# Radiation produced by collision of two fluxons in a long Josephson junction

Yu. S. Kivshar' and B. A. Malomed

*Research Institute for Biological Testing of Chemical Compounds*

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The collision of two fluxons of opposite polarity in a long (linear or annular) Josephson high-current dissipative junction is investigated. The total radiated collision energy and its spectral composition are obtained.

## 1. INTRODUCTION

One important application of Josephson junctions is their use to generate microwave radiation (see, e.g., Refs. 1 and 2). In particular, the radiation accompanies various dynamic processes in which magnetic-flux quanta (fluxons) participate. In the case of one fluxon, the generation can be due to inhomogeneity of the junction (e.g., in the presence of reflecting end points[1] or of distributed inhomogeneity,[3–5] and also to the action of an alternating external field on the fluxon.[6]

If two fluxons are present, wave generation is made possible even in a homogeneous medium by interaction between them.

The dynamics of a pair of fluxons in a long Josephson junction was investigated theoretically in Refs. 3 and 7–11. It was shown in Refs. 9 and 12, in particular, that in the presence of reflecting edges a pair of fluxon with arbitrary relative polarity is transformed as a result of the reflection into a "bunched" (bound) pair of like polarity.

We consider the interaction of fluxon of opposite polarity in a system without end points (in an annular or infinitely long linear junction), described by the equation[3,8–11]

$$\Phi_{tt}-\Phi_{xx}+\sin\Phi=f-\gamma\Phi_t, \tag{1}$$

where $\Phi$ is the normalized magnetic flux, $f$ the dimensionless density of the extraneous current, and $\gamma$ the dissipative coefficient. The coordinate $x$, directed along the junction, and the time $t$ are normalized respectively to the characteristic space and time scales (see, e.g., Ref. 6). At $f=\gamma=0$, Eq. (1) goes over into an exactly integrable sine-Gordon equation (SGE). Its elementary solution, which describes one fluxon, is of the form

$$\Phi(x,t)=4\,\mathrm{arctg}\left\{\exp\left[\sigma\frac{x-vt}{(1-v^2)^{1/2}}\right]\right\}, \tag{2}$$

where $v$ is the fluxon velocity and $\sigma=\pm 1$ is its polarity. At small $f$ and $\gamma$ on can apply to Eq. (1) the perturbation theory for solitons (Refs. 3, 7, 12, 13). In particular, this equation has as before a stable solution in the form of a kink whose velocity $v_*$ is no longer arbitrary, but is uniquely expressed in terms of the perturbation parameters[3,14]

$$v_*/(1-v_*^2)^{1/2}=\pi\sigma f/4\gamma. \tag{3}$$

In addition, small perturbations distort slightly the form of the kink, and the asymptotic values of the wave field $\Phi_0$ as $x\to\pm\infty$ are determined by the relation

$$\sin\Phi_0=f. \tag{4}$$

We calculate in this paper the total energy radiation upon collision of fluxons of opposite polarity and moving with velocities $\pm v_*$ and the spectral composition of the radiation. Note that collision of a fluxon with an antifluxon is accompanied, besides by collision radiative loss, also by direct energy loss due to the presence of dissipation in Eq. (1). Owing to these losses, the collision leads to coalescence of the fluxon and antifluxon into a bound state called bion (breather), if the extraneous-current density $f$ is much less than the critical value[7,10]:

$$f_{cr}=4\gamma^{1/2}. \tag{5}$$

We consider only the case $f>f_{cr}$, when the fluxon-antifluxon pair does not annihilate into a bion.

## 2. RADIATED ENERGY

Expressions for the total radiated energy and for its spectral density can be obtained analytically by a perturbation theory based the method of the inverse scattering problem,[7–13] under the conditions

$$f\ll 1, \tag{6}$$

$$f/\gamma\gg 1. \tag{7}$$

The condition (6) and the condition $\gamma\ll 1$ that follows from (6) and (7) permits the use of perturbation theory in the small parameters $f$ and $\gamma$, while the second condition means that $(1-v_*)^{1/2}\sim\gamma/f\ll 1$, i.e., the fluxons are "relativistic"; this simplifies substantially the subsequent calculations.

