Time evolution of a quantized anharmonic oscillator and its spin analogue

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(Submitted 10 November 1985)
Zh. Eksp. Teor. Fiz. **91**, 156–164 (July 1986)

The time evolution of a quantized anharmonic oscillator is investigated when the initial state of the oscillator is coherent. Averaging the Heisenberg equation of motion over such states leads to a closed equation for the average value of any operator which can represent a physical quantity. A distinguishing feature of the quantized system is the presence of terms containing higher-order derivatives in the coherent-state parameters, which are interpreted as quantum-mechanical generalizations of the Lagrangian coordinates in the evolution equation. We develop a perturbation theory in the weak anharmonicity based on these equations which does not require us to find either wave functions or energy levels. There are qualitative differences between the quantized and classical systems: for example, the "phase" trajectories of the former system in the space defined by the coordinate and momentum averages are open. The approach we propose allows us to study the dynamics of a single spin in a weak anisotropic field, again by using coherent (spin) states. In a reference system rotating at the precession frequency of the spin in a transverse magnetic field, the spin executes purely quantum-mechanical oscillations. The number of these oscillations within a corresponding period is determined only by the magnitude of the spin, and does not depend on the direction of the initial polarization.

The anharmonic oscillator is the simplest nonlinear system, and finds a multitude of applications. A large number of papers have been devoted to studying its quantum-mechanical properties (see, e.g., the review in Ref. 1). In this paper we will study the dynamics of such a quantum system (the oscillator is everywhere considerd to be one-dimensional, while the anharmonicity is assumed to be weak). Before we turn to a discussion of this problem let us recall briefly how things stand in classical mechanics.

The solution to the classical equation of motion is a periodic function of time, and is found in the form of an expansion in powers of a small parameter; the value of the frequency (in any given approximation) is determined from the condition that secular terms be absent.[2] As a result, the nonlinearity leads to renormalization of the "bare" frequency and the appearance of higher harmonics; however, the solution remains periodic.

An investigation of the time-dependent anharmonic oscillator was undertaken in Ref. 3. A coherent state was chosen as an initial condition; as is well known, coherent states are the most "classical."[4,5] This naturally made it possible to compare the temporal behavior of the classical and quantum systems.

According to the results of Ref. 3, the evolution of the quantum system is in essence analogous to that of the classical system: the average values of the dynamic variables are found to be periodic functions of time, while the "quantization" appears only in the fact that expressions for the frequencies and amplitudes of corresponding harmonics contain quantum corrections. Even at this level, it follows from the incommensurability of the transition frequencies for the anharmonic oscillator that the solution, generally speaking, need not be periodic. The method used in Ref. 3 consisted of operator generalizations of the classical procedure applied to the Heisenberg equation of motion for the Bose creation and annihilation operators. However, no allowance was made for the operator nature of what corresponds in the quantum system to the classical frequency. As we show below, the results given in Ref. 3 are correct only for sufficiently short times.

On the other hand, in Refs. 6 and 7 it was emphasized that, with the passage of time, the nonlinearity leads to the appearance of the essential quantum behavior of the system. This circumstance was illustrated by the example of an exactly soluble model: it was shown that a quantum modulation appears in the harmonic time dependences which correspond to classical mechanics (an analogous result for spin systems was obtained in Refs. 8 and 9).

However, the Hamiltonian corresponding to this model had a very special form: the anharmonic term in the Hamiltonian was quadratic, i.e., the square of the harmonic term. The time evolution of this system could be studied because it was possible to solve the corresponding Heisenberg equation of motion exactly; thus, these results do not allow a direct generalization. At the same time, an investigation of the dynamics of a more realistic system—the anharmonic oscillator—is of interest in its own right.

In this article we propose a quantum generalization of the Krylov-Bogolyubov method, whose applicability is not limited to a particular model. This allows us to study in detail the characteristic ways in which the evolution of the quantum system differs from that of the corresponding classical system (which it approaches as $\hbar \to 0$). We note that in contrast to Refs. 6 and 7, where the time behavior of the classical system was determined by one frequency, in our case the modulation affects also the higher harmonics.

