itself is not conserved, but only its combination with a gauge transformation $U_{\pi/2}$, which also changes the sign of the order parameter:

$$\tilde{P}=PU_{\pi/2},\quad \tilde{P}\Delta^{(0)}=\Delta^{(0)}. \qquad (A.4)$$

The variables χ and $\mathbf{A}$ transform in the following manner:

$$\tilde{P}\chi=i\tau_3\chi(-\mathbf{r}), \qquad (A.5a)$$

$$\tilde{P}\mathbf{A}=\mathbf{A}(-\mathbf{r}). \qquad (A.5b)$$

In (2 + 1)-QED (3.5) the transformation $\tilde{P}$ is a special case of $U^{comb}(1)$, namely a rotation by an angle π. The lack of correspondence between the definition of parity for $\mathbf{A}$ in ^{3}He-A and in QED is compensated by a change of sign of the charge in Eq. (3.6):

$$\tilde{P}e=-e. \qquad (A.5c)$$

3. Charge conjugation C,

$$C\chi=\tau_1\chi^*, \qquad (A.6)$$

does not change the order parameter (2.2), according to Eq. (2.4). This transformation is completely equivalent to charge conjugation in QED, since under complex conjugation the momentum k_z of the spinor χ changes into its opposite, which corresponds to a change of sign of the charge (3.6):

$$Ce=-e. \qquad (A.7)$$

4. Combined temporal parity. In ^{3}He-A the symmetry with respect to the operation T of time reversal is violated, since the vacuum state (A.1) changes under this transformation into its complex conjugate. However, time-reversal combined with a rotation O_π^x by π around the x axis:

$$T_1=TO_\pi^x,\quad T_1\Delta^{(0)}=\Delta^{(0)}. \qquad (A.8)$$

Under the action of O_π^x and T the variables χ and $\mathbf{A}$ undergo the following transformations:

$$T\chi(\mathbf{r},t)=i\tau_2\chi^*(\mathbf{r},-t), \qquad (A.8a)$$

$$T\mathbf{A}(\mathbf{r},t)=-\mathbf{A}(\mathbf{r},-t), \qquad (A.8b)$$

$$Te=e, \qquad (A.8c)$$

$$O_\pi^x\chi(\mathbf{r},t)=i\tau_1\chi(O_\pi^x\mathbf{r},t), \qquad (A.9a)$$

$$O_\pi^x\mathbf{A}(\mathbf{r},t)=O_\pi^x\mathbf{A}(O_\pi^x\mathbf{r},t). \qquad (A.9b)$$

5. Particle-hole symmetry. In going over the quantum electrodynamics (3.5) we can make use of yet another symmetry in ^{3}He, which holds only approximately: the symmetry between particles and holes near the Fermi surface where $\delta\varepsilon$ is small. We denote the operation interchanging ε and $-\varepsilon$ (or M by $-M$) by P_M; then for small $|\varepsilon|\ll\varepsilon_F$ the symmetry T_1 breaks up into two separately conserved combined symmetries: $T_2=TP_M$ and $P_2=O_\pi^xP_M$. Under the action of the transformation

$$P_2=O_\pi^xP_M \qquad (A.10)$$

the variables transform according to Eq. (A.9). For (2 + 1)-QED moreover, the fermion mass but not its charge is subject to a transformation (the latter being a scalar):

$$P_2M=-M, \quad P_2e=e. \tag{A.11}$$

For those quantities which are invariant under $M\to -M$ the symmetry O_π^x is exact. Under the transformation T_2

$$T_2=TP_M, \tag{A.12}$$

χ and **A** transform according to the laws (A.8), and in QED we have in addition

$$T_2M=-M, \quad T_2e=e. \tag{A.13}$$

6. Spatial and temporal parities for (2+1)-QED. For massless fermions ($M=0$), O_π^x and T are separately conserved, and O_π^x represents ordinary parity in (2 + 1)-QED, since it corresponds to a sign change of one of the two space components. In the chiral anomaly it is exactly the parity O_π^x which is violated (this violation is also accompanied by a violation of T or P_M, since the combined forms $O_\pi^x T$ and $O_\pi^x P_M$ are always conserved). Although for massive fermions O_π^x is not a symmetry of the Dirac equation, one might expect that in the fermion vacuum parity is conserved, i.e., that the spectrum of the fermions is invariant with respect to $eB\to -eB$ [T-invariance, see (A.8)], and separately with respect to $E\to -E$ (CT-invariance). As can be seen from Fig. 1, the zeroth Landau level does not support this symmetry. This proves that the separate symmetries are violated in the vacuum, among them P_M. Indeed, such characteristics of the vacuum as the susceptibility of the chiral current (3.18), $(e^2/4\pi)\,\mathrm{sign}\,M$ changes sign under the substitution $M\to -M$.

