Stochastic web and diffusion of particles in a magnetic field

G. M. Zaslavskiĭ, M. Yu. Zakharov, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov

Institute of Space Research, Academy of Sciences of the USSR, Moscow
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A mapping describing the dynamics of charged particles in the field of a wave packet in a transverse magnetic field is obtained and analyzed. It is shown that for rational ratios of the frequencies of the waves to the Larmor frequency the phase plane of the system is covered by a stochastic web, inside which the dynamics of the particles is chaotic. The structure of the stochastic web is fractal. For waves of small amplitude the thickness of the web is exponentially small. The existence of a stochastic web for an arbitrarily small amplitude of the wave packet leads to universal diffusion of the particles, analogous to Arnol'd diffusion in the multidimensional case. It is accompanied by stochastic heating of the particles, leading to a new mechanism of damping of the waves propagating across the magnetic field. The symmetry properties of the stochastic web are discussed, and typical bifurcations of the phase trajectories of the particle are considered.

1. INTRODUCTION

The problem and the results that will be described in this article have at least two aspects that it makes sense to discuss in detail immediately in the Introduction. The first of these is an applied aspect and is connected with the existence of universal diffusion of particles in a magnetic field and the field of a wave packet. The second is a formal aspect, consisting in the fact that the diffusion is stochastic and analogous to Arnol'd diffusion, although the system in which it occurs has one and one half degrees of freedom.

The resonance interaction of particles with a wave in a plasma placed in an external magnetic field has numerous applications. There is particular interest in this problem in the case when the wave propagates in a direction perpendicular to the magnetic field. In particular, the case of a strong magnetic field was considered in Refs. 1 and 2, and the case of a weak magnetic field was considered in Refs. 3 and 4. A review of many results in this field of investigation is contained in Ref. 5. One of the results, which turns out to be important for us in the following, is connected with the fact that the dynamics of a particle can become stochastic. This leads to stochastic heating of the particle and to the appearance of a nonlinear mechanism of damping of the waves.

Another side of this problem turns out to be no less important. It is known that chaos in Hamiltonian dynamical systems can arise even in the case of one and one half degrees of freedom, i.e., in the case when a system with one degree of freedom is under the action of a time-dependent external force. In this area, several systems with equations of motion that can be called standard, since they are typical, have been studied (see, e.g., Ref. 6). The interaction of a particle with a plane wave in a transverse magnetic field leads to one of these standard equations, which has still not been sufficiently studied:

$$\ddot{x}+\omega_H^2 x=-\frac{e}{m}E_0\sin(kx-\omega t), \qquad (1.1)$$

where x is the direction of propagation of the wave, E_0 is the amplitude of the wave, $\omega_H = eB_0/mc$ is the cyclotron frequency, and B_0 is the magnetic field, oriented along z. Equation (1.1) is supplemented by the law of conservation of the generalized momentum along y:

$$p_y+\omega_H m x=\text{const}, \quad p_y=m\dot{y}. \qquad (1.2)$$

Because of this conservation law, the problem of the motion of the particle is reduced to the single equation (1.1), and the phase plane $(x,\dot{x})$ is equivalent, to within constants, to the (p_x,p_y) plane. One of the important features of Eq. (1.1) is its degeneracy in the absence of the perturbation, when it becomes linear. Because of this the Kolmogorov-Arnol'd-Moser theory is not directly applicable to it.

In fact, to consider only one harmonic in the right-hand side of (1.1) is an approximation, since usually in a plasma not one wave but a wave packet is excited. In this case, in place of Eq. (1.1) we have

$$\ddot{x}+\omega_H^2 x=-\frac{e}{m}\sum_k E_k\sin(kx-\omega_k t)\equiv\frac{e}{m}E(x,t), \qquad (1.3)$$

with the relation (1.2) holding as before. We shall assume that the wave packet in (1.3) is sufficiently broad and uniform. An investigation of this case forms the content of this paper.

We shall show that for rational ratios between the frequencies ω_k and ω_H the phase plane $(x,\dot{x})$ of the system (1.3) is covered by a mesh of finite thickness inside which the dynamics of the particle is stochastic and outside which, i.e., in its cells, the dynamics is regular. This mesh, called a stochastic web below, exists for arbitrarily small fields E_k. It has a definite symmetry, determined by the ratios of ω_k to ω_H. For small values of E_k the thickness of the web is exponentially small. However, the fact that it covers the entire phase plane implies that particles can diffuse arbitrarily far into the region of high energies. This phenomenon is analogous to Arnol'd diffusion. The difference is connected with the origin of the separatrix mesh on which the stochastic web is formed.

Arnol'd diffusion[7] occurs when there are more than two

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degrees of freedom. The separatrix mesh is formed as a consequence of the intersection of resonance tori, which, for purely topological reasons, do not divide the phase space in this case. In Eq. (1.3) the infinite separatrix mesh is due to the form of the perturbation and exists even when there is only one harmonic in the right-hand side, as in (1.1).

A stochastic web, except in certain special cases that admit covering of the plane by regular figures, has weakly pronounced local chaos, analogous to the structures of liquids or amorphous solids.

In the paper we describe these structures, consider different bifurcations of the trajectories upon increase of the amplitudes E_k of the waves, and give estimates of the diffusion of the particles and of the rate of their stochastic heating.

One of the important physical consequences of the formation of a stochastic web is the following. For arbitrarily small fields E_k some of the particles experience stochastic heating along the channels of the stochastic web. In the self-consistent problem the increase in the energy of the particles occurs on account of loss of this energy by the waves. Thus, we arrive at the existence of a universal mechanism of damping of waves in a magnetic field—a mechanism due to pumping of energy from waves to particles accelerating in the channels of the stochastic web.