In this paper the properties of coherent states are used in an essential way. These states make it possible for us to

 0038-5646/86/070090-06$04.00 © 1987 American Institute of Physics

pass directly from the operator Heisenberg equations of motion to differential equations for the average value of any dynamic variable.[10–12] These equations are quantum generalizations of the corresponding classical evolution equations. Perturbation theory in the weak anharmonicity can be constructed directly with their help. The important peculiarity of the quantum equations is the presence in them of higher derivatives which also give rise to the special behavior of the quantum system as compared to the classical system.

The approach developed in this paper also allows us to investigate the dynamics of an anisotropic nonlinear spin system (by applying coherent spin states[13]), whose quantum properties are essentially manifested in the form of spin oscillations.

1. DYNAMIC EQUATIONS IN THE COHERENT-STATE REPRESENTATION

Let us first examine a quantum system described by a time-independent Hamiltonian of general form

$$\mathcal{H}=\mathcal{H}(a^+, a); \tag{1}$$

the creation and annihilation operators are linear combinations of the coordinate and momentum operators.

Let $\hat{f}=f(a^+, a)$ be an operator for some physical quantity in the Heisenberg representation. The time evolution of this quantity averaged over a coherent state $f=\langle z|\hat{f}|z\rangle$ (the vector state $|z\rangle$ is an eigenvector of the operator a: $a|z\rangle=z|z\rangle$, where z is a complex number), is determined by the equation

$$\frac{\partial f}{\partial t}=\frac{i}{\hbar}\langle z|\mathcal{H}(a^+, a)\hat{f}|z\rangle-\frac{i}{\hbar}\langle z|\hat{f}\mathcal{H}(a^+, a)|z\rangle. \tag{2}$$

Using the relations[4]

$$\begin{aligned}\langle z|f(a^+, a)a^+|z\rangle&=e^{-|z|^2}\frac{\partial}{\partial z}[\langle z|f(a^+, a)|z\rangle e^{|z|^2}],\\ \langle z|af(a^+, a)|z\rangle&=e^{-|z|^2}\frac{\partial}{\partial z^*}[\langle z|f(a^+, a)|z\rangle e^{|z|^2}],\end{aligned} \tag{3}$$

we obtain a closed dynamic equation for the average value f:

$$\partial f/\partial t=i\hat{K}f, \tag{4}$$

where the operator $\hat{K}$ equals

$$\hat{K}=\frac{1}{\hbar}e^{-|z|^2}\left[\mathcal{H}\left(z^*, \frac{\partial}{\partial z^*}\right)-\mathcal{H}\left(\frac{\partial}{\partial z}, z\right)\right]e^{|z|^2}. \tag{5}$$

The notation (5) implies that in the Hamiltonian (1) we must substitute $a^+\to z^*$, $a\to\partial/\partial z^*$ in the first term, and $a\to z$, $a^+\to\partial/\partial z$ in the second. It is important to note that the dynamic equation (4) for f is linear, despite the fact that the original system is, generally speaking, nonlinear.

Equation (4) describes the time variation of any function which characterizes the system. To obtain from the multitude of solutions to this equation that solution which corresponds to the physical quantity of interest to us, we must define its value as a function of the average values of position and momentum (or, what is the same thing, of z and z^*) at the initial time point. The problem of describing how the nonlinear quantum system behaves reduces to a Cauchy problem for Eq. (4).

It must be noted that in the case of a Hamiltonian which does not depend explicitly on time, Eq. (4) differs formally from the equation for the density matrix in this representation only by the sign on the right side. However, for nonconservative systems the differences are more significant: the equation for the density matrix in this case preserves the form (4) (with a change of sign), while the exact closed equation for the dynamic variable can no longer be cast in this form. This difference is connected with the fact that the Hamiltonian which enter into the equation for the density matrix is in the Schrödinger representation, while in the equation for f it is in the Heisenberg representation.

