Solitons in the two-dimensional hydrodynamic model of a collisionless cold plasma

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We consider, in the hydrodynamic model of a collisionless cold plasma, two-dimensional force-free configurations of an electron flow against the uniform background of fixed ions. We show that the charge-density distribution of the electrons can have the shape of a two-dimensional soliton.

When there are strong self-consistent fields present in a rarefied cold plasma, quasistationary localized formations may exist, such as electron bunches or rings, in which balance between the forces is guaranteed simultaneously in the longitudinal and transverse directions. A large amount of energy can be accumulated in such configurations and this determines their effective use for solving a number of problems in plasma physics.[1,2]

We show in the present paper that, for an axially symmetric system with a plane geometry, the two-dimensional stationary hydrodynamic equations for a collisionless cold plasma with immobile ions ($N_i = \text{const}$) can be reduced, under specially chosen conditions for the formation of an electron flow, to the sinh-Gordon or sine-Gordon equations of a linear massive field.

1. When describing a stationary electron flow in self-consistent crossed fields we shall start from the relativistic collisionless hydrodynamic equations of a cold plasma:

$$(\mathbf{p}\nabla)\mathbf{p}=\mathbf{E}\gamma+[\mathbf{pB}], \tag{1.1}$$

$$[\nabla\mathbf{B}]=\mathbf{p}n/\gamma, \tag{1.2}$$

$$\nabla\mathbf{E}=n-1, \tag{1.3}$$

$$[\nabla\mathbf{E}]=0, \quad \nabla\mathbf{B}=0, \tag{1.4}$$

where the hydrodynamic momentum is $\mathbf{p} = \gamma\mathbf{v}/c$; the Lorentz factor is $\gamma = (1+\mathbf{p}^2)^{1/2}$;

$$\mathbf{E}=e\mathbf{E}'/mc\omega_p, \quad \mathbf{B}=e\mathbf{B}'/mc\omega_p;$$

$\mathbf{E}'$, $\mathbf{B}'$ are the electric and magnetic field strengths; $n = N_e/N_i$; the characteristic size of the system is $\sim c/\omega_p$; $\omega_p^2 = 4\pi e^2 N_i/m$; e, m are the electron charge and mass, and c is the light velocity. Using the vector identity

$$(\mathbf{p}\nabla)\mathbf{p}=\gamma\nabla\gamma-[\mathbf{p}[\nabla\mathbf{p}]],$$

we write Eq. (1.1) in the form

$$\mathbf{E}=\nabla\gamma-\nabla f, \quad \mathbf{B}=-[\nabla\mathbf{p}]+[\nabla\boldsymbol{\Phi}], \tag{1.5}$$

where $f(\mathbf{r})$, $\boldsymbol{\Phi}(\mathbf{r})$ are auxiliary functions which are linked to $\mathbf{p}$ by the relations

$$\nabla f=[\mathbf{p}[\nabla\boldsymbol{\Phi}]]/\gamma. \tag{1.6}$$

When f, $\boldsymbol{\Phi} = 0$ we have $\gamma + \varphi = \text{const}$, $\mathbf{p} + \mathbf{A} = \text{const}$, where $(\varphi, \mathbf{A})$ is the 4-potential. Solutions of (1.2) to (1.4) for f, $\boldsymbol{\Phi} = 0$, $N_i = 0$ were considered in Refs. 3 and 4. Substituting (1.5) into (1.2) to (1.4) we get

$$n=1+\nabla^2\gamma-\nabla^2 f, \tag{1.7}$$

$$-[\nabla[\nabla\mathbf{p}]]+[\nabla[\nabla\boldsymbol{\Phi}]]=\mathbf{p}(1+\nabla^2\gamma-\nabla^2 f)/\gamma. \tag{1.8}$$

When there is no directed motion in the system with a constant-velocity in a crossed-field geometry (i.e., $\mathbf{p}\perp\mathbf{E}, \mathbf{B}$; $\boldsymbol{\Phi}\|\mathbf{p}$, i.e., only one component of $\mathbf{p}$ and $\boldsymbol{\Phi}$ is non-vanishing) and in the direction of the motion the system is uniform. Two of the three Eqs. (1.8) are then satisfied identically and the problem reduces to finding solutions which satisfy Eq. (1.6) for the remaining equation.