2. DERIVATION OF THE MAPPING WITH "TWISTING"

The starting equations of motion of the particle have the form

$$\ddot{\mathbf{r}}=\frac{e}{m}\mathbf{E}(x,t)+\frac{e}{mc}[\dot{\mathbf{r}}\mathbf{B}_0], \tag{2.1}$$

where $\mathbf{B}_0$ points along z and $\mathbf{E}$ along x. From this, after elimination of the y component, Eqs. (1.2) and (1.3) follow. For the wave packet we shall adopt the approximation of uniformity and sufficiently large width:

$$k=k_0,\quad \omega_k=\omega_0+n\Delta\omega,\quad E_k=\text{const}=E_0,$$

where n is an integer running from $-\infty$ to $+\infty$. In these conditions (for more detail, see Ref. 8)

$$E(x,t)=-E_0\sum_{n=-\infty}^{+\infty}\sin(k_0x-\omega_0t-n\Delta\omega t)$$

$$=-E_0T\sin\theta\sum_{n=-\infty}^{+\infty}\delta(t-nT), \tag{2.2}$$

where the characteristic time interval

$$T=2\pi/\Delta\omega \tag{2.3}$$

is determined by the frequency interval between the harmonics of the packet and we have introduced the phase of the central mode of the packet:

$$\theta=k_0x-\omega_0t. \tag{2.4}$$

The equation of motion (2.1) or (1.3) can be reduced to the following:

$$\ddot{x}+\omega_H{}^2x=-\frac{e}{m}TE_0\sin\theta\sum_{n=-\infty}^{+\infty}\delta(t-nT), \tag{2.5}$$

where, without loss of generality, the constant in (1.2) has been set equal to zero, i.e.,

$$v_y=p_y/m=-\omega_Hx. \tag{2.6}$$

In place of the differential equation (2.5) we shall write a finite-difference equation. Between two successive actions of the δ-functions the trajectory of the particle satisfies the equation $\ddot{x}+\omega_H^2x=0$. Its solution at passage through the δ-function at the time $t_n=nT$ should satisfy the boundary conditions

$$x(t_n+0)=x(t_n-0),\quad \dot{x}(t_n+0)=\dot{x}(t_n-0)-\frac{e}{m}TE_0\sin\theta(t_n).$$

With the help of these, from (2.5) and (2.6) we obtain

$$v_{x,(n+1)}=v_{y,(n)}\sin\omega_HT+\left[v_{x,(n)}+\frac{e}{m}E_0T\sin\left(\omega_0nT+\frac{k_0}{\omega_H}v_{y,(n)}\right)\right]\cos\omega_HT,$$
$$v_{y,(n+1)}=v_{y,(n)}\cos\omega_HT-\left[v_{x,(n)}+\frac{e}{m}E_0T\sin\left(\omega_0nT+\frac{k_0}{\omega_H}v_{y,(n)}\right)\right]\sin\omega_HT, \tag{2.7}$$

where we have denoted

$$v_{x,(n)}=v_x(t=nT-0),\quad v_{y,(n)}=v_y(t=nT-0),$$

i.e., the index n corresponds to the time immediately preceding the action of the δ-function at $t=nT$.

The mapping (2.7) preserves a measure, and, according to (2.6), as $\omega_H\to0$ goes over into the standard mapping $\hat{T}$:

$$v_{x,(n+1)}=v_{x,(n)}+\frac{e}{m}E_0T\sin(k_0x_{(n)}),$$
$$x_{(n+1)}=x_{(n)}+Tv_{x,(n+1)}-\frac{\omega_0}{k_0}T, \tag{2.8}$$

the connection of which with the kinetic description of the dynamics of a particle has been sufficiently well studied (see, e.g., Refs. 6 and 8). In particular, Eqs. (2.8) correspond to the equation of motion (1.3) with $\omega_H=0$, and the condition for stochasticity arises in this case when

$$K=eE_0k_0T^2/m=\Omega_0{}^2T^2\gtrsim1, \tag{2.9}$$

where we have introduced the frequency of vibrations of particles captured by the central wave:

$$\Omega_0=(eE_0k_0/m)^{1/2}. \tag{2.10}$$

The condition (2.9) shows that stochastic dynamics in the phase space (x,v_x) begins at field amplitudes E_0 greater than a certain critical value.

The presence of the term ω_0nT in the argument of the sine in Eqs. (2.7) has a simple physical meaning. It is sufficient to make the replacement

$$\tilde{v}_y=\omega_0nT+k_0v_y/\omega_H,$$

to see clearly that acceleration of the particles arises. The change of velocity of the particles is proportional to

$$\Delta v \sim \omega_H \omega_0 t / k_0 = \omega_H U_0 t,$$

where $U_0 = \omega_0 / k_0$ is the phase velocity of the central harmonic of the wave packet. This acceleration is due to the fact that the moving wave "shoves" the particle regularly. It is described in detail in Ref. 9, and will not be considered here. Therefore, below we set $\omega_0 = 0$.

We introduce the more convenient dimensionless variables

$$k_0 v_x / \omega_H = u, \quad k_0 v_y / \omega_H = v. \tag{2.11}$$

Then the mapping (2.7) with $\omega_0 = 0$ takes the form $\hat{M}_\alpha$:

$$u_{n+1} = (u_n + K_H \sin v_n) \cos \alpha + v_n \sin \alpha,$$
$$v_{n+1} = -(u_n + K_H \sin v_n) \sin \alpha + v_n \cos \alpha, \tag{2.12}$$

where we have denoted

$$\alpha = \omega_H T, \quad K_H = K / \omega_H T = K / \alpha. \tag{2.13}$$

The mapping (2.12) will be called in $\hat{M}_\alpha$ mapping, or mapping with twisting, where α is the angle of twist.[1)]

By resonance twisting we shall mean cases in which α is rational, i.e.,

$$\alpha_{p,q} = 2\pi p / q, \tag{2.14}$$

where p and q are integers and $p < q$. The case $p = 1$ will be denoted as $\alpha_q = 2\pi / q$. The simplest example of a mapping with twisting is realized for $q = 4$ ($\alpha_4 = \pi/2$) and has the form $\hat{M}_4$:

$$u_{n+1} = v_n, \quad v_{n+1} = -(u_n + K_H \sin v_n). \tag{2.15}$$

The physical meaning of resonance twisting can be understood from Fig. 1. On the phase plane (u,v) the straight lines correspond to the front of waves moving with velocity $U_k = \omega_k / k_0$. At the points of intersection of the wave front with the circular trajectory of the particle in the magnetic field there occurs an intense interaction of the particle with the wave. Passing through one wave, the particle falls into resonance with the next wave, and so on. The frequency shift from wave to wave is the same ($\Delta\omega = 2\pi/T$). Therefore, rational values of α correspond to phased collisions of the particle with the waves. This leads to special properties of the phase portrait of the particle.