APPENDIX B

Qualitative theory of differential equations for the Bogolyubov Hamiltonian

In this Appendix we make use of the qualitative theory of differential equations to prove that there exists a solution of the Bogolyubov equations (4.4) with $E=0$ for some value of $\tilde\varepsilon$, and that its solution exists for any twist texture. It will be seen from the arguments that this level is infinitely degenerate, since $\tilde\varepsilon$ depends on two well-defined quantum numbers k_y and k_z: $\tilde\varepsilon(k_y,k_z)$. The equation (4.4) has the form

$$H\begin{pmatrix}u\\v\end{pmatrix}=E\begin{pmatrix}u\\v\end{pmatrix},\quad \begin{array}{l}H=\tau_3(\tilde\varepsilon+p^2/2)-\eta\tau_2 i\partial_p-\zeta\tau_1 p,\\ \eta=c_\perp eB\ \ (eB>0),\quad \zeta=c_\perp>0.\end{array} \tag{B.1}$$

We introduce the complex-valued function $w=u+iv$. Then the eigenvalue equation takes the form

$$\eta w'=Eiw+(\zeta p+iW)\bar w, \tag{B.2}$$
$$w'=\partial w/\partial p,\quad \bar w=u-iv,\quad W(p)=\tilde\varepsilon+p^2/2.$$

Writing Eq. (B.2) separately for the real and imaginary parts of the function $w=r\exp(i\varphi)$ we obtain the system of two equations:

$$\eta r'=\zeta pr\cos 2\varphi+Wr\sin 2\varphi, \tag{B.3a}$$

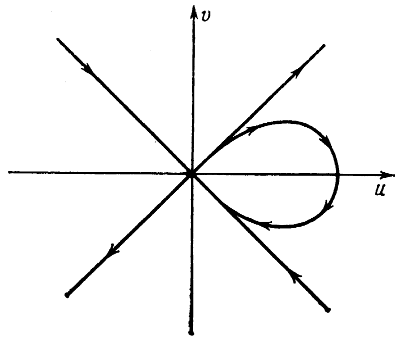

FIG. 5. The phase portrait of the system (4.4) as $p^2\to\infty$. The phase trajectory $(u(p), v(p))$ corresponding to a normalizable wave function χ is shown schematically: $\arg(u+iv)=\pi/4$ as $p\to-\infty$ and $\arg(u+iv)=-\pi/4$ as $p\to+\infty$.

$$\eta\varphi'=E-\zeta p\sin 2\varphi+W\cos 2\varphi. \tag{B.3b}$$

In what follows we investigate Eq. (B.3b) since it involves the energy E. As regards the absolute value of the function w: $|w|=r$, the requirement that the wave function should be normalizable can be satisfied by setting, e.g., $\varphi(p=-\infty)=\pi/4$, $\varphi(p=+\infty)=-\pi/4$; then $r(\pm\infty)=0$. Indeed, the phase portrait of the dynamical system (B.1) for $p^2\to\infty$ has the form represented in Fig. 5. We also note that Eq. (B.3b) is invariant under the transformation $\varphi\to-\varphi, p\to-p$.

We consider qualitatively the behavior of the solution of Eq. (B.3b), in its dependence on the parameters. For this purpose we introduce the function $\varphi_0(E,\tilde\varepsilon)=\varphi(p=0)$. Since $\varphi(p=-\infty)=\pi/4$ is an unstable point, it follows that the solution of (B.3b) satisfying this boundary condition is unique. One can prove the following obvious properties of φ_0.

1) For fixed $\tilde\varepsilon$ the function φ_0 increases monotonically with E.

2) For $E=0$, with growing $\tilde\varepsilon$ the function φ_0 increases monotonically. It is easy to show that for $E=0$

$$\varphi_0=\begin{cases}\pi/4, & \tilde\varepsilon\to\infty\\ -\pi/4, & \tilde\varepsilon\to-\infty\end{cases},\quad \zeta\neq0, \tag{B.4a}$$

$$\varphi_0=\eta/4,\quad \tilde\varepsilon\to\pm\infty,\quad \zeta=0, \tag{B.4b}$$

whence, on account of the single-valuedness of the solution of (B.3b) and the continuity of the dependence on parameters, it follows that for $\zeta\neq0$ there exists a single trajectory which for some $\tilde\varepsilon=\varepsilon^*$ passes through the point $\varphi_0=0$. On account of the symmetry of Eq. (B.3b) with respect to the substitution $\varphi\to-\varphi, p\to-p$ this trajectory automatically satisfies the boundary conditions at $p=+\infty$: $\varphi(+\infty)=-\pi/4$ (See Fig. 6).