If the Hamiltonian of the system is represented in the form of a power series in a^+ and a, one can verify easily with the help of (5) that terms will be introduced into (4) which contain derivatives with respect to z^* and z. The order of these derivatives is determined by the powers of a^+ and a in the Hamiltonian. In the case of the normally-ordered nth power of a Bose operator, only nth derivatives are generated, without any lower-order derivatives.

When we transform to the variables x and p with formulas such as

$$x=(\hbar/2)^{1/2}(z+z^*), \quad p=i(\hbar/2)^{1/2}(z^*-z)$$

powers of Planck's constant will appear as multipliers in front of $\partial/\partial x.\,\partial/\partial p$; the exponent of these powers will equal the order of the corresponding derivative. In this case, equation (4) will appear as a series of powers of Planck's constant. Going to the limit $\hbar\to 0$, we obtain the classical equation, which takes the form

$$\frac{\partial f}{\partial t}=\frac{\partial\mathcal{H}}{\partial p}\frac{\partial f}{\partial x}-\frac{\partial\mathcal{H}}{\partial x}\frac{\partial f}{\partial p}, \tag{6}$$

where $\mathcal{H}=\langle z|\hat{\mathcal{H}}|z\rangle$ coincides with the classical Hamilton's function. The right side of (6) is a Poisson bracket written in Lagrangian coordinates. In the general case of a quantum system described by equation (4), the variables x and p (or z and z^*) are quantum generalizations of the Lagrangian coordinates.

Thus, in the classical limit the system is described by a first-order partial differential equation, while the quantum properties manifest themselves in the higher order of the exact equation (for a power-law Hamiltonian the order is determined by the highest power of a^+ and a appearing in the Hamiltonian). If $\mathcal{H}$ is a transcendental function, equation (4) is a functional equation.

It follows from the above that the operator $\hat{K}$ can be cast in the form of a sum of two terms: a classical ($\hat{K}_c$) term and quantum ($\hat{K}_q$) term

$$\hat{K}=\hat{K}_c+\hat{K}_q, \tag{7}$$

where $\hat{K}_c\sim\hbar^0$, $\hat{K}_q\sim\hbar^\alpha$, $\alpha\geqslant 1$.

We note that the characteristics of the classical equation (6) are given by Hamilton's equation. In the case of a harmonic oscillator, the classical and quantum equations coincide.

Equation (4) allows us to describe the quantum system with the help of an analogue of phase space, and in this sense can be compared with the Wigner-function method, since the Wigner function also obeys a linear equation.[14] We emphasize, however, that in contrast to the Wigner method the approach developed in this paper makes it possible for us to investigate a dynamic equation which directly describes the averaged, i.e., physical quantities rather than a quasiprobability which has no direct physical meaning. In the formalism we describe below, the averaged operator can be arbitrary but the choice of representation is specific. In the Wigner-function method, a definite representation is used to construct an operator which then can be averaged over an arbitrary state.

Let us now proceed directly to the anharmonic oscillator. In this case, the Hamiltonian takes the form

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + m\frac{\omega_0^2 x^2}{2} + \gamma x^3 + \frac{\beta}{4}x^4 \qquad (8)$$

(here, β differs from the corresponding coefficient in Ref. 3 by a factor of 1/4).

Let us introduce Bose operators as follows:

$$a=(m\omega_0/2\hbar)^{1/2}(\hat{x}+i\hat{p}/m\omega_0), \quad a^+=(m\omega_0/2\hbar)^{1/2}(\hat{x}-i\hat{p}/m\omega_0). \qquad (9)$$

For further convenience we will make use of polar coordinates, in which equation (4) for the system under investigation takes the form

$$\dot{f}+\omega_0\frac{\partial f}{\partial\varphi} = \frac{i}{\hbar}\left[\gamma(\tilde{x}^3-\text{h.c.}) + \frac{\beta}{4}(\tilde{x}^4-\text{h.c.})\right]f, \qquad (10)$$

where

$$\dot{f}=\frac{\partial f}{\partial t}, \quad \tilde{x}=r\cos\varphi+\lambda^2\hat{L}, \quad \lambda=\left(\frac{\hbar}{2m\omega_0}\right)^{1/2}$$

$$\hat{L}=e^{i\varphi}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}\right), \quad r=2\lambda\rho \quad (z=\rho e^{i\varphi}).$$