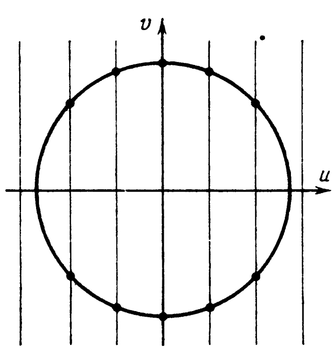

FIG. 1. Regions of interaction of the particle with the waves of the packet.

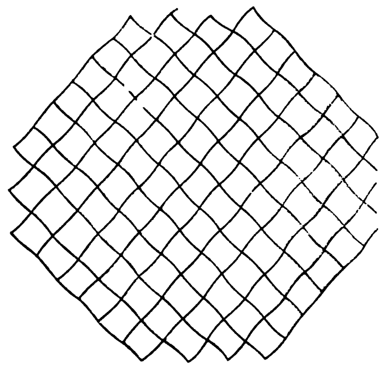

FIG. 2. Phase plane for $\alpha = \alpha_4, K_H = 0.9$.

3. THE PHASE PLANE FOR RESONANCE TWISTINGS

In order to understand how the phase plane is constructed in the case of resonance twistings, we turn first to the results of the numerical analysis, which are given in Figs. 2 and 3 for small values of K_H and $\alpha = \alpha_4$ and $\alpha = \alpha_3$. On the phase plane there is a regular separatrix mesh. Inside the central island are closed phase trajectories. All the other cells of the mesh are filled by closed curves, four cells at a time in Fig. 2, and three cells at a time in Fig. 3. The filling of the cells has the corresponding rotational symmetry. Thus, the entire phase plane is tiled by a regular "parquet" due to the symmetry of the twisting through $\alpha_4 = 2\pi/4$ and $\alpha_3 = 2\pi/3$.

The separatrix mesh has a finite width and, in reality,

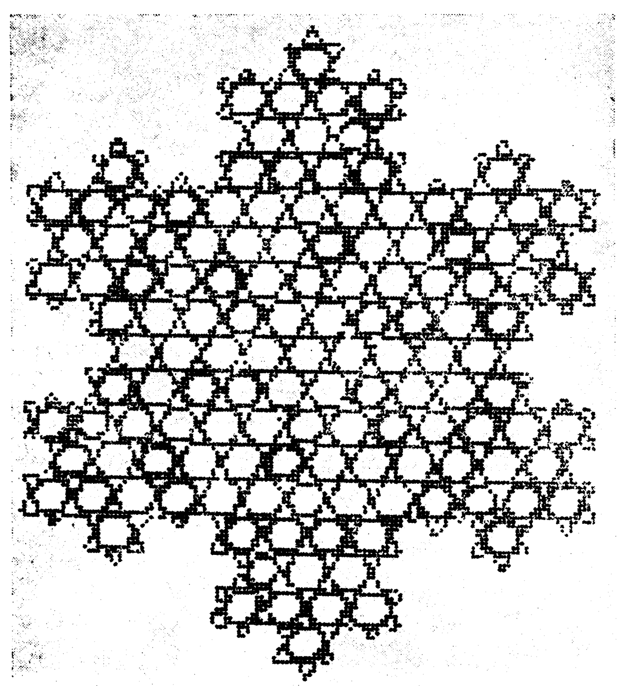

FIG. 3. Phase plane for $\alpha = \alpha_3, K_H = 0.4$.

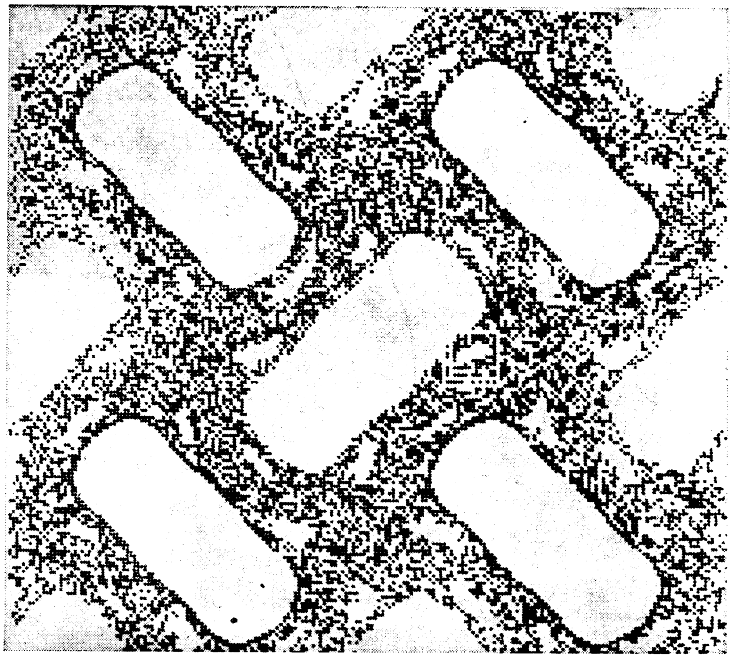

FIG. 4. Part of the phase plane for $\alpha = \alpha_4, K_H = 2$.

consists in thin stochastic layers, inside which the particle executes random walks. We shall call this mesh a stochastic web, and shall discuss it in more detail below.

With increase of K_H successive bifurcations occur, and we shall also discuss these a little later. Simultaneously, the stochastic web expands, forming wide channels of random walks of the particle (see Fig. 4). Inside the region of stochastic dynamics are islands corresponding to resonances of higher order in the interaction of the particle with the wave.

As an example of the calculation of the trajectories for resonance twisting and small values of K_H we shall consider the case $\alpha = \alpha_4 = \pi/2$. The mapping (2.12) reduces to (2.15). We shall eliminate the twisting from it. For this we construct the mapping $\hat{M}_4^4$, i.e., we integrate (2.15) four times and take into account only the leading terms in $K_H \ll 1$. As a result we obtain $\hat{M}_4^4$:

$$\bar{u} = u + 2K_H \sin \bar{v}, \quad \bar{v} = v - 2K_H \sin u. \tag{3.1}$$

The mapping (3.1) conserves the measure. It is possible to write a Hamiltonian system for which the points of the trajectory are connected by the relations (3.1).

