Effect of level quantization on the lifetime of metastable states

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The positions of the quantum levels and the transition matrix elements connecting them are found for superconducting tunneling junctions in the semiclassical approximation. The location of the upper levels affects the coefficient of the exponential in the expression for the probability for decay of the metastable voltage state. A variable current with frequency close to the difference between the energies of two levels increases the decay probability. The theoretical results are in good agreement with the experimental data.

1. INTRODUCTION

The voltage states of superconducting tunneling junctions are metastable. The investigation of the lifetime of such states is of interest in its own right.

Furthermore, this problem is a convenient model for the study of the metastable states in other physical systems. Below we investigate how the quantization of the levels in the potential well affects the lifetime of a metastable state. For current strengths close to the critical value, the potential energy has the form of a cubic parabola. In such a potential the width of the well is of the same order of magnitude as the barrier width. One would have thought that the probability for decay of a metastable state would be small only in a semiclassical potential, in which the number of levels is large. But this is not so for numerical reasons. The difference in effective action between two neighboring levels is equal to 2π. The tunneling probability is determined by an imaginary action. In potentials with equal well and barrier widths the ratio of the probabilities for decay from two neighboring levels is close to $\exp(-2\pi) = 0.00187$. Therefore, the lifetime of the metastable state is long even when the number of levels is small. In the experiments in which quantum tunneling was investigated,[1–4] the number of levels was not large (ranging from 1 to 10).

The discreteness of the levels has the greatest effect on the lifetime of the metastable state when the temperatures T are of the order of the level spacing. In this case the coefficient exponential in the expression for the probability for decay of the metastable state is an oscillating function of the depth of the potential well. The oscillation amplitude decreases as the viscosity increases. The quantization of the levels at small value of viscosity leads to a resonant dependence of the metastable-state lifetime on the pump frequency.

2. DISTRIBUTION FUNCTION

Normally, it is assumed in the investigation of the tunneling of particles through a potential barrier that the particle motion in the classically accessible region can be described with the aid of wave packets. This approximation is valid for sufficiently broad potential wells. In Ref. 5 the present authors used this approximation for a potential in the form of a cubic parabola. The basis for this approximation is the fact that at energies close to the potential barrier height, the period of the classical motion is long, the level spacing is small, and we can construct wave packets. But for numerical reasons this approximation holds only for very high potential barriers. The level spacing δE in the vicinity of the top of the barrier is equal to

$$\frac{\delta E}{\Omega_p} = \left(1 + \frac{1}{2\pi}\ln N\right)^{-1}, \qquad (1)$$

where N is the number of levels in the potential well, Ω_p is the frequency of the classical oscillations in the vicinity of the bottom of the well in the inverted potential, $m\Omega_p^2 = -\partial^2 U/\partial\varphi^2$ at the maximum of $U(\varphi)$.

Quantum tunneling can be observed only when the number N of levels is not large. Level crowding does not occur in such potentials. On the other hand, the probability that a particle will tunnel through a potential barrier depends very strongly on the particle energy E, and for energies close to the barrier height U,

$$\gamma(E) = (\delta E/2\pi)\exp(-(U-E)/T_0), \quad T_0 = \Omega_p/2\pi. \qquad (2)$$

Therefore, we cannot go over to the continuous energy distribution, and at small viscosity values must write down the kinetic equation for the probabilities ρ for finding the particle at the j-th level:

$$\frac{\partial \rho_j}{\partial t} = \sum_k (W_{jk}\rho_k - W_{kj}\rho_j) - \gamma_j\rho_j, \qquad (3)$$

where W_{jk} is the probability for transition from the state k into the state j due to the interaction of the particle with the heat bath. For an equilibrium heat bath the matrix elements W_{jk} satisfy the condition

$$W_{jk} = W_{kj}\exp[(E_k - E_j)/T], \qquad (4)$$

where T is the temperature of the heat bath.