2. PERTURBATION THEORY FOR DYNAMIC VARIABLES: SPECIFICS OF THE BEHAVIOR OF A QUANTUM SYSTEM

Equation (10) cannot be solved exactly. We construct a perturbation theory to investigate it, in powers of the weak anharmonicity. Let us emphasize at once that, generally speaking, we do not have at our disposal any systematic method of solving this equation approximately based on a direct transcription of the approach used in the classical case. In point of fact, if by analogy with Ref. 2 we seek a solution in the form

$$f=f^{(0)}+f^{(1)}, \qquad (11)$$

where $f^{(0)}=r\exp[i(\varphi-\omega t)]$ while $f^{(1)}\sim\beta$ (for simplicity, we assume for the present that $\gamma=0$), then from the first-order equation we can determine both $f^{(1)}$ and the correction to the oscillator frequency (from the condition that no secular terms be present). The expressions obtained agree with the results of Ref. 3.

However, an attempt to carry this same scheme one step further would lead to contradictions, since secular terms of several types would then appear, which, in contrast to the classical case, cannot all be reduced to zero at once by choosing one parameter—the frequency. In reality, already at the zero-order approximation, a function containing an infinite number of frequencies is superimposed on the harmonic dependence, and this circumstance must be taken into account at the outset where solving Eq. (10).

Let us investigate Eq. (10) with an initial condition of general form, corresponding to an arbitrary function of the operator position and momentum:

$$f|_{t=0} = \sum_{n=-\infty}^{+\infty} f_n(\rho)e^{in\varphi}.$$

We will seek its solution in the form

$$f(t,\rho,\varphi) = \sum_{n=-\infty}^{+\infty} g_n(\rho,t)\exp[in(\varphi-\omega_0 t)] + \tilde{f}^{(1)} + f^{(1)}, \qquad (12)$$

where $\tilde{f}^{(1)}\sim\gamma$ while $f^{(1)}$ contains terms of order β and γ^2 (we neglect terms of higher order). This notation reflects the fact that the anharmonicity has an effect even in the zero-order (adiabatic) approximation, which leads to slowly-varying functions of time g_n (compared to the rapid variation at a frequency $\omega_0^{(1)}$). In essence, the model we are developing is analogous to the Krylov-Bogolyubov method[15] (let us recall, however, that here we are dealing not with a nonlinear equation but with a linear partial differential equation).

To first order in γ the secular terms are absent, and the term $\tilde{f}^{(1)}$ can be expressed directly in terms of the g_n. The condition that secular terms be absent in the equation for $f^{(1)}$ lets us obtain from (10) an equation for g_n:

$$\dot{g}_n = \frac{in\hbar}{m\omega_0}\left(\frac{15}{4}\frac{\gamma^2}{m^2\omega_0^3} - \frac{3}{8}\frac{\beta}{m\omega_0}\right)\left[g_n(1+2\rho^2) + \rho\frac{\partial g_n}{\partial\rho}\right]. \qquad (13)$$

The equation can be solved exactly, and its solution with the initial condition $g_n|_{t=0}=f_n(\rho)$ takes the form

$$g_n = f_n(\rho e^{-i\tau_n/2})\exp[\rho^2(e^{-i\tau_n}-1) - i\tau_n/2]. \qquad (14)$$

Here

$$\tau_n = n\Omega t \equiv n\tau, \quad \Omega = {}^3/_4\Omega_\beta - {}^{15}/_2\Omega_\gamma,$$
$$\Omega_\beta = \beta\hbar/m^2\omega_0^2, \quad \Omega_\gamma = \gamma^2\hbar/m^3\omega_0^4.$$

