Parametric magnons as a source of singularities in the spectra of other quasiparticles

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It is shown that the singular character of the distribution of parametrically excited magnons in **k**-space (points, lines, surfaces) leads to singularities in the spectra of other quasiparticles. For antiferromagnets the positions and types of the singularities arising in the spectra of the magnons of the antiferromagnetic branch and of the phonons upon parametric excitation of magnons of the ferromagnetic branch are found.

Under the action of a microwave field of frequency ω_p and power above threshold, in a magnet spin waves with wave vectors concentrated near the constant-frequency surface $\omega_{\mathbf{k}} = \omega_p/2$ in **k**-space are parametrically excited.[1] At power levels slightly above threshold the packet of parametrically excited spin waves (PSW) is rather narrow,[2] and the occupation numbers of the PSW are such as to correspond to spectral temperatures of millions of degrees.[1] This implies that in **k**-space there is a strongly superheated constant-frequency surface. Thanks to this singular character of the distribution of PSW their interaction with other quasiparticles should lead to singularities in the dispersion law of the latter.[3] As noted in Ref. 3, in this respect there is a certain analogy with the situation in metals, where at $T = 0$ there exists a distinct constant-energy Fermi surface $\varepsilon(k) = \varepsilon_F$, and the interaction of electrons with other quasiparticles (e.g., phonons) leads to anomalies of one kind or another at $\mathbf{k} = 2\mathbf{k}_F$ in the spectrum of these quasiparticles.[4]

For definiteness we shall consider an antiferromagnet with two branches of the spin-wave (magnon) spectrum[5]: a ferromagnetic branch (f)

$$\omega_{\mathbf{k}} = (\omega_0^2 + v_m^2 k^2)^{1/2} \tag{1}$$

and an antiferromagnetic branch (a)

$$\omega_{a\mathbf{k}} = (\omega_{a0}^2 + v_m^2 k^2)^{1/2},$$

and also with an acoustic branch whose dispersion law we shall assume to be linear:

$$\omega_{s\mathbf{k}} = v_s k$$

(here v_s and v_m are the sound velocity and the "limiting" velocity of the magnons).

We suppose that parametric pumping with frequency ω_p excites f-magnons whose wave vectors lie on the constant-frequency surface $\omega_{\mathbf{k}} = \omega_p/2$. We shall discuss first the distortion which then arises in the spectrum of the a-magnons. At small levels of excitation, and if the external magnetic field is not too small, the chief interaction process between magnons of different branches is the coalescence-decay process described by the Hamiltonian

$$H = \hbar \sum_{\mathbf{q}\mathbf{k}_1\mathbf{k}_2} \Psi_{\mathbf{q}12} a_{\mathbf{q}} f_{\mathbf{k}_1}^* f_{\mathbf{k}_2}^* \Delta(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2) + \text{c.c.},$$

where a and f are the amplitudes of the waves, and the matrix element (see Ref. 3)

$$\Psi \propto (\omega_{a\mathbf{q}} + \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})(\omega_{a\mathbf{q}}\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2})^{-1/2}$$

is a smooth function of the frequencies and wave vectors.

The nonlinear frequency shift $\delta\omega_{a\mathbf{q}}$ is determined by the real part of the polarization operator. Calculating it for small $f_{\mathbf{k}}$ ($|\Psi f| \ll \omega_{\mathbf{k}}$), we obtain, in second order of perturbation theory,

$$\delta\omega_{a\mathbf{q}} = \lim_{\eta\to 0} \operatorname{Re} \sum_{\mathbf{k}} \frac{|\Psi_{\mathbf{q}\mathbf{k}(\mathbf{q}-\mathbf{k})}|^2 N_{\mathbf{k}}}{\omega_{a\mathbf{q}} - \omega_{\mathbf{k}} - \omega_{\mathbf{q}-\mathbf{k}} + i\eta}, \tag{2}$$

where $\langle f_{\mathbf{k}} f_{\mathbf{k}'}^* \rangle = N_k \Delta(\mathbf{k}-\mathbf{k}')$, the angular brackets denote averaging over the random phases (or over the random external force in the diagram technique of Wild[6]), and $N_{\mathbf{k}}$ is the number of excited f-magnons.