As will be seen below, in a potential in the form of a cubic parabola

$$U(\varphi) = 3U(\varphi/\varphi_0)^2[1 - {}^2/_3\varphi/\varphi_0] \qquad (5)$$

only the transitions between the nearest levels are important. For sufficiently deep levels, i.e., for $j < n$, the tunneling prob-

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ability γ_j is small compared to the transition matrix elements $W_{j-1,j}$, and it can be neglected. In this approximation the steady-state solution to Eq. (3) has the form

$$\rho_j = \exp(-E_j/T) - C\sum_{k\leq j}\exp[-(E_j-E_k)/T]/W_{k-1,k},\ j\leq n, \quad (6)$$

where C is a constant that is found from the solution of the system of equations (3) for $n\leq j<n+\nu$.

For these states both the tunneling and dissipation processes are important. For states with $j\geq n+\nu$ tunneling is much more probable than the dissipative-transition process, and the probabilities ρ_j for $j\geq n+\nu$ can be set equal to zero. Because of the rapid growth of the tunneling probability γ_j as the number j increases, we can limit ourselves to small values of the quantity ν. The solutions to the system of equations (3) for $\nu=2$ and $\nu=3$ give practically the same result.

The metastable-state decay probability Γ in this approximation is equal to

$$\Gamma=\Big\{\sum_{j<n+\nu}\gamma_j\rho_j+W_{n+\nu,n+\nu-1}\rho_{n+\nu-1}\Big\}\Big/\sum_{j<n+\nu}\rho_j. \quad (7)$$

3. THE WEAK VISCOSITY LIMIT

We can follow the qualitative pattern of oscillation of the coefficient exponential under the simplest assumption that $\nu=1$. This means that it is only at level $j=n$ that both the tunneling and the dissipative-transition processes are important. From Eq. (3) with $j=n$ we find

$$\rho_n=W\exp\left(-\frac{E_n}{T}\right)\frac{1-\exp(-\delta E/T)}{W+\gamma_n}. \quad (8)$$

In deriving the formula (8) we use the expression (6) for the quantity ρ_n and the assumption that $\rho_{n+1}=0$. Furthermore, we assumed that the matrix elements W_{j-1} do not depend on the number j, and that the levels are equally spaced, with $\delta E=E_j-E_{j-1}$.

If the temperature T is not very close to the temperature T_0 (specifically, if $T-T_0>T_0/2\pi$), then it is sufficient to retain in the expression (7) for the quantity Γ only the terms with ρ_n:

$$\Gamma=2\,\mathrm{Sh}(\Omega_p/2T)\exp(-U/T)[1-\exp(-\delta E/T)]\mathscr{F}, \quad (9)$$

$$\mathscr{F}=W\exp\left(\frac{U-E_n}{T}\right)\frac{\gamma_n+W\exp(-\delta E/T)}{W+\gamma_n}.$$

It follows from the formula (2) that the quantity γ_n depends critically on the level energy E_n. Therefore, the expression (9) is an oscillating function of the position of the level E_n.

In the region $W\exp(-\delta E/T)<\gamma_n<W$ the process of tunneling from the nth level is a "bottleneck" and the quantity $\mathscr{F}$ depends weakly on the dissipation W and is equal to

$$\mathscr{F}=(\delta E/2\pi)\exp[-(U-E_n)(1/T_0-1/T)]. \quad (10)$$

In this region function $\mathscr{F}$ increases with increasing E_n.

In the region $\gamma_n>W$ the transition of the system from the $(n-1)$-th to the n-th level as a result of the interaction with the heat bath is the bottleneck. In this case the function $\mathscr{F}$ decreases as the energy E_n increases:

$$\mathscr{F}=W\exp\left(\frac{U-E_n}{T}\right). \quad (11)$$

In the region $\gamma_n<W\exp(-\delta E/T)$ the function $\mathscr{F}$ is given by the formula (11) with n replaced by $n+1$.