As a result, the expression for the adiabatic approximation becomes equal to

$$f^{(0)} = \sum_{n=-\infty}^{+\infty} f_n(\rho e^{-i\tau_n/2})\exp[\rho^2(e^{-i\tau_n}-1) - i\tau_n/2] \times\exp[in(\varphi-\omega_0 t)]. \qquad (15)$$

In particular, for the average values of position and momentum we obtain from this

$$z(t) = \langle a(t)\rangle = \rho\exp[-2\rho^2\sin^2(\tau/2)] \times \exp[i(\varphi-\omega t-\rho^2\sin\tau)], \qquad (16)$$
$$\langle x\rangle = 2\lambda\,\text{Re}\,z(t), \quad \langle p\rangle = 2\lambda m\omega_0\,\text{Im}\,z(t), \quad \omega=\omega_0+\Omega.$$

From (15) and (16) evidently arises a modulation of the harmonic dependences of $\langle x\rangle$ and $\langle p\rangle$, having a quantum origin. As a result, the system dynamics differs significantly

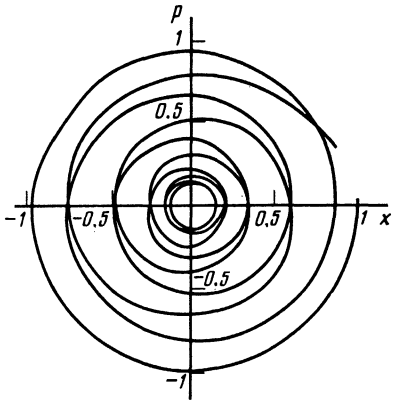

FIG. 1.

at sufficiently long times from the results of Ref. 3. This difference appears particularly striking in the "phase trajectory" of the system in $\langle x\rangle$-$\langle p\rangle$ space. In the classical case, the phase trajectory is a circle. Within the framework of the approach in Ref. 3, the curves remain circles (in the present approximation); only the frequency of uniform rotation of a point going around them changes. Including the quantum modulation significantly changes the character of the phase trajectory: the curve now has the form of a spiral, bounded by two circles of radius ρ and $\rho\exp(-2\rho^2)$ (in dimensionless units)—see Fig. 1. Although in the special case of commensurability of the frequencies ω and Ω the curves are closed, the corresponding "periodicity" has an approximate character and disappears when we include the time dependence due to higher approximations in the anharmonicity. For $n\sim\omega^2/2\pi\Omega\Omega_{\beta,\gamma}$ the formulae obtained for the rotation are no longer applicable (see details below).

The quantum properties of the system are apparent also in the fact that the product of the position and momentum uncertainties introduced by the Heisenberg relation becomes significantly different from its initial minimum value with the passage of time

$$\begin{gathered}\Delta x^2\Delta p^2=\tfrac{1}{4}\hbar^2(\mu_1^2-\mu_2^2),\quad \Delta x^2=\langle x^2\rangle-\langle x\rangle^2,\\ \Delta p^2=\langle p^2\rangle-\langle p\rangle^2,\\ \mu_1=1+2\rho^2\{1-\exp[-4\rho^2\sin^2(\tau/2)]\},\\ \mu_2=\rho^2\exp[2i(\varphi-\omega t)]\{\exp[\rho^2(e^{-2i\tau}-1)\\ -i\tau]-\exp[2\rho^2(e^{-i\tau}-1)]\}+\text{h.c.}\end{gathered}\tag{17}$$

Let us now turn to evaluating the next term in the expansion in the anharmonicity. In avoiding excessively tedious calculations we limit ourselves to the case of a pure quartic anharmonicity ($\gamma=0$) and calculate average values of position and momentum (since $f=z(t), f|_{t=0}=\rho e^{i\varphi}$). The equation for $f^{(1)}$, in which secular terms are now absent due to our choice of $g(\rho,t)$ according to (16), can be directly integrated. It must be noted that in the expressions so obtained terms proportional to $\exp[i(\omega_0 t-\varphi)]$, $\exp[\pm 3i(\varphi-\omega_0 t)]$ are present. It is not hard to see by analogy with the previous discussion that elimination of the secular terms in the equation for $f^{(2)}$ results in modulation of these terms corresponding to (15), which also applies to the next approximation in which terms appear containing $\exp[in(\varphi-\omega_0 t)]$. The requirement that secular terms of the type $\exp[i(\varphi-\omega_0 t)]$ be absent in the equation for $f^{(2)}$ makes it necessary to find the expression for $f^{(0)}$ to higher accuracy than is given by (14) and (15). Thus, the corresponding equation analogous to (13) will have a more complicated form. We present here the explicit expression for the first correction:

$$\begin{gathered}z^{(1)}(t)=(\beta\rho/m\omega_0^2)\{\xi_1\exp[i(3\varphi-\omega_0 t)]\\ +\xi_2\exp[i(\omega_0 t-\varphi)]+\xi_3\exp[3i(\varphi-\omega_0 t)]\\ +\xi_4\exp[3i(\omega_0 t-\varphi)]+\xi_5\exp[-i(\varphi+\omega_0 t)]\\ +\xi_6\exp[-i(3\varphi+\omega_0 t)]+\xi_7\exp[i(5\varphi-\omega_0 t)]\},\\ \xi_1=(\eta r^2/32)[3\varkappa^2+6\varkappa-2+(r/\lambda)^2\varkappa(1+3\varkappa/2+\varkappa^2/2)],\\ \xi_2=-3(\lambda^2\eta^*+r^2\nu/4)/4,\quad \xi_3=r^2\alpha/16,\\ \xi_4=-r^2\alpha^*/32,\quad \xi_5=\eta[3(\varkappa+1)^2\lambda^2/4+3r^2(1+3\varkappa+3\varkappa^2/2)/16\\ +(r^4\varkappa/32\lambda^2)(1+3\varkappa/2+\varkappa^2/2)],\\ \xi_6=(\eta r^2/32)[(\varkappa+1)^3+(r^2/4\lambda^2)\varkappa(1+3\varkappa/2+\varkappa^2+\varkappa^3/4)],\\ \xi_7=(\eta r^4\varkappa/128\lambda^2)(1+3\varkappa/2+\varkappa^2+\varkappa^3/4),\\ \eta=\exp(\rho^2\varkappa-i\tau),\ \nu=\exp(\rho^2\varkappa+2i\tau),\\ \alpha=\exp[\rho^2(e^{-3i\tau}-1)-6i\tau],\quad \varkappa=\exp(-i\tau)-1.\end{gathered}\tag{18}$$

Formula (8) of Ref. 3 is obtained from (16) and (18) when $\Omega t\ll 1$, which implies that it is possible to neglect the variation of the modulating factor $g(\rho,t)$ over the first of a hierarchy of time scales, over which this modulation does not appear. The region of applicability of results (15)–(18) is wider, and encompasses the second time scale:

$$(\Omega_{\beta,\gamma}/\omega_0)\Omega t\ll 1,\quad (\omega_{\beta,\gamma}/\omega_0)\Omega t\ll 1,$$

where

$$\omega_\gamma=\gamma^2 r^2/m^2\omega_0^3,\quad \omega_\beta=\beta r^2/m\omega_0.$$

(In addition, in both cases—just as in classical mechanics—the conditions $(\omega_\beta/\omega_0)\omega^{(1)}t\ll 1$, $(\omega_\gamma/\omega_0)\omega^{(1)}\ll 1$, $\omega^{(1)}=3\omega_\beta/8-15\omega_\gamma/4$ must be fulfilled.)