As can be seen from (2), the character of the singularity of $\delta\omega_a(\mathbf{q})$ depends on the dimensionality of the distribution of PSW in **k**-space. The simplest case is considered in Ref. 3, viz., the excitation of a monochromatic standing wave (i.e., of a pair $\pm\mathbf{k}_0$, where $\omega_{\mathbf{k}_0} = \omega_p/2$), and in this case two terms remain in (2):

$$\delta\omega_{a\mathbf{q}} = \frac{|\Psi_{\mathbf{q}\mathbf{k}_0(\mathbf{q}-\mathbf{k}_0)}|^2 N_{\mathbf{k}_0}}{\omega_{a\mathbf{q}} - \omega_{\mathbf{k}_0} - \omega_{\mathbf{q}-\mathbf{k}_0}} + \frac{|\Psi_{\mathbf{q}\mathbf{k}_0(\mathbf{q}+\mathbf{k}_0)}|^2 N_{\mathbf{k}_0}}{\omega_{a\mathbf{q}} - \omega_{\mathbf{k}_0} - \omega_{\mathbf{q}+\mathbf{k}_0}}. \tag{3}$$

For $\Delta\omega_{\mathbf{q}}^{\pm} \equiv \omega_{a\mathbf{q}} - \omega_{\mathbf{k}_0} - \omega_{\mathbf{q}\pm\mathbf{k}_0} \to 0$ formula (3) displays a singularity of the form $1/\Delta\omega_{\mathbf{q}}$. The resonance surfaces specified by the equalities $\Delta\omega_{\mathbf{q}}^{\pm} = 0$ are ellipsoids of revolution:

$$\frac{v_m^2(q_y^2+q_z^2)}{\omega_0} + \frac{v_m^2(q_x \pm k_0\omega_{a0}/2\omega_0)^2}{\omega_{\mathbf{k}_0}} = \frac{\omega_{a0}}{\omega_0^2}(\omega_{a0}^2 - 4\omega_0^2) \tag{4}$$

(here the x axis is directed along $\mathbf{k}_0$). As can be seen from (4), these surfaces exist for $\omega_{a0} \geqslant 2\omega_0$.

We turn now to the case when the wave vectors of the f-magnons lie on a line (usually, this is a circle on the surface $\omega_{\mathbf{k}} = \omega_p/2$). Letting the z axis be perpendicular to the plane in which this circle lies, we write (2) in the form

$$\delta\omega_{a\mathbf{q}} = \int_0^{2\pi} \frac{|\Psi_{\mathbf{q}\mathbf{k}(\mathbf{q}-\mathbf{k})}|^2 |f_{\mathbf{k}}|^2 \, d\varphi}{\omega_{a\mathbf{q}} - \omega_{\mathbf{k}} - [\omega_0^2 + v_m^2(q^2 + k^2 - 2kq \sin\theta \cos\varphi)]^{1/2}}.$$

Here θ is the angle formed by the vector **q** and the z axis. Since the matrix element Ψ is a smooth function of φ it can be

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taken outside the integral, after which the integral can be calculated. For

$$\Delta\omega_{\mathbf{q}}\equiv\omega_{a\mathbf{q}}-\omega_{\mathbf{k}}-[\omega_0^2+v_m^2(q^2+k^2+2kq\sin\theta)]^{1/2}\geqslant 0$$

we obtain ($\mathcal{N}=\Sigma_{\mathbf{k}} N_{\mathbf{k}}$ is the total number of PSW)

$$\delta\omega_{a\mathbf{q}}=\frac{4|\Psi|^2\mathcal{N}}{(2v_m^2kq\sin\theta)^{1/2}}\left\{-\frac{K([2/(b+1)]^{1/2})}{(b+1)^{1/2}}+\left[\frac{\Delta\omega_{\mathbf{q}}K(r)}{(2v_m^2kq\sin\theta)^{1/2}}+2(b+1)^{1/2}\Pi\left(\frac{\pi}{2},\frac{\omega_{a\mathbf{q}}-\omega_{\mathbf{k}}+\omega_{\mathbf{q}+\mathbf{k}}}{\Delta\omega_{\mathbf{q}}}r,r\right)\right]\times\frac{1}{(b+1)^{1/2}+(b-1)^{1/2}}\right\},$$

$$b=(\omega_0^2+v_m^2(k^2+q^2))/2v_m^2kq\sin\theta,\quad r=b-(b^2-1)^{1/2},\tag{5}$$

where K and Π are complete elliptic integrals of the first and third kinds, respectively. Since $\Pi(\pi/2,x,r)\to\pi x^{-1/2}$ as $x\to\infty$, the singularity is of the square-root kind: $\delta\omega_{a\mathbf{q}}\propto(\Delta\omega_{\mathbf{q}})^{-1/2}$ as $\Delta\omega_{\mathbf{q}}\to 0$. The resonance surface $\Delta\omega_{\mathbf{q}}=0$ is a torus, one of whose sections is the ellipse (4), while the other is two circles with their centers at the origin.