We note first that the time interval between two successive steps of the mapping $\hat{M}_4^4$ is equal to $4T$. We set

$$H_4 = -\frac{K_H}{2T}\cos v - 2K_H \cos u \sum_{n=-\infty}^{+\infty} \delta(t-4nT). \tag{3.2}$$

The equation of motion for the Hamiltonian H_4 has the form

$$\dot{u} = \frac{\partial H_4}{\partial v} = \frac{K_H}{2T}\sin v,$$
$$\dot{v} = -\frac{\partial H_4}{\partial u} = -2K_H \sin u \sum_{n=-\infty}^{+\infty} \delta(t-4nT). \tag{3.3}$$

We consider the time interval

$$4nT - 0 \leqslant t \leqslant 4(n+1)T - 0,$$

and let (u,v) and $(\bar{u},\bar{v})$ be the values of the variables at the ends of the interval. Then, integrating (3.3), we arrive exactly at the mapping (3.1). The dynamical system (3.3) with the Hamiltonian (3.2) will be said to be equivalent to the mapping $\hat{M}_4^4$.

The different resonances in the system (3.3) are easily obtained if we represent (3.2) in the form of the expansion

$$H_4 = -\Omega_4(\cos v + \cos u) - \Omega_4 \cos u \sum_{n=-\infty,\ n\neq 0}^{+\infty} \cos(n\Delta\omega t/4), \tag{3.4}$$

where

$$\Omega_4 = K_H/2T \tag{3.5}$$

is the frequency of the nonlinear resonance. The period $2\pi/\Omega_4$ is the time required for complete passage around a small closed orbit in any cell of the parquet in Fig. 2 except the central cell. The perturbation terms contained in the sum in (3.4) have the same amplitude, but are rapidly oscillating, since $\Omega_4 \ll \Delta\omega$ for small values of K_H. Therefore, the averaged Hamiltonian has the simple form

$$\bar{H}_4 = -\Omega_4(\cos v + \cos u). \tag{3.6}$$

We shall consider trajectories of the averaged motion that describe the motion inside one cell of the parquet with symmetry α_4. The Hamiltonian (3.6) leads to the equations of motion

$$\dot{u} = \Omega_4 \sin v, \quad \dot{v} = -\Omega_4 \sin u$$

or

$$\ddot{u} + \Omega_4^2 (C \sin u - {}^1/_2 \sin 2u) = 0, \tag{3.7}$$

where the dimensionless integral is equal to

$$C=\cos v+\cos u. \qquad (3.8)$$

The values

$$|C|=2, \quad v=\pi n, \quad u=\pi m, \quad m+n=2l \quad (l=0, \pm 1, \ldots) \quad (3.9)$$

correspond to stable positions of equilibrium (elliptic points). The values

$$C=0, \quad v=\pi n, \quad u=\pi m, \quad m+n=2l+1 \qquad (3.10)$$

correspond to the unstable (hyperbolic) points through which the separatrices pass.

The trajectory of the averaged motion is easily found from Eq. (3.7), and for $|C|<2$ has the form

$$\cos v = -\frac{C}{2}+\left(1-\frac{C}{2}\right)\mathrm{cd}\left[\left(1+\frac{C}{2}\right)\frac{K_H}{2T}t, \varkappa\right],$$

$$\cos u = \frac{C}{2}-\left(1-\frac{C}{2}\right)\mathrm{cd}\left[\left(1+\frac{C}{2}\right)\frac{K_H}{2T}t, \varkappa\right],$$

where cd = cn/dn is the ratio of elliptic functions with modulus

$$\varkappa=(2-C)/(2+C).$$

4. THE STOCHASTIC WEB

The results of the preceding section are approximate, since they describe only the averaged motion. In reality, the perturbation in the Hamiltonian (3.4) of the equivalent dynamical system destroys the separatrix and gives rise to a stochastic layer in its place. We shall consider the formation of this layer. For this we turn first to the approximate mapping $\hat{M}_4^4$, to which corresponds the equivalent Hamiltonian (3.6). The equations of the separatrices should be obtained from the condition $C=0$, i.e., according to (3.8),

$$\cos u+\cos v=0. \qquad (4.1)$$

The solutions of Eq. (4.1) have the form of straight lines:

$$v=\pm(u+\pi)+2\pi n. \qquad (4.2)$$

These are two families of straight lines on the plane, forming a square mesh. It is obvious that allowance for the following, discarded terms, proportional to $K_H \sin u$ or $K_H \sin v$, leads to the appearance of a periodic modulation of the mesh obtained (see Fig. 2).

A qualitative explanation of the appearance of an infinite separatrix mesh consists in the following. We indicate on the phase plane, e.g., two nonoverlapping separatrices, due to motion near each of two plane waves of the wave packet (Fig. 5). Then the rotation of the particle in the magnetic field joins these separatrices, which was impossible before. An important feature of these two separatrices is the fact that they move relative to each other. If one of them corresponds to the plane wave

$$E_0 \sin(k_0 x-n\Delta\omega t),$$

where n is some integer, then in the other, neighboring wave we have $n_1=n\pm 1$. Consequently, the relative velocity of the displacement of the separatrices in Fig. 5 is equal to

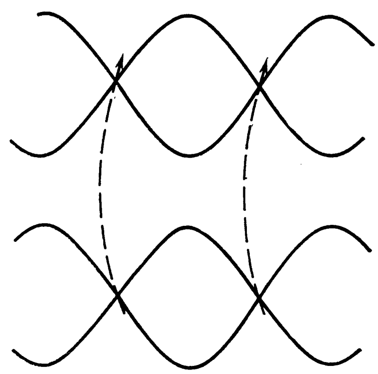

FIG. 5. The formation of an infinite separatrix mesh is due to the rotation of the particle in the magnetic field (dashed lines).