We now discuss passage to the classical limit (for $\gamma=0$) and trace how we obtain a harmonic dependence with renormalized frequency from the formulae given above. From the exact expression for the period of oscillation of a classical anharmonic oscillator, using the initial conditions it is not hard to obtain an expansion of the frequency in powers of β:

$$\omega_{\text{кл}}=\omega_0+\omega^{(1)}+\omega^{(2)}+\ldots,\tag{19}$$

where

$$\frac{\omega^{(1)}}{\omega_0}=\frac{3}{8}\frac{\beta r^2}{m\omega_0^2},$$

$$\frac{\omega^{(2)}}{\omega_0}=\frac{3}{32}\left(\frac{\beta r^2}{m\omega_0^2}\right)^2\left(\cos 2\varphi+\frac{\cos 4\varphi}{4}-\frac{17}{8}\right).$$

From (13) or (14) it is clear that as $\hbar\to 0$ the first-order correction to ω_0 follows directly from the expression for the modulating factor. For the correction $\omega^{(2)}$ things are somewhat more complicated. The part of $\omega^{(2)}$ which is independent of φ comes from the expression for g when the second time scale is taken into account (which corresponds to retaining terms $\sim\beta^2$ in the transition frequencies). As regards the part of (19) which depends on φ, it can be derived using a rearrangement of the perturbation series in the classical limit.

Let us keep in mind that the method we have adopted

does not require us to find either wave functions or energy levels, provided that we deal directly with averaged, i.e., integrated, quantities.[2] Thus, it allows us to obtain the average of an arbitrary function of a and a^+ in one operation (and, correspondingly, for any initial condition).

3. SPIN DYNAMICS IN THE PRESENCE OF A WEAK ANISOTROPIC FIELD

Up until now, the discussion has dealt with properties of the quantum anharmonic oscillator. Another nonlinear system, in which a modulation of the classical time dependence due to quantum corrections turns out to be significant, is a spin located in an anisotropic field (from here on we assume the anisotropy to be weak). The corresponding physical object is a paramagnetic ion in a nonmagnetic crystal[17] (analogous pseudo-spin models are systems of interacting fermions[18]).

We limit ourselves for simplicity to the uniaxial case, and assume that the magnetic field is directed perpendicular to the axis we select. In this case the Hamiltonian has the form

$$\mathcal{H}=-HS_z-DS_x^2. \tag{20}$$

Here, H plays the role of the magnetic field and D is an anisotropy constant; S_i is the operator for the ith projection ($i=x,y,z$) of the spin, which equals S.

We assume a spin-coherent state[13,19] as our initial condition, i.e., the average spin is assumed polarized along a direction n characterized in spherical polar coordinates by the angles θ and φ:

$$\begin{aligned}&|\Psi_0\rangle=(1+|\xi|^2)^{-S}\exp(\xi S_-)|S\rangle,\\ &\xi=e^{i\varphi}\operatorname{tg}(\theta/2),\qquad \langle\Psi_0|\mathbf{S}|\Psi_0\rangle=S\mathbf{n}.\end{aligned} \tag{21}$$

Here, $S_-=S_x-iS_y$ is a lowering operator and $|S\rangle$ is the state with maximum possible S_z projection. Rather than repeat the discussion of the previous paragraphs, we present the results.

The dynamics of this system[3] is described by the equations

$$\begin{aligned}&\dot f+\omega_0\partial f/\partial\varphi=(D/\hbar)\,(2S\sin\theta\cos\varphi+\hat e)\,\hat m f,\\ &\hat e=\cos\varphi\cos\theta\partial/\partial\theta-(\sin\varphi/\sin\theta)\partial/\partial\varphi,\\ &\hat m=\sin\varphi\partial/\partial\theta+\cos\varphi\operatorname{ctg}\theta\partial/\partial\varphi,\qquad \omega_0=H/\hbar.\end{aligned} \tag{22}$$

We limit ourselves to determining the average value of the component $S_+=S_x+iS_y$, so that $f|_{t=0}=Se^{i\varphi}\sin\theta$. To lowest order in D, including the quantum modulation,

$$\begin{aligned}&f^{(0)}=S\sin\theta\exp[i(\varphi-\omega_0t)]\,(\cos\tau+i\sin\tau\cos\theta)^{2S-1},\\ &\tau=Dt/2\hbar.\end{aligned} \tag{23}$$