In the case when the wave vectors of the PSW fill the whole constant-frequency surface $\omega_{\mathbf{k}}=\omega_p/2$, the singularity in the dispersion law of the other quasiparticles becomes logarithmic. We shall illustrate this for the example of the interaction of f-magnons with phonons. If the velocities of the PSW exceed the sound velocity, then for $2v_m^2k_0\geqslant v_s\omega_p$ a Cerenkov process of emission of a phonon by magnons is allowed. The Hamiltonian of this process has the form

$$H=\sum_{\mathbf{kq}}V_{(\mathbf{k}+\mathbf{q})\mathbf{kq}}f^*_{\mathbf{k}+\mathbf{q}}f_{\mathbf{k}}(b_{\mathbf{q}}+b_{\mathbf{q}}^*),\tag{6}$$

where $b_{\mathbf{q}}$ are the amplitudes of the sound waves. In this case the expression for the polarization operator in second order of perturbation theory[7] contains the difference $N_{\mathbf{k}}-N_{\mathbf{k}-\mathbf{q}}$:

$$\Pi(\mathbf{q},\omega)=\sum_{\mathbf{k}}\frac{|V_{(\mathbf{k}+\mathbf{q})\mathbf{kq}}|^2(N_{\mathbf{k}}-N_{\mathbf{k}-\mathbf{q}})}{\omega_{\mathbf{k}-\mathbf{q}}-\omega_{\mathbf{k}}+\omega}.$$

The renormalized dispersion law $\tilde{\omega}_{s\mathbf{q}}$ is expressed in terms of the phonon Green function $D(\mathbf{q},\omega)$, which is related to the polarization operator by the Dyson equation

$$D^{-1}(\mathbf{q},\omega)=D_0^{-1}(\mathbf{q},\omega)+\Pi(\mathbf{q},\omega).$$

Here the bare Green function is

$$D_0(\mathbf{q},\omega)=(\omega-\omega_{s\mathbf{q}}+i\eta)^{-1}-(\omega+\omega_{s\mathbf{q}}-i\eta)^{-1},\quad \eta\to+0.$$

Assuming that the occupation numbers of the PSW are the same for all points on the constant-frequency surface, we obtain

$$\operatorname{Re}\Pi(\mathbf{q},\omega)=-\frac{|V_{(\mathbf{k}+\mathbf{q})\mathbf{k}_0\mathbf{q}}|^2\mathcal{N}}{4v_s^2k_0q}\left[2\omega_{\mathbf{k}_0+\mathbf{q}}-2\omega_{\mathbf{k}_0-\mathbf{q}}+\left(\frac{\omega_p}{2}+\omega\right)\times\ln\frac{\omega_p/2+\omega-\omega_{\mathbf{k}_0+\mathbf{q}}}{\omega_p/2+\omega-\omega_{\mathbf{k}_0-\mathbf{q}}}+\left(\frac{\omega_p}{2}-\omega\right)\ln\frac{\omega_p/2-\omega-\omega_{\mathbf{k}_0+\mathbf{q}}}{\omega_p/2-\omega-\omega_{\mathbf{k}_0-\mathbf{q}}}\right],$$

$$\omega_{\mathbf{k}_0}=(\omega_0^2+v_m^2k_0^2)^{1/2}=\omega_p/2,\qquad \omega_{\mathbf{k}_0\pm\mathbf{q}}=[\omega_0^2+v_m^2(k_0\pm q)^2]^{1/2}.\tag{7}$$

Putting $\omega=\omega_{s\mathbf{q}}$ in the argument of the logarithm, we see that the renormalized dispersion law of the phonons,

$$\tilde{\omega}_{s\mathbf{q}}^2=\omega_{s\mathbf{q}}^2-2\omega_{s\mathbf{q}}\operatorname{Re}\Pi(\mathbf{q},\omega_{s\mathbf{q}})$$

has logarithmic singularities on two surfaces in $\mathbf{q}$-space:

$$[\omega_0^2+v_m^2(k_0\pm q)^2]^{1/2}=(\omega_0^2+v_m^2k_0^2)^{1/2}\pm v_sq/v_m\tag{8}$$