Thus, for any $\zeta\neq0$ there exists a solution of equation (B.3b) with $E=0$. For $\zeta=0$ the point ε^* of intersection of the line representing the spectrum for $n=0$ with the $\tilde\varepsilon$ axis goes off to $-\infty$, as shown in Fig. 4.

Starting with Eq. (B.3b) we now calculate the point ε^* for which $E=0$. We shall assume that ζ is sufficiently small and that $\tilde\varepsilon\gg\eta^{2/3}$ (see sec. 4). Then, for $p<0$ we consider two regions in Eq. (B.3b).

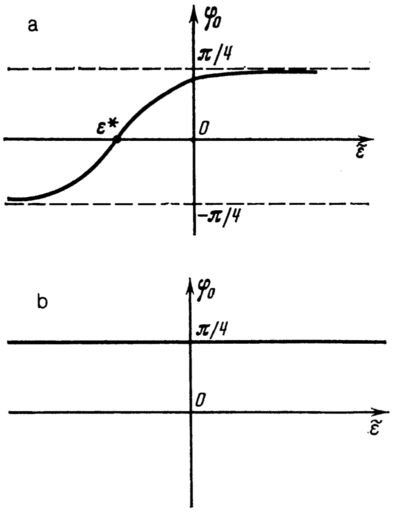

FIG. 6. a) $\zeta \neq 0$. There exists a point ε^* where $\varphi_0 = 0$. This implies the existence of a level with $E = 0$. b) The point $\zeta = 0$ is a bifurcation point for the Hamiltonian (B.1). In this limit there is no level with $E = 0$ (see Fig. 4, a).

1) $p \ll -(2|\tilde{\varepsilon}|)^{1/2}$. In this case we have

$$\varphi(p) \approx \pi/4 + \zeta/p. \qquad \text{(B.5)}$$

2) $-(2|\tilde{\varepsilon}|)^{1/2} \ll p < 0$, then solving (B.3b) and neglecting p^2 and ζp compared to $\tilde{\varepsilon}$, we obtain

$$\tilde{\varepsilon} p = {}^1/_4 \eta \ln |(1+\sin\varphi)/(1-\sin\varphi)|. \qquad \text{(B.6)}$$

Joining the solutions (B.5) and (B.6) at $p \approx -(2|\tilde{\varepsilon}|)^{1/2}$ we obtain

$$\varepsilon^* \sim -\eta^{2/3} [\ln (\eta^{1/3}/\zeta)]^{2/3}, \qquad \text{(B.7)}$$

whence, substituting $\eta = c_\perp eB$ and $\zeta = c_\perp$, we obtain the desired result (4.18).

We now consider the case $\zeta = 0$. Then the equation (B.3b) takes the form

$$\eta\varphi' = E + W\cos 2\varphi. \qquad \text{(B.8)}$$

For $p \to 0$ one can solve the equation (B.8), assuming $\tilde{\varepsilon} \gg p^2$. Then

$$(\operatorname{tg}\varphi + 1 + E/\tilde{\varepsilon})(\operatorname{tg}\varphi - 1 - E/\tilde{\varepsilon})^{-1} = e^{2\tilde{\varepsilon}p/\eta}. \qquad \text{(B.9)}$$

Then, for $E \ll \tilde{\varepsilon}$, $p \sim -(2|\tilde{\varepsilon}|)^{1/2}$, joining the solution (B.9) to the asymptotic solution $\varphi(p \to -\infty) = \pi/4$ we obtain

$$E \sim |\tilde{\varepsilon}| \exp[-|\tilde{\varepsilon}|^{3/2}\eta^{-1}O(1)]. \qquad \text{(B.10)}$$

In the argument of the exponential of (B.10) we have written the order of magnitude $O(1)$ since one cannot determine the constant in (B.10) by this method. For $\zeta = 0$ the energy does not vanish anywhere, although it is exponentially small, which points to a spontaneous breaking of the supersymmetry. Thus, the point $\zeta = 0$ is a bifurcation point for the equation (B.3b). However, this point corresponds to $B = \infty$ and consequently, in a real texture the bifurcation is absent, and the anomalous current exists for all twist-textures.

Note added in proof (April 17, 1986). A more exact discussion, making use of the Atiyah-Singer index theorem for an elliptic operator, shows that both curl terms in the current (1) are consequences of the chiral anomaly.

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