In a plane geometry (coordinates x, y, z) when $\partial/\partial y = 0$ (z,x have the meaning of the longitudinal and transverse coordinates) the momentum component p_y and the field components E_x, E_z, B_x, B_z are non-vanishing. The equations for p_y and for the functions f, Φ_y have in this case the form

$$\frac{\partial f}{\partial x}=\frac{p_y}{(1+p_y^2)^{1/2}}\frac{\partial\Phi_y}{\partial x}, \quad \frac{\partial f}{\partial z}=\frac{p_y}{(1+p_y^2)^{1/2}}\frac{\partial\Phi_y}{\partial z}, \tag{1.9}$$

$$\frac{\partial^2 p_y}{\partial x^2}-\frac{\partial^2\Phi_y}{\partial x^2}+\frac{\partial^2 p_y}{\partial z^2}-\frac{\partial^2\Phi_y}{\partial z^2}=\frac{p_y}{(1+p_y^2)^{1/2}}\left\{1+\frac{\partial^2(1+p_y^2)^{1/2}}{\partial x^2}-\frac{\partial}{\partial x}\left[\frac{p_y}{(1+p_y^2)^{1/2}}\frac{\partial\Phi_y}{\partial x}\right]+\frac{\partial^2(1+p_y^2)^{1/2}}{\partial z^2}-\frac{\partial}{\partial z}\left[\frac{p_y}{(1+p_y^2)^{1/2}}\frac{\partial\Phi_y}{\partial z}\right]\right\}. \tag{1.10}$$

We put $p_y = \sinh u$, $\Phi_y = \Phi(u)$, $u = u(x,z)$. Equations (1.9) are then compatible for any $\Phi(u)$ and determine f, while (1.10) reduces to the form

$$\sigma_{xx}+\sigma_{zz}=\operatorname{sh} u, \tag{1.11}$$

where the field function

$$\sigma(x,z)=u-\int_0^u\frac{d\Phi}{du'}\operatorname{ch}^{-1}u'\,du', \quad \sigma_x=\frac{\partial\sigma}{\partial x}, \quad \sigma_z=\frac{\partial\sigma}{\partial z}$$

and so on. By specifying the functions $\Phi(u)$ or $\sigma(u)$ we determine the type of the interactions in the system. These functions can, in turn, be determined either from the boundary conditions or, if the system has no boundaries, from the conditions for the formation of a flow. For the charge density and the electric and magnetic field components we have in this case

$$n=\operatorname{ch} u(\operatorname{ch} u+u_x\sigma_x+u_z\sigma_z), \tag{1.12}$$

$$E_x=\sigma_x\operatorname{sh} u, \quad E_z=\sigma_z\operatorname{sh} u, \quad B_x=\sigma_z\operatorname{ch} u, \quad B_z=-\sigma_x\operatorname{ch} u. \tag{1.13}$$

When $N_i = 0$ Eq. (1.11) changes into the Laplace equation and describes a vacuum E-layer.

We consider some integrable models of $\sigma(u)$ with $N_i \neq 0$ in which Eq. (1.11) can have localized solutions.

2. When the relations $\gamma + \varphi = \text{const}$, $\mathbf{p} + \mathbf{A} = 0$ are satisfied, it turns out that $\Phi = 0$, $\sigma = u$ and (1.11) changes into the sinh-Gordon equation. The "repulsive" non-linear field model which is then realizable has no bounded solutions. We put $\sigma = -u/\lambda$; here positive and negative signs of λ correspond to "attraction" and "repulsion," and $|\lambda|$ is a size factor. We then have from (1.11)

$$u_{xx}+u_{zz}=-\lambda\,\mathrm{sh}\,u. \tag{2.1}$$

We shall look for the solution of (2.1) in the form[5]

$$u=2\ln[(X(x)+Z(z))/(X(x)-Z(z))]. \tag{2.2}$$

Substituting (2.2) into (2.1) we get

$$X'^2=k^2X^4-(\lambda+\mu^2)X^2+\nu^2,\quad Z'^2=-k^2Z^4+\mu^2Z^2-\nu^2. \tag{2.3}$$