To allow for the perturbation-induced shift (4) of the asymptotic values of $\Phi$, it is convenient to transform to a new variable $u$: $\Phi=\Phi_0+u$, for which Eq. (1) takes approximately the form

$$u_{tt}-u_{xx}+\sin u=f(1-\cos u)-\gamma u_t. \tag{8}$$

In terms of the inverse problem method,[15] the radiation is described by a complex coefficient $b(\lambda)$ that constitutes the scattering data for the continuous spectrum. Here $\lambda$ is a real spectral parameter connected with the radiation wave number $k$:

$$k=\lambda-1/4\lambda. \tag{9}$$

The spectral energy density $E_{rad}$ takes in the case of small $|b(\lambda)|^2$ the form[7]

$$\rho(\lambda)\equiv dE_{rad}/d\lambda\approx(1+4\lambda^2)\,|b(\lambda)|^2/\pi\lambda^2. \tag{10}$$

The physical spectral density is easily expressed in terms of (10) with allowance for (9):

$$\mathscr{E}(k)\equiv\frac{dE_{rad}}{dk}=\left[\frac{dE_{rad}}{d\lambda}\frac{4\lambda^2}{1+4\lambda^2}\right]\Bigg|_{\lambda=\frac{1}{2}[k+(1+k^2)^{1/2}]}. \tag{11}$$

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The perturbation governed evolution of $b(\lambda)$ is determined by known equation of soliton perturbation theory[10,13,16]

$$\frac{\partial b(\lambda,t)}{\partial t}=-i\left(\lambda+\frac{1}{4\lambda}\right)b(\lambda,t)+\frac{i}{4}\int_{-\infty}^{\infty}dx[f(1-\cos u)-\gamma u_t]$$

$$\{[\Psi^{(2)*}(x,t;\lambda)]^2-[\Psi^{(1)*}(x,t;\lambda]^2\},\qquad(12)$$

where $\Psi^{(1,2)}(x,t;\lambda)$ are called Jost functions and are used in the inverse-problem method. Since a solitary fluxon is an exact solution of Eq. (1) and does not radiate energy, it is natural to assume that no radiation exists prior to the collision. This means that Eq. (12) can be supplemented by the initial condition $b(\lambda, t=-\infty)=0$. The total density of the energy radiated as a result of the collision is determined by expression (10), in which $b(\lambda)$ must be replaced by $b(\lambda, t=+\infty)$. Obtaining this value from (12) with (7) taken into account, we obtain after rather laborious calculations

$$\mathscr{E}(k(\lambda))=\frac{f}{\nu^2}\left\{|J(\lambda)|^2+\left|J\left(\frac{1}{4\lambda}\right)\right|^2+2\operatorname{Re}\left[J(\lambda)J\left(\frac{1}{4\lambda}\right)\right]\right\},\qquad(13)$$

where

$$J(\lambda)=\frac{\pi^2}{8\nu^2(\lambda-i\nu)^2(\lambda-i/4\nu)^2}\left[\frac{Q_1(\lambda)}{\operatorname{sh}a\operatorname{sh}b}-\frac{Q_2(\lambda)}{\operatorname{ch}a\operatorname{ch}b}+\frac{Q_3(\lambda)}{\operatorname{sh}a\operatorname{ch}b}\right],\qquad(14)$$

$$a=\pi\lambda/2\nu,\quad b=\pi/8\lambda\nu,$$

while $Q_m$ $(m=1, 2, 3)$ are the complex polynomials

$$Q_1(\lambda)=\frac{i\lambda^3}{\nu}-2\lambda^2\nu^2-\frac{i\lambda^2}{4\nu}+i\nu-\frac{1}{8}+\frac{i}{32\lambda\nu},$$

$$Q_2(\lambda)=\frac{2i\lambda^5}{\nu}+\lambda^4+2i\lambda^3\nu+\lambda^2\nu^2+\frac{i\lambda}{16\nu}+\frac{i}{64\lambda\nu},$$

$$Q_3(\lambda)=\pi\left(i\lambda^4+4\nu^3\lambda^3-i\nu^2\lambda^2+\frac{\lambda\nu}{4}-\frac{i}{16}\right).$$

We have introduced here a quantity which is designated as

$$\nu=(1-v_*^2)^{-1/2}\approx\pi f/4\gamma,\qquad(15)$$

and is a large parameter by virtue of (7).

The dependence of the physical density of the radiated energy on the wave number is shown in the figure. It can be seen from (13) that $\mathscr{E}(k)=\mathscr{E}(-k)$ (reversal of the sign of $k$ is equivalent, according to (9), to replacement of $\lambda$ by $1/4\lambda$). This symmetry is perfectly natural, since the waves emitted to the left and to the right should carry away equal energies.

It is readily understandable that the value $k_{\max}$ corresponding to the maximum of the spectral density (Fig. 1) is of the order of the reciprocal width (2) of the fluxon: $k_{\max}\sim\nu$. An investigation of expressions (13) and (4) yields the corresponding value

$$\mathscr{E}_{\max}\equiv\mathscr{E}(\pm k_{\max})\sim f^2/\nu^4\sim\gamma^4/f^2.$$

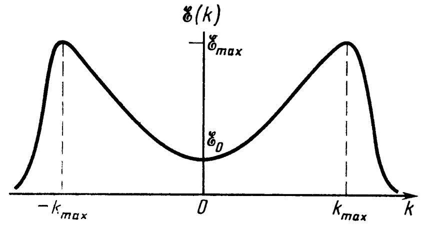


FIG. 1. Spectral density of radiated energy.