$\Delta v_{ph}=\Delta\omega/k_0$. If the Larmor rotation shifts the particle, e.g., from a hyperbolic point on one separatrix exactly to a hyperbolic point on the other separatrix, a distinctive resonance arises. It consists in the fact that the entire phase plane is covered by a separatrix mesh. The resonance condition, obviously, has the form

$$\omega_H=(p/q)\Delta\omega, \qquad (4.3)$$

which coincides with (2.14) and elucidates the meaning of resonance twisting (p and q are integers, and $p<q$).[2)]

The separatrix mesh that was formed is destroyed, and in its neighborhood a thin stochastic layer is formed. We shall estimate the thickness of this layer for the above-considered case of the α_4 resonance. For this, in Eq. (3.4) we shall keep two terms of the perturbation (those with $n=\pm 1$). We have

$$H_4\approx-\Omega_4(\cos v+\cos u)-2\Omega_4 \cos u \cos(\Delta\omega t/4).$$

According to a known estimate,[10] for the destruction of a separatrix by nonresonance terms we have

$$\delta H/\max H_4\sim\delta H/\Omega_4\sim\exp(-\pi\Delta\omega/8\Omega_4)=\exp(-\pi^2/2K_H), \quad (4.4)$$

where we have used the expression (3.5) for Ω_4.

Thus, the entire phase plane is covered by an almost square mesh of finite thickness, inside which the particle executes a random walk. A system of stochastic paths of the form described will be called a stochastic web.

Appendix 1. For any resonance twisting, i.e., for a rational $\alpha_{p,q}$, the phase plane is covered by a stochastic web for arbitrarily small perturbations K_H. The thickness of the web decreases exponentially with decrease of K_H.

The appearance of a stochastic web and its consequences for the dynamics of particles are practically entirely analogous to Arnol'd diffusion, which arises in the multidimensional case (with more than two degrees of freedom) and leads to unbounded diffusional drift of the particle in the phase space. The difference is that now the stochastic web was formed for Eq. (2.5), which describes the dynamics of a system with one and one half degrees of freedom. Another important difference between the stochastic web in the present case and that which arises in Arnol'd diffusion is connected with the symmetry properties of the phase plane. We shall discuss this in the next section.

With increase of K_H the thickness of the stochastic web increases, and for $K_H > 1$ the width of the channels of stochastic dynamics becomes comparable to the size of the cells. An example of such a structure can be seen in Fig. 4.

A stochastic web was observed numerically for different values of q, including $q = 191$. With increase of q the structure of the mesh becomes strongly deformed. Its cells decrease in size, and the size of the central island increases. The latter can be understood from the following considerations. For a given q the first necklace of cells of the stochastic web should correspond to a regular polygon with q sides. The length of a side of the polygon is fixed and equal to $\delta v = \Delta\omega/k_0$. Therefore, the radius R_q of the polygon is of the order $R_q \sim q\Delta v = q\Delta\omega/k_0$.

The formation of parquets from closed orbits on the phase plane was also observed in Ref. 2 in an analysis of the motion of a particle in a magnetic field and in the field of only one wave (see Eq. (1.1)). This parquet was of the square type, which is a specific feature of resonances in this case. In our problem, with the wave packet (2.2), the symmetry turns out to be richer, and we now turn to a consideration of it.

5. SYMMETRY OF THE PHASE PLANE AND STOCHASTIC DESTRUCTION OF THE SHORT-RANGE

We shall consider the initial mapping $\hat{M}_\alpha$ defined by formula (2.12) and specifying the trajectory of the particle. We represent this mapping in the form

$$\hat{M}_\alpha a = R_\alpha (1 + K_H \hat{S}) a, \qquad (5.1)$$

where

$$a = \begin{pmatrix} u \\ v \end{pmatrix}, \quad R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad \hat{S}a = \begin{pmatrix} \sin v \\ 0 \end{pmatrix}; \qquad (5.2)$$

here a is the state vector of the particle, R_α is the matrix of rotation through angle α, and $\hat{S}$ is a nonlinear operator. For a rotational twist angle $\alpha_{p,q}$ we have

$$R_\alpha^q = 1. \qquad (5.3)$$

The fixed points of the mapping $\hat{M}_\alpha^q$ are determined from the condition

$$\hat{M}_\alpha^q a = a. \qquad (5.4)$$

For small values of K_H it is possible in Eq. (5.4) to confine ourselves to the terms of first order in K_H. This leads to the following simplification of Eq. (5.1):

$$\hat{M}_\alpha^q a = a + K_H[\hat{S}a + R_\alpha^{q-1}\hat{S}(R_\alpha a) + S_\alpha^{q-2}\hat{S}(R_\alpha^2 a) + \ldots + R_\alpha \hat{S}(R_\alpha^{q-1}a)], \qquad (5.5)$$

where we have used the rationality condition (5.3). By virtue of (5.4) for the fixed points, it follows from (5.5) that

$$\hat{S}a + R_\alpha^{q-1}\hat{S}(R_\alpha a) + \ldots + R_\alpha \hat{S}(R_\alpha^{q-1}a) = 0. \qquad (5.6)$$

The main question concerning Eq. (5.6) is: What separatrix-mesh structure does it determine? Using the relation (5.3), we note that if a_0 is a solution of Eq. (5.6), then the points

$$R_\alpha a_0, \quad R_\alpha^2 a_0, \ldots, \quad R_\alpha^{q-1} a_0, \qquad (5.7)$$

i.e., points obtained from a_0 by rotations through angles $m\alpha_q$ ($m = 1,\ldots,q-1$), are also solutions. Therefore, to within terms $O(K_H^2)$, the separatrix mesh should possess rotational symmetry with angle of rotation α_q. This symmetry is approximate and should be broken by the discarded terms.

Here, however, the following question arises. It is known that the plane can be covered by a regular parquet consisting of figures of the same type: either triangles, squares, or hexagons. This corresponds to the values

$$\alpha_q = 2\pi/3, \quad 2\pi/4, \quad 2\pi/6. \qquad (5.8)$$

Since the introduction of values $p < q$ does not change the symmetry, the first and last cases produce the same separatrix mesh. Thus, the cases $\alpha_q = 2\pi/4$ and $\alpha_q = 2\pi/6$ account for all "simple" symmetries. The symmetry for all other values of α_q can be ensured only by facing the plane with figures of different shapes. This is clearly seen from Figs. 6 and 7, in which a system of islands and a separatrix mesh are depicted for $\alpha = \alpha_5 = 2\pi/5$. As can be seen from the figures, the shapes of the islands have a weakly pronounced structural spread. We shall discuss this in more detail.