Here and henceforth the prime indicates differentiation with respect to the appropriate independent variable, and k^2, μ^2, and ν^2 are arbitrary parameters. Using (2.1) to (2.3) we have in this case from (1.12) and (1.13) for the charge density and the components of the electromagnetic field and p_y and γ

$$n=\frac{1+6X^2/Z^2+X^4/Z^4}{(1-X^2/Z^2)^4}\left\{1+\frac{6X^2}{Z^2}+\frac{16X^2}{\lambda Z^2}\times\left[(Z^2-X^2)\left(k^2-\frac{\nu^2}{X^2Z^2}\right)+\lambda\right]+\frac{X^4}{Z^4}\right\},$$

$$E_x=-\frac{16}{\lambda}\frac{X'X(1+X^2/Z^2)}{Z^2(1-X^2/Z^2)^3},\qquad E_z=\frac{16}{\lambda}\frac{X^2Z'(1+X^2/Z^2)}{Z^3(1-X^2/Z^2)^3},$$

$$B_x=\frac{4}{\lambda}\frac{XZ'(1+6X^2/Z^2+X^4/Z^4)}{Z^2(1-X^2/Z^2)^3}, \tag{2.4}$$

$$B_z=\frac{4}{\lambda}\frac{X'(1+6X^2/Z^2+X^4/Z^4)}{Z(1-X^2/Z^2)^3},$$

$$p_y=4\frac{X(1+X^2/Z^2)}{Z(1-X^2/Z^2)^2},\qquad \gamma=\frac{1+6X^2/Z^2+X^4/Z^4}{(1-X^2/Z^2)^2}.$$

In the general case $k^2, \mu^2, \nu^2 \neq 0$ the solutions (2.3) can be expressed in terms of elliptic functions. However, by choosing these parameters specially one can obtain values of X and Z expressed in terms of elementary functions. For a "repulsive" field with $k^2 = 0, \mu, \nu^2 \neq 0$ we have from (2.3)

$$X=\nu(\lambda-\mu^2)^{-1/2}\,\mathrm{sh}[x(\lambda-\mu^2)^{1/2}],\quad Z=(\nu/\mu)\,\mathrm{ch}(\mu z),\quad \mu^2<\lambda;$$
$$X=\nu(\mu^2-\lambda)^{-1/2}\sin[x(\mu^2-\lambda)^{1/2}],\quad Z=(\nu/\mu)\,\mathrm{ch}(\mu z),\quad \mu^2>\lambda; \tag{2.5}$$

for an "attractive" field

$$X=\nu(\lambda+\mu^2)^{-1/2}\sin[x(\lambda+\mu^2)^{1/2}],\quad Z=(\nu/\mu)\,\mathrm{ch}(\mu z), \tag{2.6}$$

where the functions in (2.5) and (2.6) are chosen such that

$$p_y(-x)=-p_y(x),\quad p_y(-z)=p_y(z).$$

One cannot realize solutions with negative λ in an infinite system, as the denominators in (2.4) vanish for $\lambda < 0$ on the surfaces $X^2(x) = Z^2(z)$. From (2.4) we have for the charge density

$$n(0,z)=1-16\mu^2/\lambda\,\mathrm{ch}^2(\mu z),$$

i.e., in "repulsive" fields an excess negative charge is concentrated in the near-axial region of the system, and for "attractive" fields $n(0,z) < 1$ and the equilibrium has the shape of an electron ring.

We consider in greater detail the solution with $\lambda > 0$. Expressions (2.4) and (2.6) describe an equilibrium configuration with a charge density distribution which has the shape of a soliton in the longitudinal direction and is periodic in x, and the integrated charge in each z-section and with $-\pi(\lambda+\mu^2)^{-1/2} \leqslant x \leqslant \pi(\lambda+\mu^2)^{-1/2}$ is:

$$Q=\int_{-\pi(\lambda+\mu^2)^{-1/2}}^{\pi(\lambda+\mu^2)^{-1/2}}(n-1)\,dx=0.$$

When $x = \pm\pi(\lambda+\mu^2)^{-1/2}$ we have

$$p_y=0,\quad \gamma=1,\quad E_x=E_z=B_x=0,$$

$$B_z=-4\mu/\lambda\,\mathrm{ch}(\mu z),\quad n=1-16\mu^2/\lambda\,\mathrm{ch}^2(\mu z)$$