In view of the condition (7), the dependence of the principal term in expression (13) or (14) for the density of the radiated energy on the parameters $f$ and $\gamma$ and on the wave number $k$ (in the region $k\sim k_{\max}$) can be represented in the "self-similar" form

$$\mathscr{E}(k)\approx(\gamma^4/f^2)F(k\gamma/f).\qquad(16)$$

For large $k$, $|k|\gg k_{\max}$, the energy density falls off exponentially. Its value at $k=0$ is $\mathscr{E}_0\sim\gamma^6/f^4$.

We obtain the total radiated energy by integrating over all $k$:

$$E_{rad}=\int_{-\infty}^{\infty}\mathscr{E}(k)dk=\int_0^{\infty}\rho(\lambda)d\lambda.\qquad(17)$$

The condition (7) facilitates greatly the calculation of the integral in (17). We ultimately get

$$E_{rad}=2^6D\gamma^2/\pi^3f+O(\gamma^5/f^3).\qquad(18)$$

The coefficient $D$ can be expressed rather cumbersomely in terms of numerical values of non-elementary functions. We have approximately $D=0.568$.

We note in conclusion that both perturbed terms in (1) make contributions of the same order to (17). In fact, using (3) we obtain the simple estimate $\gamma u_t\sim\gamma v_*/(1-v_*^2)^{1/2}\sim f$.

## 3. CONCLUSION

The calculations above made use essentially of condition (7). The radiated energy $E_{\rm rad}$ and its spectral composition can in principle be determined also for the case $f\lesssim\gamma$ (Ref. 7). Although the expressions obtained turn out to be exceedingly unwieldy, it is easy to write down in this case an estimate for the total radiated energy: $E_{\rm rad}\sim\gamma^2$. (In contrast to the situation $f\gg\gamma$ considered above, the radiation energy is concentrated in a region with wavelengths larger than or of the order of unity.) Note that this energy is always less than the "direct" energy loss due to dissipation: $E_{\rm dis}\sim\gamma$ (Ref. 13). In particular, the radiation loss should lead to a relatively small increase of the quantity $f_{\rm cr}$ defined in (5). Reasoning similar to that developed in the preceding section yields readily a corresponding estimate for the relative change of this quantity: $\delta f_{\rm cr}/f_{\rm cr}\sim\gamma^2$.

So far, all our calculations pertained strictly speaking to an infinite straight line. From the physical point of view, it is more realistic to consider a circle, meaning a narrow annular Josephson junction.[3,5,17] Using the results of the numerical calculations of Ref. 17 we can conclude that the equations obtained by us for an infinitely long Josephson junction are applicable also to this problem, if the length $L$ of the ring is large enough compared with the fluxon dimension: $L \gtrsim 10(1-v_*^2)^{1/2}$. The continuous spectrum shown in Fig. 1 is then transformed into a discrete one, consisting of many individual lines spaces $\sim L^{-1}$ apart. The envelope of this spectrum is close to the curve shown in the figure. The total power $P$ of this radiation can be expressed in obvious manner by (17) and (18): $P = 2E_{\rm rad}/L$ (where it is recognized that in the intervals between the collisions the fluxon velocity is close to unity by virtue of condition (7)).

We note that an "annular generator" of microwave radiation based on a Josephson junction was proposed in Ref. 3, in which is calculated the radiation generated by one fluxon that is successively scattered by a system of "microshorts" (microinhomogeneities) built into the junction. In the present paper we conclude in fact the feasibility of a "two-fluxon generator" based on a homogeneous annular junction. According to (16), the spectral composition of the radiation can be easily varied by changing the density of the extraneous current $f$. To assess the real possibility of creating such a generator, however, it is necessary to take into account also at least the influence of the perturbation on the generated radiation.

*Note added in proof (16 April 1986).* A. Davidson, B. Dueholm, B. Krygger, and N. F. Pederson report in a recent experimental paper [Phys. Rev. Lett. **55**, 2059 (1985)] the first direct observation of one and several fluxons in a long annular Josephson junction. This circumstance makes quite feasible an experimental realization of the problem considered in the present paper. We note furthermore that according to Lin Lei, Shu Changqing, Shen Juelin, P. M. Lam, and Huang Yun [Phys. Rev. Lett. **49**, 1335 (1982)] our Eq. (1) describes also soliton motion in a liquid-crystal (nematic) layer with shear flow.

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