4. POSITIONS OF THE LEVELS AND THE TRANSITION PROBABILITIES

For the purpose of making a quantitative comparison with the experimental data we must find the positions and widths of the levels. If a level is not very close to the top of the potential barrier (i.e., if $U-E>0,\ 4\Omega_p$), then its position and width can be found with the aid of the semiclassical formulas:

$$S(E_n)=\pi(n+{}^1/_2),$$

$$\gamma(E_n)=\frac{\delta E\exp(-2S_0(E_n))}{(2\pi)^{1/2}\Gamma(n+1)}\times\exp\left\{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-1\right]\right\}, \quad (12)$$

where $S(E)$ is the action in the classically accessible region and $S_0(E)$ is the action in the classically inaccessible region. For a potential of the form (5) we find

$$S(E)=\frac{\pi}{4}\varphi_0(mU)^{1/2}(x_2-x_3)^2\times(x_1-x_3)^{1/2}F\left(-\frac{1}{2};\frac{3}{2};3;\frac{x_2-x_3}{x_1-x_3}\right),$$

$$S_0(E)=\frac{\pi}{4}\varphi_0(mU)^{1/2}(x_1-x_2)^2\times(x_1-x_3)^{1/2}F\left(-\frac{1}{2};\frac{3}{2};3;\frac{x_1-x_2}{x_1-x_3}\right); \quad (13)$$

where F is the hypergeometric function and $x_3<x_2<x_1$ are the roots of the cubic equation

$$x^3-3x^2/2+E/2U=0. \quad (14)$$

The interaction of the normal excitations with the heat bath is determined by the resistance R of the junction to normal current. The probability $W_{j-1,j}$ for transition from the state j into the state $j-1$ is given in Ref. 6:

$$W_{j-1,j}=\frac{\delta E}{Re^2}[1+\mathrm{cth}(\delta E/2T)]\,|\langle j|\varphi|j-1\rangle|^2, \quad (15)$$

where

$$\langle j|\varphi|j-1\rangle=-\frac{\pi^2\varphi_0(x_2-x_3)}{2k^2K^2(k)\,\mathrm{Sh}[\pi K'(k)/K(k)]},\qquad k=\left(\frac{x_2-x_3}{x_1-x_3}\right)^{1/2}; \quad (16)$$

$K(k)$ and $K'(k)$ are complete elliptic integrals. That value of the energy E in the formula (14) at which the roots $x_{1,2,3}$ in the formula (16) are computed is equal to $E=(E_j+E_{j-1})/2$. A level lying close to the top of the barrier is quite wide. Therefore, the probability for transition to it is described by quantum-mechanical formulas for transitions into the continuum. The transition probability density in an interval dE is given as before by the formula (15), in which we must now replace the wave function of the upper level by

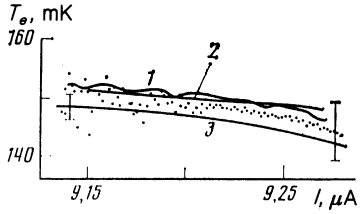

FIG. 1.

the continuum wave function normalized as a δ-function in energy. Using the semiclassical approximation for the wave functions, we obtain for the matrix element the expression

$$|\langle j|\varphi|E\rangle|^2=\frac{9\varphi_0^2}{\Omega_p^4}\frac{\pi(E_j-E_{j-1})}{2m}\frac{(E-E_j)^2}{\mathrm{Sh}^2[\pi(E-E_j)/\Omega_p]}|C|^2, \qquad (17)$$

where

$$|C|^2=\frac{2m}{\pi}\exp(2\pi y)\left|1+\frac{(2\pi)^{1/2}}{\Gamma(1/2+iy)}\times\exp\left[\frac{\pi}{2}y+i\frac{6\cdot 6^{1/2}}{5}\varphi_0(mU)^{1/2}+2iy\ln(6\varphi_0(2m\Omega_p)^{1/2})\right]\right|^{-2};\quad y=(E-U)/\Omega_p. \qquad (18)$$

The expression (17) has a sharp maximum. The position of this maximum determines the level energy E_{j+1}, while its width determines the tunneling probability $\gamma(E_{j+1})$. These expressions are used in the computation of the decay probability Γ with the aid of Eq. (7). Expression (17) can also be used in the case of transitions into states with energies $E>U$. It is found that there exists in this energy region one virtual level whose width is small (of order $\Omega_p/2\pi$). This level should be taken into account in Eq. (7) along with the sub-barrier levels.

