Types of unbound geodesics in the Kerr metric

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Sets of constants of motion of a particle that correspond to different types of r-motion are considered. The topology of these sets is determined and a number of constants characterizing these sets are found.

INTRODUCTION

An important problem in the study of unbound motion of particles in the Kerr metric is the description of the set of constants of motion for which a particle traveling from infinity goes below the horizon of a black hole. We shall give a qualitative description of this set and also of the set of constants of motion for which the particle asymptotically approaches a sphere placed around the black hole, and the sets of constants of motion for which the particle departs to infinity. The solution of this problem is important in connection with the accretion of noninteracting particles on a rotating black hole.

It is well-known that Kepler orbits are characterized by two constants (E and L), since we can identify orbits that can transform into one another by rotations through the Euler angles. Hence, orbits in the Schwarzschild metric are also characterized by two constants. It turns out that a change in the radial coordinate in the Kerr metric is determined by only three constants in the case of moving particles (because the particle mass characterizes the connection between the affine parameter and the proper time of the particle, and the affine parameter can be chosen to be the proper time of the particle), and two constants in the case of the motion of photons (because of the photon energy characterizes the set of different affine parameters in the equation for the change in the r coordinate.)

1. BASIC NOTATION

The equation of motion for the radial variable in the Kerr metric is[1]

$$\rho^4 (dr/d\tau)^2 = R(r), \tag{1}$$

$$R(r) = r^4 + (a^2 - \xi^2 - \eta) r^2 + 2M[\eta + (\xi - a)^2] r - a^2\eta \quad \text{(Photons)},$$

$$R(r) = r^4 + (a^2 - \xi^2 - \eta) r^2 + 2M[\eta + (\xi - a)^2] r - a^2\eta - r^2\Delta/E \quad \text{(Particles)},$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad a = S/M. \tag{2}$$

The constants S and M refer to the black hole, namely, S is the angular momentum and M the mass of the black hole. The constants E, ξ, and η refer to the particle, namely, E is its energy at infinity, $\xi = L_z/E$ (L_z is the angular momentum of the particle about the axis of rotation of the black hole), and $\eta = Q/E^2$ (Q is given by

$$Q = p_\theta^2 + \cos^2\theta[a^2(\mu^2 - E^2) + \sin^{-2}\theta L_z^2],$$

and μ is the mass of the particle). It is readily verified that the radial motion of the particle depends on the following constants:

$$\hat{a} = a/M, \quad \hat{E} = E/\mu, \quad \hat{\xi} = \xi/M, \quad \hat{\eta} = \eta/M^2.$$

The radial motion of photons does not depend on the constant $\hat{E}$. Instead of the coordinate r, we now introduce $\hat{r} = r/M$. (The symbol $\wedge$ will be omitted henceforth.) Thus, the character of motion in the r-coordinate for given value of a is determined by the three constants E, ξ, η in the case of a moving particle, and by the two constants ξ and η in the case of photons.

Depending on the multiplicities of the roots of the polynomial $R(r)$ (for $r \geqslant r_g$), we can have three types of motion in the r-coordinate,[2] namely:

(1) the polynomial $R(r)$ has no roots (for $r \geqslant r_g$). The particle then falls into the black hole;

(2) the polynomial $R(r)$ has roots and $r_{\max} > r_g$ ($r_{\max}$ is the maximum root); for $(\partial R/\partial r)(r_{\max}) \neq 0$ we then have, $(\partial R/\partial r)(r_{\max}) > 0$, and the particle departs to infinity after approaching the black hole;

(3) the polynomial $R(r)$ has a root and $R(r_{\max}) = (\partial R/\partial r)(r_{\max}) = 0$; the particle now takes an infinite proper time to approach the sphere of radius $r_{\max}$.