(the signs can be changed independently). Here it is pertinent to note again that the singularity under discussion is a direct analog of the Kohn effect in metals.[7] In 1959 Kohn predicted the presence of a logarithmic singularity of the phonon group velocity $\partial\tilde{\omega}_{s\mathbf{q}}/\partial\mathbf{q}$ at $\hbar q=2p_0$, where p_0 is the Fermi momentum. This singularity is connected with the electron-phonon interaction of the form (6). Since the distribution of magnons has a δ-function dependence on the energy, and the Fermi distribution at $T=0$ (i.e., a θ-function) is an integral of the δ-function, it is clear why in our case a logarithmic singularity is observed not in $\partial\tilde{\omega}_{s\mathbf{q}}/\partial\mathbf{q}$ but in the frequency $\tilde{\omega}_{s\mathbf{q}}$ itself. For an electron-phonon system, $v_s/v_{p_0}\cong(m/M)^{1/2}\ll 1$, where v_{p_0} is the electron velocity at the Fermi surface, and m and M are the electron and ion masses, respectively. In our case, for $v_s\ll v_m$, from (8) we find that in the present case of a magnet with pumping the singularities will be at $q\cong 2k_0$, as in the Kohn effect. The dependence of the character of the singularity on the dimensionality of the distribution of PSW on the resonance surface is analogous to the predicted (by Afanas'ev and Kagan[4]) dependence of the character of the singularity on the geometry of the Fermi surface (sphere, cylinder, or plane). If the wave vectors of the PSW lie on a line, then, as for a-magnons, the singularity in the phonon dispersion law is a square-root singularity and occurs at

$$\sin\theta[q\sin\theta\pm(4k_0^2-q^2\cos^2\theta)^{1/2}]=\pm v_s\omega_0/v_m^2.$$

In the case $\theta=\pi/2$ (i.e., when the wave vectors of the PSW and phonon lie in the same plane) the singularity should be observed at $q=\pm 2k_0\pm v_s\omega_0/v_m^2$.

Of course, the formulas (3), (5), and (7), obtained under the assumption that $\delta\omega_{\mathbf{q}}\ll\omega_{\mathbf{q}}$, are not valid in the immediate vicinity of the corresponding resonance surface. To describe the behavior of $\delta\omega_{\mathbf{q}}$ as $\Delta\omega_{\mathbf{q}}\to 0$ it is necessary to analyze the whole perturbation-theory series, and this should transform $\delta\omega_{\mathbf{q}}$ into a regular function of $\mathbf{q}$. For example, for the case of a pair $\pm\mathbf{k}_0$ (more precisely, for the case of a pair of narrow packets of waves concentrated near $\pm\mathbf{k}_0$), it can be shown that in each order of perturbation theory the main diagrams for $\Pi(\mathbf{q},\omega_{a\mathbf{q}})$ as $\Delta\omega_{\mathbf{q}}\to 0$ are the diagrams of the "skeleton" and "ladder" series,[8] which can be summed, leading to an algebraic equation for $\delta\omega_{\mathbf{q}}$ of the form

$$\delta\omega_{\mathbf{q}}=\frac{|\Psi|^2\mathcal{N}}{(\delta\omega_{\mathbf{q}}+\Delta\omega_{\mathbf{q}})\{1+[|\Psi|/(\delta\omega_{\mathbf{q}}+\Delta\omega_{\mathbf{q}})]^2\}}.\tag{9}$$

From (9) it is easy to see that on the resonance surface itself (i.e., for $\Delta\omega_{\mathbf{q}}=0$) there is no nonlinear frequency shift: $\delta\omega_{\mathbf{q}}=0$. This fact can be perceived directly from an analysis of the diagrammatic series, from which it follows that $\delta\omega_{\mathbf{q}}$ for $\Delta\omega_{\mathbf{q}}\to 0$ is an odd function of $\Delta\omega_{\mathbf{q}}$. The regularity of $\delta\omega_{\mathbf{q}}$, in combination with the oddness, leads to the conclusion that $\delta\omega_{\mathbf{q}}|_{\Delta\omega_{\mathbf{q}}=0}=0$. The frequency shift is a maximum at

$\Delta\omega_{\mathbf{q}} \cong |\Psi|$. For $\Delta\omega_{\mathbf{q}} \gg |\Psi|$ we have $\delta\omega_{\mathbf{q}} \cong |\Psi|^2 \mathcal{N}/\Delta\omega_{\mathbf{q}}$, i.e., (9) goes over into (3).