5. COMPARISON WITH THE EXPERIMENTAL DATA

The experimental data reported in Ref. 1 for the decay probability Γ are represented in the form

$$\Gamma=\frac{\Omega_p}{2\pi}\exp(-U/T_e).$$

for $T>T_0$ the quantity T_e differs from the temperature T of the heat bath because of the coefficient of the exponential in the expression for the decay probability. Therefore, the quantity T_e, like this coefficient is an oscillatory function of the depth of the potential well. In the experiment reported in Ref. 3 the depth U of the potential well was varied by varying the current J through the junction. Figure 1 shows the theoretical and experimental data. The points in Fig. 1 indicate the experimental data reported in Ref. 3. The curve 1 is the theoretical curve obtained with the aid of the formulas (6), (7), (12), and (14)–(16). It was computed with the following junction parameters: $R=190$ Ω, $C=6.35$ pF, $J_c=9.489$ μA, and $T=0.151$.

The parameters φ_0 and U of the potential (5) are given in terms of the critical current J_c and the current J flowing through the junction by the formulas

$$\varphi_0=[1-(J/J_c)^2]^{1/2},\quad U=\hbar J_c\varphi_0^3/3e. \qquad (19)$$

The curve 2 was constructed with the aid of the formula given in Ref. 5, in which it is assumed that level crowding occurs at an energy close to the top of the barrier. The curve 2 is a smooth, nonoscillatory curve, but it lies very close to the curve 1 obtained in the present paper. The curve 3 was taken from Ref. 3, and reproduces the theoretical results obtained in Ref. 7. By varying the junction parameters within the limits of possible experimental errors ($R=190\pm100$ Ω and $C=6.35\pm0.4$ pF), we can obtain an even better agreement between the theoretical and experimental results. The amplitude of the oscillations (curve 1) turned out to be small. This is due to the fact that the shunting resistance was not high (the case of intermediate viscosity: the viscosity η is of the order of[5] η_1), while the temperature T was high in comparison with T_0 ($T\sim3T_0$). Under these conditions the decay proceeds largely via a resonance level with $E>U$ whose width depends weakly on the energy. The amplitude of the oscillations increases as the temperature T is lowered in the region $T-T_0>T_0/2\pi$. It also increases when the shunting resistance is increased, since at low viscosity values the system decays from deeper-lying levels whose lifetime depends exponentially on the energy (the formula (2)). The quantity T_e/T for current strengths close to the critical value is a universal function of three dimensionless parameters:

$$T_e/T=\psi(z;\ T/T^*;\ R/R^*), \qquad (20)$$

where

$$z=(J_c-J)/J_c^*,\quad T^*=k^{-1}(\hbar J_c e/C)^{1/2}(e^3/\hbar J_c C)^{1/10},$$
$$R^*=(\hbar J_c C/e^3)^{1/10}(\hbar/eJ_cC)^{1/2},\quad J_c^*=J_c(e^3/\hbar J_c C)^{2/5}, \qquad (21)$$

k being the Boltzmann constant. Figure 1 shows a plot of T_e for the parameters $T/T^*=0.017251$ and $R/R^*=13.8252$. Figure 2a shows a plot of T_e/T for the junction parameters used by Martinis *et al.*[4]: $R=135.45$ Ω, $C=47$ pF, $T=28$ mK, $J_c=30.572$ μA, $T/T^*=0.006658$, and $R/R^*=35.048$.

The quantity z is connected with the number N of levels in the potential well by the relation

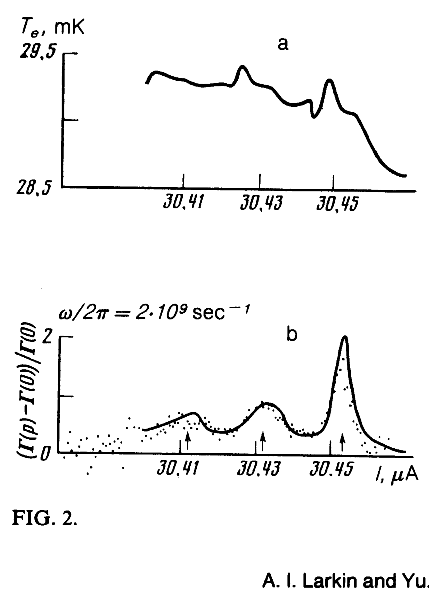

FIG. 2.