For this we rewrite Eq. (1.3), substituting into it the form of the wave packet (2.2) with $\omega_0 = 0$ and replacing the time by $\tau = \omega_H t$:

$$\frac{d^2x}{d\tau^2} + x = -\frac{e}{m} E_0 \sin k_0 x \sum_{n=-\infty}^{+\infty} \cos\frac{2\pi n\tau}{\alpha}. \qquad (5.9)$$

From this it can be seen that the equation is invariant under a shift in the time variable by the amount

$$\Delta t = \Delta\tau/\omega_H = \alpha/\omega_H.$$

Since on the phase plane the particle executes one rotation in

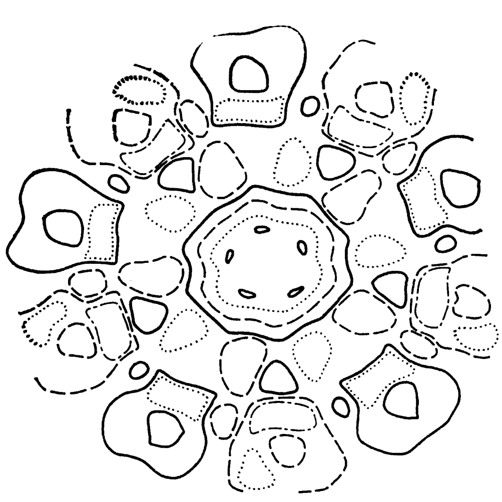

FIG. 6. Phase plane for $\alpha = \alpha_5, K_H = 0.5$.

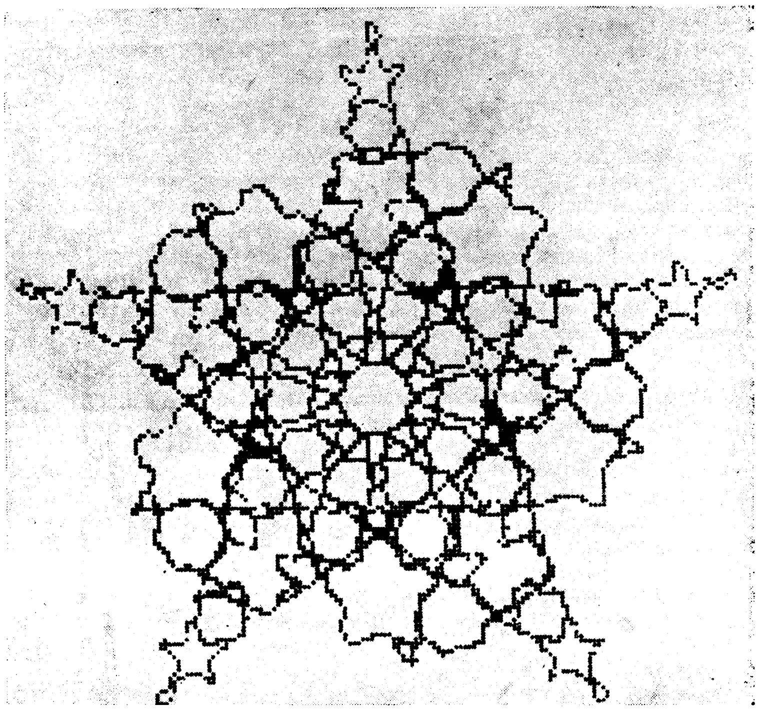

FIG. 7. Separatrix mesh for $\alpha = \alpha_5, K_H = 0.7$.

the time $2\pi/\omega_H$, this means that for rational values of $\alpha_q < 1$ the rotational-symmetry property for the particle trajectory and for the separatrix mesh should be global. This means, in particular, that the rotational symmetry of the separatrix mesh for rational values of α_q can be made arbitrarily accurate by decrease of K_H. A new question now arises: How can one cover the phase plane by a mesh with almost exact rotational symmetry for arbitrary values of α_q? All the cases studied numerically have possessed this symmetry. The answer can be formulated as follows.

Appendix 2. The global rotational symmetry of the system specified by the mapping $\hat{M}_\alpha$ for rational $\alpha_{p,q}$ ($\alpha_{p,q} = 2\pi p/q, p < q$) determines the long-range order on the phase plane. However, the island shapes corresponding to one and the same invariant (e.g., the area of the islands) but to different positions on the phase plane have a small random scatter, and so weakly chaotic short-range order exists in the system. It is precisely because of this scatter that it is possible to cover a plane with rotational symmetry by arbitrary q-gons. The mapping $\hat{M}_q$ (with integer q) can be regarded as the generator of such a covering.

Increase of the parameter K_H leads to loss of the short-range order. On the phase plane there appears an "amorphous" structure of small islands that are so deformed that it becomes meaningless to speak of q-sided polygons. The islands are separated by large regions of stochasticity, through which diffusion of the particles occurs. However, the symmetry of the twisting is preserved and is manifested in the character of the particle diffusion.

The transition to the destruction of the short-range order occurs through a sequence of bifurcations, some of which are mentioned below.

6. BIFURCATIONS

The mapping $\hat{M}_\alpha$ always has the fixed point $u = v = 0$. We shall consider the tangent matrix to the mapping at the point (0,0):

$$\hat{M}_\alpha' = \begin{bmatrix} \cos\alpha & K_H \cos\alpha + \sin\alpha \\ -\sin\alpha & \cos\alpha - K_H \sin\alpha \end{bmatrix}. \qquad (6.1)$$

Its eigenvalues λ satisfy the equation

$$\lambda^2 - \lambda \,\mathrm{Sp}\hat{M}_\alpha' + 1 = 0.$$

The point (0,0) becomes unstable for $|\mathrm{Sp}\hat{M}_\alpha'| > 2$ or for

$$K_H > 2\,\mathrm{ctg}(\alpha/2). \qquad (6.2)$$

In particular, for $\alpha_4 = 2\pi/4$ the instability condition has the simple form $K_H > 2$.

The instability consists in the fact that the elliptic point (0,0) is transformed into a hyperbolic point. At the same time, two new elliptic points are created. This is the usual island-doubling bifurcation (Fig. 8). Inside the island one can see the new stochastic layer that is formed on the separatrix passing through the saddle point (0,0).