and the solution satisfies the "matching" conditions with an infinite unperturbed plasma in an external shaped magnetic field $B_z(z)$. As $z \to \pm\infty$ the momentum and the electric and magnetic field components tend to zero and $\gamma \to 1$. In each of the ranges $-\pi(\lambda+\mu^2)^{-1/2} \leqslant x \leqslant 0$ and $0 \leqslant x \leqslant \pi(\lambda+\mu^2)^{-1/2}$ the momentum has a single sign and B_z changes from a maximum value

$$B_z(x=0)=4\mu/\lambda\,\mathrm{ch}(\mu z)$$

to a minimum one

$$B_z[x=\pm(\lambda+\mu^2)^{-1/2}\pi]=-4\mu/\lambda\,\mathrm{ch}(\mu z),$$

changing its sign at $x = \pm\pi(\lambda+\mu^2)^{-1/2}/2$. The closing of the magnetic field lines occurs then in the points $z = \pm\infty$. The charge density is a minimum in the points $z = 0$; $x = 0$, $\pm\pi(\lambda+\mu^2)^{-1/2}$. The condition that the density be non-negative in those points determines the limiting value of the ratio: $\mu^2/\lambda \leqslant 1/16$.

For a given ion density N_i the solution considered is characterized by two parameters: λ and μ. We express these parameters in terms of the maximum values of the magnetic field $B_z(x=0, z=0) = B_0$ and the Lorentz factor $\gamma[x = \pi(\lambda+\mu^2)^{-1/2}/2, z=0] = \gamma_0$. Considering the case $\mu^2/\lambda \leqslant 1/16$ (when $\mu^2/\lambda \geqslant 1/16$ the boundary of the configuration is distorted and the "matching" condition is violated) we have from (2.4)

$$\mu/\lambda=B_0/4,\quad \mu^2/\lambda=(\gamma_0-1)/8.$$

The characteristic longitudinal and transverse dimensions then equal

$$x_0=B_0[2(\gamma_0-1)]^{-1/2},\quad z_0=2B_0/(\gamma_0-1),$$

and the maximum value of $\gamma_0 \leqslant 1.5$.

As $\mu^2/\lambda \ll 1$, we get from (2.4) and (2.6) up to the first non-vanishing terms in μ^2/λ for the functions of the system

$$
\begin{aligned}
&p_y=[2(\gamma_0-1)]^{1/2}\sin(x/x_0)/\mathrm{ch}(z/z_0),\\
&\gamma=1+(\gamma_0-1)\sin^2(x/x_0)/\mathrm{ch}^2(z/z_0),\\
&E_x=-B_0[(\gamma_0-1)/2]^{1/2}\sin(2x/x_0)/\mathrm{ch}^2(z/z_0),\\
&E_z=B_0(\gamma_0-1)\sin^2(x/x_0)\,\mathrm{sh}(z/z_0)/2\,\mathrm{ch}^3(z/z_0),\\
&B_x=B_0[(\gamma_0-1)/8]^{1/2}\sin(x/x_0)\,\mathrm{sh}(z/z_0)/\mathrm{ch}^2(z/z_0),\\
&B_z=B_0\cos(x/x_0)/\mathrm{ch}(z/z_0),\\
&n=1-2(\gamma_0-1)\cos(2x/x_0)/\mathrm{ch}^2(z/z_0);
\end{aligned}
\tag{2.7}
$$

we have used here the relations $E_x+v_yB_z=0$, $E_z-v_yB_x=0$.

Evaluating the ratio B_x/B_z from (2.4) we determine the equation for the magnetic field lines:

$$\sin(x/x_0)=C_0\,\mathrm{ch}(z/z_0);$$

the constant C_0 characterizes the number of the line. We evaluate the current in the system:

$$J=\int v_y n\,d\Omega,$$

$\Omega(x,z)$ is the integration domain. Integrating over $-\infty<z<\infty$, $0\leqslant x\leqslant\pi x_0$ we get

$$J=4\pi B_0{}^2/3.$$

The average current density

$$j_0\sim J_0/x_0z_0=(\pi/3)[2/(\gamma_0-1)]^{1/2}$$

is now independent of the magnitude of B_0.

In conclusion we note that some of the properties described above are typical of the whole class of N-soliton bounded solutions of Eq. (2.1). The equilibrium configurations corresponding to such solutions have a solenoidal geometry $p_y(-x)=-p_y(x)$, $p_y(-z)=p_y(z)$ with sharp boundaries in the transverse direction and magnetic field lines which close up in infinitely removed points.