$$N=0.5+\frac{3\cdot 2^{7/4}}{5\pi}z^{5/4}. \qquad (22)$$

The period of the oscillations is determined from the requirement that the number of levels in the potential well change by unity:

$$\delta J={}^1/_3\pi J_c^*(2/z)^{1/4}. \qquad (23)$$

6. RESONANT DECREASE OF THE LIFETIME UNDER THE ACTION OF AN EXTERNAL CURRENT

The existence of levels is most strikingly manifested in the resonant decrease of the lifetime of the metastable state under the action of a variable current with frequency equal to the level spacing. Such resonances have been experimentally observed by Martinis *et al.*[4] This phenomenon is investigated theoretically in Ref. 8. In the present paper we investigate two limiting cases: the case of high Q, when the width of the resonances is smaller than their spacing, and the case of arbitrary resonance width in the presence of a large number of almost equidistant levels in the system.

For the purpose of carrying out a quantitative comparison with the experimental data obtained by Martinis *et al.*,[4] let us use the following system of equations for the density matrix[8]:

$$\frac{\partial \rho_f^{\,j}}{\partial t}=\frac{iJ_1}{e}\cos(\omega t)\sum_m\{\langle j|\varphi|m\rangle\exp[-i(E_m-E_j)t]\rho_f^{\,m}$$

$$-\langle f|\varphi|m\rangle\exp[i(E_m-E_f)t]\rho_m^{\,j}\}$$

$$+\sum_{m,n}W_{fn}^{\,jm}\rho_n^{\,m}-\frac{1}{2}\sum_m(W_{mj}^{\,mj}+W_{mf}^{\,mf})\rho_f^{\,j}. \qquad (24)$$

In the experiment performed by Martinis *et al.*[4] the temperature T was small compared to the level spacing. The off-diagonal density-matrix elements, to within terms exponentially small with respect to this parameter, are equal to

$$\rho_{j+1}^{\,j}=-\frac{J_1}{2e}\frac{\langle j|\varphi|j+1\rangle(\rho_j-\rho_{j+1})}{\omega-E_{j+1}+E_j-{}^1/_2 i\Gamma_j}, \qquad (25)$$

where

$$\Gamma_j=\gamma_j+\gamma_{j+1}+W_{j+1,j}+W_{j-1,j}+W_{j,j+1}+W_{j+2,j+1}. \qquad (26)$$

The system of equations (24) for the diagonal density-matrix elements has, when allowance is made for the formula (25), the form

$$\sum_k(W_{jk}\rho_k-W_{kj}\rho_j)-\gamma_j\rho_j-\frac{J_1^2}{4e^2}\langle j|\varphi|j+1\rangle^2$$

$$\times(\rho_j-\rho_{j+1})\frac{\Gamma_j}{(\omega-E_{j+1}+E_j)^2+\Gamma_j^2/4}$$

$$+\frac{J_1^2}{4e^2}\langle j|\varphi|j-1\rangle^2$$

$$\times(\rho_{j-1}-\rho_j)\Gamma_{j-1}((\omega-E_j+E_{j-1})^2+\Gamma_{j-1}^2/4)^{-1}=0. \qquad (27)$$

Let us solve the system of equations (27) in the approximation used to solve the system (3). Two equations of this system, namely, the equations for the virtual level lying above the barrier and the top subbarrier level, were solved exactly. For the deeper-lying levels the tunneling probability, γ_i was set equal to zero. In this approximation the solution to the system of equations (27) can be represented in the form

$$\rho_j=\Big(\prod_{\nu<j}G_\nu\Big)\Big\{1-C\sum_{k<j}\Big[(1+b_k)W_{k,k+1}\prod_{\nu<k+1}G_\nu\Big]^{-1}\Big\}, \qquad (28)$$

where

$$G_\nu=\frac{b_\nu+\exp[-(E_{\nu+1}-E_\nu)/T]}{1+b_\nu},$$

$$b_\nu=\frac{J_1^2}{4e^2}\frac{\langle\nu|\varphi|\nu+1\rangle^2}{W_{\nu,\nu+1}}\frac{\Gamma_\nu}{(\omega-E_{\nu+1}+E_\nu)^2+\Gamma_\nu^2/4}. \qquad (29)$$