We shall consider the mapping $\hat{M}_4^2$:

$$\bar{\bar{u}} = -u - K_H \sin v, \quad \bar{\bar{v}} = -v + K_H \sin(u + K_H \sin v). \qquad (6.3)$$

Its fixed points correspond to a cycle with period $2T$ and

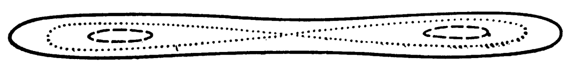

FIG. 8. Island-doubling bifurcation.

satisfy the equations

$$u_0=(K_H/2)\sin(K_H u_0/2),\quad v_0=(K_H/2)\sin(K_H v_0/2). \qquad (6.4)$$

Investigation of these points for stability leads to the following condition: $|\mathrm{Tr}(M_4^2)'|<2$, which corresponds to the inequality

$$0<(K_H^2/4)\cos\left(\frac{K_H}{2}\sin v_0\right)\cos v_0<1. \qquad (6.5)$$

Here K_H and v_0 are connected by the relation (6.4). It follows from (6.5) that the cycle of period $2T$ loses stability at the value of K_H satisfying the equation $(K_H/2)\sin(K_H/2)=\pi/2$, i.e., at $K_H=4.88665\ldots$. This bifurcation corresponds to the creation of a cycle of period $4T$. We note also that under the condition

$$\frac{K_H^2}{4}\cos\left(\frac{K_H}{2}\sin v_0\right)\cos v_0=1$$

and intermediate bifurcation occurs in the system at $K_H\cong 4.54$. At the same time the two elliptic points with period $2T$ lose their stability. However, this bifurcation leads not to the appearance of a cycle with period $4T$ but to the appearance of four elliptic points with the former period $2T$.

We shall construct the mapping $\hat{M}_4^4$:

$$\begin{aligned}\bar{u}&=u+K_H\sin v+K_H\sin[v-K_H\sin(u+K_H\sin v)],\\ \bar{v}&=v-K_H\sin\bar{u}-K_H\sin(u+K_H\sin v).\end{aligned} \qquad (6.6)$$

For $K_H\ll 1$ this mapping goes over into (3.1). The coordinates of the fixed points of period $4T$ are given by the solutions of the following equations:

$$\begin{aligned}v&=(-1)^m\frac{K_H}{2}\sin\left(\frac{K_H}{2}\sin v\right)+\pi n,\\ u&=(-1)^n\frac{K_H}{2}\sin\left(\frac{K_H}{2}\sin u\right)+\pi m,\end{aligned} \qquad (6.7)$$

where n and m are integers. It follows from (6.7) that the phase plane is covered in checkerboard fashion by alternating elliptic and hyperbolic points, and, as shown in Sec. 3, for $K_H\ll 1$ all the hyperbolic points $(m+n=2l+1,\ l=0,\pm 1,\ldots)$ belong to the separatrix mesh forming the square parquet on the phase plane. A stability investigation of the elliptic points with period $4T$ shows that for $m+n=2l\,(l\neq 0)$ they lose their stability simultaneously at $K_H=2$ and cycles with period $8T$ appear. In each of the elements of the parquet the bifurcations occur analogously to the bifurcations in the central element. The loss of stability of the cycle with period $4T$ and the formation of the cycle with period $8T$ in the central element occur at $K_H=4.92934\ldots$.

The numerical analysis shows that with increase of K_H successive period-doubling bifurcations occur. At $K_H=K_H^{(n)}$ the elliptic points with period 2^nT lose their stability and a cycle with period $2^{n+1}T$ appears. The sequence of bifurcation values $K_H^{(n)}$ converges rapidly to the limit $K_H^{(\infty)}$:

$$2=K_H^{(1)}<K_H^{(2)}<\ldots<K_H^{(\infty)}=4.93488\ldots \qquad (6.8)$$

We shall determine the ratio

$$\delta_n=(K_H^{(n-2)}-K_H^{(n-1)})/(K_H^{(n-1)}-K_H^{(n)}),\qquad n=3,4,\ldots. \qquad (6.9)$$

It follows from the numerical analysis that for sufficiently large values of n the sequence of $K_H^{(n)}$ converges as a geometric progression. This means, in particular, that

$$\lim_{n\to\infty}\delta_n=\delta=\text{const}.$$

It was found that $\delta\cong 8.72$. It should be noted that in a number of papers[11] devoted to the numerical study of doubling bifurcations in mappings that preserve the phase volume a constant $\delta=8.72109720\ldots$ has been found. Our numerical results are evidence that this constant is universal for Hamiltonian systems.

To conclude this section we note that doubling bifurcations by no means exhaust the variety of the bifurcation pattern in the mappings (2.12) and (2.15). In particular, in the interval of variation of the parameter K_H between two values corresponding to the sequence of doubling bifurcations, in the vicinity of elliptic points necklaces of islands corresponding to higher-order resonances continuously appear and split off. Figure 9a illustrates the phase portrait of the mapping (2.15) in the region of an elliptic point for $K_H=3.15$; the splitting off of a necklace of four islands is shown. Figure 9b corresponds to the appearance of a necklace of three islands at $K_H=4.7815$.

7. DIFFUSION OF PARTICLES

The diffusion of particles on the phase plane (v_x, v_y) can be represented conventionally in the form of two different

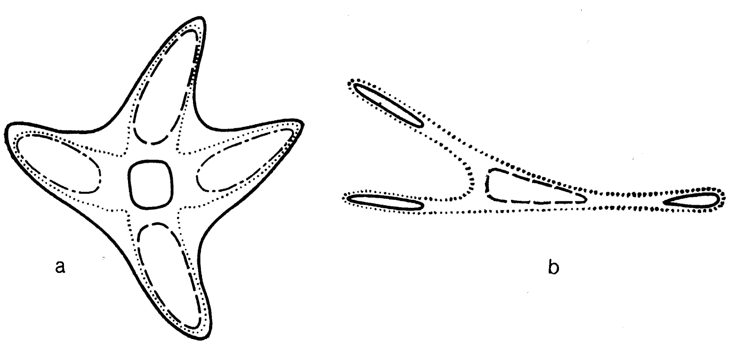

FIG. 9. a) Splitting off of a necklace of four islands for $\alpha=\alpha_4, K_H=3.15$; b) splitting off of a necklace of three islands for $K_H=4.7815$.

limiting cases. The first of these correspond to small values of K_H. Then this diffusion is analogous to Arnol'd diffusion. Only those particles which are inside the stochastic web diffuse. Their number is small and proportional to the phase occupied by the stochastic web. The other particles execute regular oscillations inside the islands.