It is interesting to compare the results given here with the solutions of the linear field model realizable when $\sigma=-\sinh u/\lambda$. In that case we have from (1.11)

$$\sigma_{xx}+\sigma_{zz}=-\lambda\sigma. \tag{2.8}$$

Bounded solutions of Eq. (2.8) are possible only when $\lambda>0$:

$$\sigma=-(A_0/\lambda)\sin(\mu x)\cos[(\lambda-\mu^2)^{1/2}z], \tag{2.9}$$

A_0 and μ are arbitrary constant. The configuration described by (2.9) has the form of a doubly periodic standing space-charge wave. The characteristic dimensions and the constant A_0 are determined by the values of the magnetic field $B_z(x=0,z=0)=B_{z0}$, $B_x(x=\pi x_0/2, z=\pi z_0/2)=B_{x0}$ and the Lorentz factor $\gamma(x=\pi x_0/2, z=0)=\gamma_0$:

$$
\begin{aligned}
&A_0=(\gamma_0{}^2-1)^{1/2},\quad x_0=B_0{}^2/B_{z0}\gamma_0(\gamma_0{}^2-1)^{1/2},\\
&z_0=B_0{}^2/B_{x0}\gamma_0(\gamma_0{}^2-1)^{1/2},\\
&B_0{}^2=B_{x0}{}^2+B_{z0}{}^2.
\end{aligned}
$$

The expressions for p_y, γ^2, the charge density, and the components of the electric and magnetic fields then take the form

$$
\begin{aligned}
&p_y=(\gamma_0{}^2-1)^{1/2}\sin(x/x_0)\cos(z/z_0),\\
&\gamma^2=1+(\gamma_0{}^2-1)\sin^2(x/x_0)\cos^2(z/z_0),\\
&n=1+[(\gamma_0{}^2-1)/2B_0{}^2][B_{x0}{}^2\cos(2z/z_0)-B_{z0}{}^2\cos(2x/x_0)\\
&\qquad-B_0{}^2\cos(2x/x_0)\cos(2z/z_0)],\\
&E_x=-B_{z0}p_y\cos(x/x_0)\cos(z/z_0),\quad E_z=B_{x0}p_y\sin(x/x_0)\sin(z/z_0),\\
&B_x=B_{x0}\gamma\sin(x/x_0)\sin(z/z_0),\quad B_z=B_{z0}\gamma\cos(x/x_0)\cos(z/z_0).
\end{aligned}
\tag{2.10}
$$

If we consider this solution in the region $|x|\leqslant\pi x_0$ and $|z|\leqslant\pi z_0/2$, the corresponding configuration will have steep boundaries in the longitudinal and transverse directions. When $x=\pm\pi x_0$

$$
\begin{aligned}
&p_y=E_x=E_z=B_x=0,\quad B_z=-B_{z0}\cos(z/z_0),\\
&n=1-(B_{z0}{}^2/2B_0{}^2)(\gamma_0{}^2-1)[1+\cos(2z/z_0)];
\end{aligned}
$$

when $z=\pm\pi z_0/2$

$$
\begin{aligned}
&p_y=E_x=E_z=B_z=0,\quad B_x=B_{x0}\sin(x/x_0),\\
&n=1-(B_{x0}{}^2/2B_0{}^2)(\gamma_0{}^2-1)[1-\cos(2x/x_0)].
\end{aligned}
$$

When $x=\pm\pi x_0$, $z=\pm\pi z_0/2$ the conditions for "matching" with an unbounded unperturbed plasma and an external shaped magnetic field have in this case two components. Integrating over the region $0\leqslant x\leqslant\pi x_0$, $-\pi z_0/2\leqslant z\leqslant\pi z_0/2$ we get for the current in the system

$$J=\frac{4B_0{}^4}{B_{x0}B_{z0}\gamma_0{}^2(\gamma_0{}^2-1)}\int_1^{\gamma_0}\frac{\ln[t+(t^2-1)^{1/2}]}{t^2-1}dt.$$