2. DESCRIPTION OF THE SET OF CONSTANTS CORRESPONDING TO DIFFERENT TYPES OF MOTION

We shall now examine the sets of constants of motion E, ξ, and η corresponding to different types of particles motion for a given black-hole rotation parameter $a = \text{const}$. Let us cut the space E, ξ, η with the plane $E = \text{const} \geqslant 1$ and describe in this slice the set of constants corresponding to different types of motion. It then turns out that the boundary of the set of constants corresponding to the second type of motion for $\eta \geqslant 0$ is the set of constants for which the motion belongs to the third type. We shall look upon this set as the graph of the function $\eta = \eta(\xi)$. We note that the set of these constants as functions $\xi(r)$ and $\eta(r)$ was examined by Chandrasekhar[1]. Let us describe some of the properties of the function $\eta(\xi)$. If the motion of the particle is of the third type, we have

$$R(r) = 0, \quad (\partial R/\partial r)(r) = 0 \tag{3}$$

for $\eta \geqslant 0, r \geqslant r_g$.

Thus, to obtain the function $\eta(\xi)$, we must eliminate r from (3). Assuming that (3) provides an implicit specification of $r(\xi)$ and $\eta(\xi)$, we find that

$$d\eta/d\xi(-\Delta) = 2\xi r^2 - 4(\xi - a) r,$$
$$d\eta/d\xi(-2r + 2) + (dr/d\xi)(\partial^2 R/\partial r^2) = 4\xi r - 4(\xi - a) \tag{4}$$

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for $r \geqslant r_g, \eta \geqslant 0$. We note that, for $\Delta > 0$ and $\partial^2 R/\partial r^2 \neq 0$, the implicit function theorem shows that $r(\xi)$ and $\eta(\xi)$ are single-valued functions. Analysis similar to that given in Ref. 3 then shows that, when $a \neq 1$ or $\xi \neq 2$, we have $\partial^2 R/\partial r^2 > 0$. When $a = 1$ and $\xi = 2$, we find from (3) that $\Delta = 0$. When $a = 1$, it is readily verified that the set corresponding to the third type of motion includes the straight segments $[\xi = 2, 0 \leqslant \eta \leqslant (3E^4 - 4E^2 + 1)/(E^2(E^2 - 1))]$ (Ref. 4) (for photons, $\xi = 2, 0 \leqslant \eta \leqslant 3$, by analogy with Refs. 5 and 6). It can also be shown that the function $\eta(\xi)$ has one maximum and $r(\xi)$ is a monotonically decreasing function.[4] Thus, the set of constants corresponding to the first type of motion is bounded by the curve $\eta(\xi)$ for $\eta \geqslant 0$, as shown in Figs. 1 and 2. It is also readily shown that, when $\eta < 0$ and when η and ξ are such that the motion of the particle is possible, i.e.,

$$-[a(E^2-1)^{1/2}/E-|\xi|]^2 \leqslant \eta < 0, \quad |\xi| \leqslant a(E^2-1)^{1/2}/E,$$

the particle is also captured[2] (this set is illustrated in Fig. 2).

3. UNBOUND MOTION OF PHOTONS

Chandrasekhar[1] has shown that the condition for capture of a particle in the equatorial plane is the inequality

$$6\cos[\arccos(-a)/3+2\pi/3]$$

$$-a \leqslant \xi \leqslant 6\cos[\arccos(-a)/3]-a. \tag{5}$$

Thus, the functions of $r(\xi)$ and $\eta(\xi)$ are defined only for values satisfying the inequalities (5). We also note that the function $\eta(\xi)$ is a maximum for $\xi = -2a, r(-2a) = 3(\eta(-2a) = 27)$. This can be verified by direct evaluation of (3) and (4). Figure 2 shows a plot of the function $\eta(\xi)$ for $a = 1$.