The constant C is determined from the condition for matching with the exact solution to the equations for the two top levels. The lifetime of the metastable state is given as before by the formula (7), and is determined in the region $T>T_0$ by the distribution function at the upper levels.

From the formula (28) we find in the approximation linear in the pump power that

$$\rho_j=\rho_j^{(0)}+\sum_{\nu<j}b_{\nu-1}(\rho_{\nu-1}^{(0)}-\rho_\nu^{(0)})\exp(-(E_j-E_\nu)/T), \qquad (30)$$

where the functions $\rho_j^{(0)}$ are given by Eq. (6).

At high Q values the function b_ν has a sharp maximum. The pump-induced change in the distribution function at high levels is equal to the sum of the resonance contributions. The linear approximation (30) breaks down for two reasons. In the case of narrow resonances the quantity b_ν can be greater than, or of the order of, unity in the vicinity of a resonance. This leads to the equalization of the populations of the ν- and $(\nu+1)$th levels. Further increase in the pump power does not cause the peak to grow. At low temperatures the magnitude of the effect is exponentially large:

$$\Delta\Gamma/\Gamma=b_\nu\exp[(E_{\nu+1}-E_\nu)/T].$$

For broad resonances the overlapping of neighboring resonances can be important. If the quantity b_ν for two neighboring levels is greater than, or of the order of, $\exp(-[E_{\nu+1}-E_\nu)/T]$, then the general formula (28) should be used. The linear approximation (30) is valid for the conditions under which Martinis and his co-workers[4] performed their experiment. The positions of the levels and the transition matrix elements connecting them were computed with the aid of the semiclassical formulas (12) and (15)–(17). Only one adjustable parameter—the pump power, which was unknown to us—was used in the comparison with the experimental data of Martinis *et al.*[4] A good agreement is obtained for a pump power P equal to

$$P=RJ_1^2/2=8{,}57\cdot 10^{-4}\hbar\omega^2, \qquad (31)$$

where ω is the pump frequency. In Fig. 2b the points indicate the experimental data obtained by Martinis *et al.*[4] and the continuous curve is the result of the numerical computation performed with the use of the formulas (7) and (30). The junction parameters are the same as those used in the computation of the curve in Fig. 2a, and the frequency satisfies $\omega/2\pi=2\times10^9\ \text{sec}^{-1}$.

7. CONCLUSION

The good agreement between the experimental and theoretical data on the resonant decrease of the lifetime of a metastable state of tunneling junctions confirms the existence of quantum levels in the potential well. The positions and widths of the level can be found with a high degree of accuracy with the semiclassical formulas. The coefficient of the exponential exhibits oscillations when the level leaves the potential well. Under the conditions of the experiment reported in Ref. 3 the oscillation amplitude was small, and was less than the spread in the experimental data. The junctions most suitable for the observation of the oscillations are those with high effective resistances at temperatures close to the transition temperature T_0. The experimental data reported in Ref. 3 are in good agreement with both the results obtained in the present paper and those obtained in Ref. 5.

In the present paper we have assumed that the system goes over into a continuum state when it decays from a metastable state. Such a situation results in the case of a junction included in a circuit with a prescribed current, or in a circuit with a high inductance. If the inductance of the external circuit is comparable to that of the junction, then the potential energy $U(\varphi)$ has a small number of wells. The coefficient of the exponential in the expression for the probability for transition from one well into another depends on the relative disposition of the levels in these wells. In the case of weak viscosity a resonant increase can occur in the probability for that transition in which the level energies in the initial and final states are equal. It is possible that such a phenomenon was observed in the experiment reported in Ref. 9. To carry out a quantitative calculation, we must know the values of the capacitance and resistance of the junction.

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