The condition that the configuration can be realized is:

$$B_{x0}{}^2(\gamma_0{}^2-1)/B_0{}^2\leqslant1,\quad B_{z0}{}^2(\gamma_0{}^2-1)/B_0{}^2\leqslant1,$$

whence for $B_{x0}=B_{z0}$ the maximum value $\gamma_0^2\leqslant3$. We show in Fig. 1 the magnetic force-line configuration. The examples of "attractive" fields given above correspond to a solenoidal geometry of the system. In conclusion we consider a model of a field in which the "attractive" regions alternate with "repulsive" regions:

$$\mathrm{sh}\,u=\lambda\sin\sigma,\quad \sigma_{xx}+\sigma_{zz}=\lambda\sin\sigma. \tag{2.11}$$

Putting

$$\sigma=4\,\mathrm{arctg}[X(z)/Z(z)],$$

we get

$$X'^2=k^2X^4+\mu^2X^2-\nu^2,\quad Z'^2=k^2Z^4+(\lambda-\mu^2)Z^2-\nu^2; \tag{2.12}$$

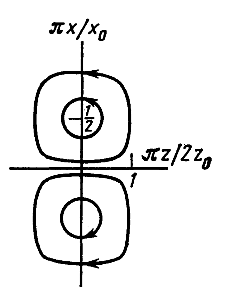

FIG. 1.

k^2, μ^2, ν^2, and λ are arbitrary parameters. The expression for the momentum, γ^2, the charge density, and the electric and magnetic field components have the form

$$
\begin{aligned}
&p_y=4\lambda XZ(Z^2-X^2)/(X^2+Z^2)^2,\\
&\gamma^2=1+16\lambda^2X^2Z^2(Z^2-X^2)^2/(X^2+Z^2)^4,\\
&n=1+16\lambda(X^2+Z^2)^{-4}[2\lambda X^2Z^2(X^4-4X^2Z^2+Z^4)\\
&\quad+(X^2+Z^2)(k^2X^2Z^2-\nu^2)(X^4-6X^2Z^2+Z^4)],\\
&E_x=4p_yZX'/(X^2+Z^2),\quad E_z=-4p_yXZ'/(X^2+Z^2),\\
&B_x=-4\gamma XZ'/(X^2+Z^2),\quad B_z=-4\gamma ZX'/(X^2+Z^2).
\end{aligned}
\tag{2.13}
$$

Monotonic solutions (2.12) leading to bounded distributions of the functions (2.13) are realized when $\lambda > 0$. When $k^2 = 0$ we have

$$X=(\nu/\mu)\operatorname{ch}(\mu x),\quad Z=\nu(\lambda-\mu^2)^{-1/2}\operatorname{ch}[z(\lambda-\mu^2)^{1/2}]. \tag{2.14}$$

The solution (2.14) corresponds to a quadrupole magnetic field configuration in the medium. We restrict the consideration to distributions of the functions which are symmetric in x and z, putting $\lambda = 2\mu^2$. In that case

$$p_y(x=\pm z)=E_x(x=\pm z)=E_z(x=\pm z)=0.$$

On the equipotential surfaces $\cosh \mu z = \alpha \cosh \mu x$ ($|\alpha|\geqslant 1$ is an arbitrary constant)

$$p_y=8\mu^2\alpha(\alpha^2-1)/(\alpha^2+1)^2,$$

$$n=1+\frac{128\mu^4}{(\alpha^2+1)^4}\left[\alpha^2(\alpha^4-4\alpha^2+1)-\frac{\alpha^2+1}{4\mu^4\tau^2}(\alpha^4-6\alpha^2+1)\right];$$

here the running coordinate $|\alpha|\leqslant\tau^2\leqslant\infty$, $\tau^2(x=0, z=0) = 1$, $\tau^2(x, z\to\infty)\to\infty$. On the surface $\alpha = \pm(2^{1/2}\pm 1)$ the momentum takes on extremal values: $p_{y\,\mathrm{extr}} = \pm 2\mu^2$, when, putting $\gamma^2_{\mathrm{extr}} = \gamma_0^2$, we get

$$4\mu^4=\gamma_0^2-1,\quad x_0=z_0=[4/(\gamma_0^2-1)]^{1/4}.$$

When $\gamma = \gamma_0$ the charge density is a maximum and is independent of τ^2:

$$n_{max}=1+4(\gamma_0^2-1);$$

when $\gamma = 1$

$$n=1-4(\gamma_0^2-1)[1-4/\tau^2(\gamma_0^2-1)]$$

and as $\tau^2\to\infty$

$$n_{min}=1-4(\gamma_0^2-1),$$

whence, putting $n_{\min}\geqslant 0$, we have $\gamma_0^2\leqslant 5/4$.