In a real situation there are always factors leading to smearing out of the singular PSW distributions: thermal noise, scattering by defects,[2] and nonlinear interaction of the PSW with each other and with thermal waves.[2,9] In the case of smooth (albeit narrow) PSW distributions the singularities in the spectra of other quasiparticles are also smeared out. If the wave vectors of the f-magnons occupy a layer of a certain width about the constant-frequency surface, with a distribution $N_{\mathbf{k}}$ in the form of, e.g., the Lorentz function $[(\omega_{\mathbf{k}} - \omega_p/2)^2 + \eta^2]^{-1}$, a spectral soliton $\cosh^{-1}[(\omega_{\mathbf{k}} - \omega_p/2)/\eta]$, or a more complicated soliton,[2,9] the correction to the phonon frequency has the form

$$\delta\omega_{s\mathbf{q}} \propto \ln[(\omega_p/2+\omega_{s\mathbf{q}}-\omega_{\mathbf{k}_0+\mathbf{q}})^2+\eta^2].$$

Here we have written out the principal term near the surface $\omega_p/2 + \omega_{s\mathbf{q}} - \omega_{\mathbf{k}_0+\mathbf{q}} = 0$. It should be noted that situations are possible in which the **k**-distributions of the PSW have finite widths in all directions but sharp boundaries in just one of the directions[10,11] (e.g., a distribution in the form of an oblong in the modulus k). In this case, as in the Kohn effect, there should be a logarithmic singularity in the group velocity $\partial\tilde{\omega}_{s\mathbf{q}}/\partial\mathbf{q}$.

A quantitative estimate of the magnitude of the effect, e.g., for $FeBO_3$, for which the exchange constant $\omega_{ex} \cong 2\pi\cdot 8\cdot 10^3$ GHz, $v_s = 4.7\times 10^5$ cm/sec, $v_m \cong 3v_s$, the magnetoelastic constant $\Theta/k_B \cong 20$ K, and the elastic-energy constant $Mv_s^2/k_B \cong 10^5$ K gives, for $\omega_p = 2\pi\cdot 30$ GHz, supercriticality $h/h_c \cong 10$, and $\eta/\omega_p \cong 10^{-3}$,

$$V_{(\mathbf{k}_0+\mathbf{q})\mathbf{k}_0\mathbf{q}} = 2\omega_{ex}^2\Theta^2\omega_{s\mathbf{q}}\hbar/Mv_s^2\omega_p(\omega_p-\omega_{s\mathbf{q}}),$$

$$\left|\frac{\delta\omega_{s\mathbf{q}}}{\omega_{s\mathbf{q}}}\right| \approx \frac{|V_{(\mathbf{k}_0+\mathbf{q})\mathbf{k}_0\mathbf{q}}|^2\mathcal{N}\omega_p}{\hbar^2 v_s^2 k_0 q\omega_{s\mathbf{q}}}\ln\frac{\omega_p}{\eta} \approx 10^{-2}.$$

The presence of ω_{ex} in the expression for V^2 gives grounds for drawing attention to the fact that the phonon renormalization that depends on the level of pumping of the magnons, while possible in principle both in ferromagnets and in antiferromagnets, is much more clearly expressed in the latter. This is yet another manifestation of the exchange enhancement of single-ion interactions in antiferromagnets.[12]

The above-described singularities in dispersion laws can be observed in neutron-scattering experiments (very difficult, since it is difficult to make $|\mathbf{q}|$ large) or in the excitation of a-magnons or phonons by an external source. For example, if the source excites these quasiparticles parametrically, then, as their frequencies approach the corresponding resonance surface, singularities will be observed both in the threshold $h_c^{(\mathbf{q})}$ and in the above-threshold behavior of the wave system. In addition, if through the magnet in which the PSW are excited one passes pulses (e.g., sound pulses) with a frequency approaching the resonance frequency, because of the presence of the anomalously large dispersion the time of spreading of the pulses should decrease with increase of the level of excitation of the PSW.

Experimental study of the singularities in the emission spectra of other quasiparticles will make it possible to determine the distribution of the initial parametrically excited spin waves in **k**-space.

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