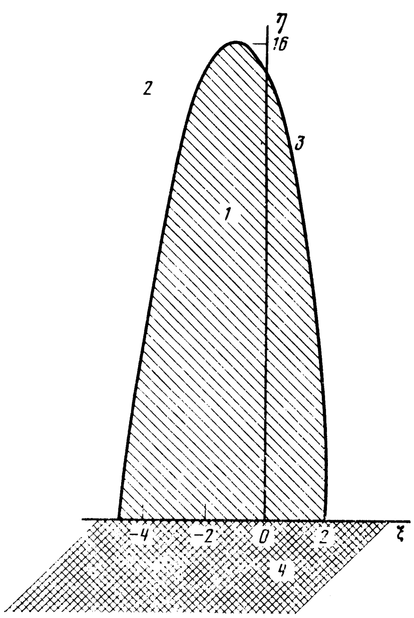

FIG. 1. Different types of particle motion for $E = 1$ and $a = 1$. Region 1—particle trapped, region 2—scattering; curve 3 corresponds to the third type of motion. Region 4 corresponds to values of the constants for which particle motion is impossible.

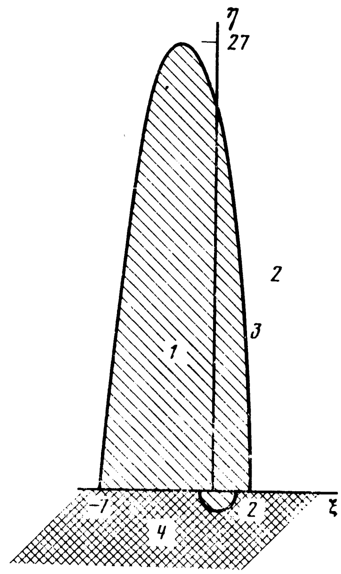

FIG. 2. Same as Fig. 1 for a massless particle and $a = 1$.

4. MOTION OF PARTICLE OF ARBITRARY ENERGY

Consider a moving particle of arbitrary energy at infinity ($E > 1$). It can be verified that, if

$$\eta_{max} = \frac{-(\alpha^2-18\alpha-27)+(\alpha^4+28\alpha^3+270\alpha^2+972\alpha+729)^{1/2}}{2E^2\alpha},$$

$$r_{max} = (8\alpha^3/27+\eta_{max}E^2\alpha(\alpha/3+1))^{1/3}-2\alpha/3, \tag{6}$$

$$\xi_{max} = 2a/(r_{max}-2),$$

where $\alpha = (E^2 - 1)^{-1}$, these values ensure that $R(r)$ and $\partial R/\partial r$ vanish, i.e., they satisfy (3). We also note that, for values chosen in accordance with (6), the right-hand side of the first equation in (4) vanishes, i.e., these values correspond to the maximum of $\eta(\xi)$. The values η_{max} and r_{max} turn out to be equal to the corresponding values of these quantities for $a = 0$ (Schwarzschild metric).[7]

5. ONE CASE OF UNBOUND PARTICLE MOTION

Consider a case of unbound particle motion for $E = 1$. If the motion takes place in the equatorial plane, $\eta = 0$ (Ref. 8) and

$$R(r) = 2r^3 - \xi^2 r^2 + 2(a-\xi)^2 r. \tag{7}$$

The motion then belongs to the third type if $\xi^4 = 16(a - \xi)^2$, and $r = \xi^2/4$. It follows that there are only two values that correspond to the third type of motion in the equatorial plane, namely, $\xi = -2 - 2(1 + a)^{1/2}$ and $\xi = 2 + 2(1 - a)^{1/2}$. Thus, the domain of definition of $\eta(\xi)$ is the segment $[-2(1 + (1 + a)^{1/2}), 2(1 + (1 - a)^{1/2})]$. The domain of variation of the function $r(\xi)$ is the segment $[(1 + (1 - a)^{1/2})^2, (1 + (1 + a)^{1/2})^2]$. This follows from the fact that $r(\xi)$ is a monotonically decreasing function of ξ. When $a = 0$, we find that $\eta(\xi) = 16 - \xi^2$. When $E \to 1$, we can show from (6) that the maximum of the function $\eta(\xi)$ is reached for $\xi = -a, r(-a) = 4, \eta(-a) = 16$ [since $\eta_{\max}(\xi) \to 16$ for $E \to 1$]. This can be demonstrated by direct verification of (3) and (4).

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