The expressions for the electric and magnetic field components on the γ = const surfaces have the form

$$E_x=2[4(\gamma_0^2-1)]^{1/4}\frac{\alpha p_y(\alpha)}{\alpha^2+1}\left(1-\frac{1}{\tau^2}\right)^{1/2},$$

$$E_z=-2[4(\gamma_0^2-1)]^{1/4}\frac{p_y(\alpha)}{\alpha^2+1}\left(\alpha^2-\frac{1}{\tau^2}\right)^{1/2},$$

$$B_x=-2[4(\gamma_0^2-1)]^{1/4}\frac{\gamma(\alpha)}{\alpha^2+1}\left(\alpha^2-\frac{1}{\tau^2}\right)^{1/2},$$

$$B_z=-2[4(\gamma_0^2-1)]^{1/4}\frac{\alpha\gamma(\alpha)}{\alpha^2+1}\left(1-\frac{1}{\tau^2}\right)^{1/2}.$$

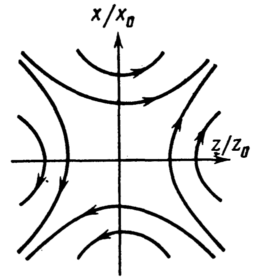

FIG. 2.

The magnetic field line configuration is shown in Fig. 2. As $\tau^2\to\infty$ the maximum value of the magnetic field on the surfaces $\gamma = 1$ is

$$B_0=(B_x^2+B_z^2)^{1/2}=2(\gamma_0^2-1)^{1/4}.$$

The expression for the current is similar to the one considered earlier. The magnitude of the current is restricted by the quantity B_0. In a force $E_x(\tau\to\infty)\neq 0$, $E_z(\tau\to\infty)\neq 0$ the given configuration cannot be realized in an unbounded system. The N-soliton solutions of Eq. (2.11) correspond to multipole configurations of the magnetic field.

3. Our analysis shows that the leeway in the choice of equilibrium configurations for the motion of a charged fluid in self-consistent crossed fields determined by the conditions that an electron flow can be formed, exists also in a longitudinally inhomogeneous system. Such configurations can be realized, for instance, through the injection of electrons from a "non-equipotential" cathode in a suitably shaped external magnetic field, or when striking a high-current discharge when the gas pressure is lowered in shaped crossed E- and B-fields. Under conditions of integral charge compensation there correspond formally to each form of interaction in the system two signs—"attraction" and "repulsion." This fact reflects the property of a solenoidal-type equilibrium configuration (the existence of a surface on which the longitudinal component of the magnetic field changes sign) and can be rigorously explained in the framework of a self-consistent kinetic description by studying the trajectories for the motion of individual particles. We note here that vanishing of B_z on some surface (an "attractive" field is then realized) is possible in an unbounded system only when there exist loops in the projections of the electron trajectories on the xy plane.

The equilibrium configurations considered above are realized for a comparatively low level of relativistic behavior of the electron flow, but there are no restrictions on the limiting current. For a solenoidal geometry the average current density in the ring is determined solely by the quantity $\gamma^2_{\max}$ and for given $\gamma^2_{\max}$ and N_i the dimensions of the ring are proportional to the magnitude of the current in the system.

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[1]V. P. Sarantsev, Usp. Fiz. Nauk **123**, 4 (1977) [*sic*!]
[2]A. A. Rukhadze, L. S. Bogdankevich, S. E. Rosinskiĭ, and V. G. Rukhlin, Fizika sil'notochnykh relyativistskikh elektronnykh puchkov (Physics

of high-current relativistic electron beams) Atomizdat, Moscow, 1980, p. 99.
[3]R. G. Lomax, J. Electron. Control **5**, 563 (1958).
[4]Yu. A. Khlestkov and Yu. B. Movsesyants, Phys.Lett. **64A**, 180 (1977).
[5]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, Teoriya solitonov (Theory of solitons) Nauka, Moscow, 1980, p. 74 [English translation published by Consultants Bureau].

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