The first-order correction has the form

$$\begin{aligned}f^{(1)}=&(D/4\hbar\omega_0)S(2S-1)\sin\theta\cdot\{C_1(\tau,\theta)\exp[i(\omega_0t-\varphi)]\\ &+C_2(\tau,\theta)\exp[-i(\omega_0t+\varphi)]+C_3(\tau,\theta)\exp[i(3\varphi-\omega_0t)]\},\end{aligned} \tag{24}$$

$$C_1(\tau,\theta)=(\cos\tau\cos\theta-i\sin\tau)\,(\cos\tau-i\sin\tau\cos\theta)^{2(S-1)},$$

$$\begin{aligned}&C_2(\tau,\theta)\\ &=-(\cos\theta-iS\sin\tau\cos\tau\sin^2\theta)\,(\cos\tau+i\sin\tau\cos\theta)^{2S-3},\end{aligned}$$

$$C_3(\tau,\theta)=i(S-1)\sin^2\theta\cos\tau\sin\tau(\cos\tau+i\sin\tau\cos\theta)^{2S-3}.$$

We remark that for spins $S=1/2$, 1, 3/2, formulae (23) and (24), in contrast to the anharmonic oscillator, allow comparison with exact solutions.

It is interesting that the modulation part of the time dependence of (23), describing the behavior of the spin in a strong (compared to the anisotropy field) magnetic field is analogous to the dynamics of a spin with the same initial conditions but in the absence of the field.[8,9] As is clear from (23), in the lowest approximation in D the spin executes oscillations in the reference system rotating with frequency ω_0 which are purely quantum-mechanical in origin (compare with Ref. 8). Thus, in the interval $0\leqslant\tau\leqslant\pi$ each of the transverse components of the spin in this system vanishes exactly $2S-1$ times, independent of the value of θ. In the limit $\theta\to\pi/2$, when the quantum properties of the system are especially striking, all of the zeroes condense down into one, which is $(2S-1)$-fold degenerate.

We now turn to a generalization of (23) for arbitrary functions of the spin operator, one example of which is the case of biaxial anisotropy where the Hamiltonian differs from (20) by the addition of a term $-ES_z^2/2$. Let us introduce the variable ζ corresponding to $\cos\theta=-\operatorname{th}\zeta$. Then, if the initial value of the quantity sought is

$$f|_{t=0}=\sum_{n=-\infty}^{+\infty}f_n(\zeta)e^{in\varphi},$$

then for $t>0$ the leading approximation for the amplitude yields

$$f=\sum_{n=-\infty}^{+\infty}f_n(\zeta-i\tau_n)\left[\frac{\operatorname{ch}(\zeta-i\tau_n)}{\operatorname{ch}\zeta}\right]^{2S}\exp[i(\varphi-\omega_0t)n],$$

$$\tau_n=\frac{n(D-E)t}{2\hbar}.$$

From this, in particular, it is clear that there is no modulation in the longitudinal component $\langle S_z\rangle$. It is also absent (in this approximation) for equality of the anisotropy constants $D=E$.

Thus, it follows from the analysis presented in Refs. 6–9 and in the present article that even when the quantum system (both for the Hamiltonian and the spin) is as close as possible to classical (due to our choice of an initial coherent state), the presence of even a weak nonlinearity leads to qualitative differences in the behavior of the quantum and classical systems. Although in the present paper the time evolution was connected with the time-dependent character of the initial state, we can suppose that these methods and results may prove to be useful also in more complicated situations—for example, in studying periodic excitation of a quantized nonlinear system.

The authors are grateful to B. Fryz for help in computations. We also acknowledge useful discussions with V. I. Man'ko.

[1] In the language of perturbation theory formula (12) corresponds to a zero-order approximation in the anharmonicity for the wave functions and first order in the energy levels.

[2] In Ref. 16, pseudo-coherent states were used, and approximately closed equations were obtained for the characteristic of the "center of gravi-

ty," corresponding to a wave packet; these equations were used for numerical analysis.

[3)]It is noteworthy that the time-independent properties of this system permit effective-potential description in terms of spin coherent states.[20]

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Translated by Frank J. Crowne