According to the estimate (4.4), the fraction of diffusing particles is of the order of

$$\delta\rho/\rho_0 \sim \exp(-\pi\Delta\omega/2\Omega_q), \tag{7.1}$$

where Ω_q is the frequency of the small oscillations for the twist angle $\alpha_q = 2\pi p/q (p<q)$, and ρ_0 is the density of the particles. If in the system there are any weak collisions of any kind, this will mean that there is a constant source filling the channels of the stochastic web with particles. Thus, a constant diffusive flux of particles that gather energy is realized. Let $\mathscr{E}_0 \sim \tilde{T}$ be the characteristic energy of the particles of a plasma with temperature $\tilde{T}$, and let ν be the characteristic frequency of the Coulomb or collective collisions. Then the time within which the particles in unit volume gather energy $\tilde{\mathscr{E}}$ is of the order of

$$\tau_{\mathscr{E}} \sim \nu^{-1} (\tilde{\mathscr{E}}/\rho_0 \tilde{T})^2 \exp(\pi\Delta\omega/\Omega_q). \tag{7.2}$$

If we take $\tilde{\mathscr{E}}$ to be the energy density of the wave packet, the estimate (7.2) essentially determines the time of damping of the wave packet. As can be seen, this time is rather long for small values of K_H, i.e., for small field amplitudes and large values of the parameter $\omega_H T$.

The situation, however, changes with increase of K_H. The stochastic web is transformed into broad regions of stochastic dynamics, in which most of the particles take part. The diffusion in this case can be described in the usual way by means of an equation of the Fokker-Planck-Kolmogorov (FPK) type.

For this we return to the mapping (2.7). We denote

$$I = v_x^2 + v_y^2 = 2\mathscr{E}/m, \quad \varphi = \arctan(v_x/v_y), \tag{7.3}$$

where $\mathscr{E}$ is the energy of the particle. Then the mapping (2.7) is equivalent to the following:

$$\begin{aligned} \bar{I} = I &+ 2(e/m) E_0 T I^{1/2} \sin\varphi \cos(k_0 r_H \cos\varphi) \\ &+ (eE_0 T/m)^2 \cos^2(k_0 r_H \cos\varphi), \end{aligned} \tag{7.4}$$

$$\begin{aligned} \tan\bar{\varphi} = \{ &I^{1/2} \sin\omega_H T \cos\varphi \\ &+ [I^{1/2} \sin\varphi + (eE_0T/m)\cos(k_0 r_H \cos\varphi)] \cos\omega_H T \} \\ \times \{ &I^{1/2} \cos\omega_H T \cos\varphi - [I^{1/2} \sin\varphi + (eE_0T/m) \\ &\times \cos(k_0 r_H \cos\varphi)] \sin\omega_H T \}^{-1}. \end{aligned}$$

The FPK-equation approximation begins to work for values of the action I large enough for the condition

$$|\Delta I| = |\bar{I} - I| \ll I$$

to be fulfilled. This means, according to (7.3) and (7.4), that we should have sufficiently large particle energies. The phase φ in this case can be assumed to be stochastic and almost uniformly distributed in the interval $(0, 2\pi)$. The condition for this is, obviously, the inequality

$$eE_0 T k_0 r_H / m I^{1/2} = \Omega_0^2 T^2 / \omega_H T = K_H \gg 1, \tag{7.5}$$

which follows immediately from Eq. (7.4) as the condition for local instability of the phases:

$$\max |\partial\bar{\varphi}/\partial\varphi - 1| \gg 1.$$

Using the first equation (7.4) we calculate

$$\begin{aligned} A &= \frac{1}{T} \langle\langle (\bar{I} - I) \rangle\rangle = \frac{1}{2\pi T} \int_0^{2\pi} d\varphi \left(\frac{eE_0 T}{m} \right)^2 \cos^2(k_0 r_H \cos\varphi) \\ &= \frac{1}{2T} \left(\frac{eE_0 T}{m} \right)^2 [1 + J_0(2k_0 r_H)], \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{T} \langle\langle (\bar{I} - I)^2 \rangle\rangle \\ &= \frac{1}{2\pi T} \int_0^{2\pi} d\varphi \left(\frac{2eE_0 T}{m} \right)^2 I \sin^2\varphi \cos^2(k_0 r_H \cos\varphi) \\ &= \frac{1}{T} \left(\frac{eE_0 T}{m} \right)^2 I \left[1 + \frac{1}{k_0 r_H} J_1(2k_0 r_H) \right], \end{aligned} \tag{7.6}$$

where $r_H = I^{1/2}/\omega_H$, and J_0 and J_1 are Bessel functions. The coefficients A and B are related by

$$\frac{1}{2} \frac{\partial B}{\partial I} = A.$$

Therefore, the diffusion equation takes the divergence form

$$\frac{\partial F}{\partial t} = \frac{1}{2} \frac{\partial}{\partial I} D(I) \frac{\partial F}{\partial I}, \tag{7.7}$$

where the diffusion coefficient D, according to (7.6), is equal to

$$D(I) = B = \omega_H K_H^2 I \left[1 + \frac{1}{k_0 r_H} J_1(2k_0 r_H) \right]. \tag{7.8}$$

Since I and the energy $\mathscr{E}$ are linearly related (see (7.3)), the expressions (7.7) and (7.8) can be rewritten finally as

$$\frac{\partial F(\mathscr{E}, t)}{\partial t} = \frac{e^2 E_0^2 T}{2m} \frac{\partial}{\partial \mathscr{E}} \mathscr{E} \left[1 + \frac{1}{k_0 r_H} J_1(2k_0 r_H) \right] \frac{\partial F(\mathscr{E}, t)}{\partial \mathscr{E}}. \tag{7.9}$$

Nowhere have we imposed restrictions on the quantity $k_0 r_H$. Therefore, Eq. (7.9) contains an oscillating diffusion coefficient for $k_0 r_H > 1$, depending on the magnitude of the magnetic field. In particular, for $k_0 r_H \ll 1$ we have

$$\frac{\partial F}{\partial t} = \frac{e^2 E_0^2 T}{m} \frac{\partial}{\partial \mathscr{E}} \mathscr{E} \frac{\partial F}{\partial \mathscr{E}}. \tag{7.10}$$

We multiply by $\mathscr{E}$ and, by integrating over $\mathscr{E}$, find the increase of the average energy with time:

$$\langle \mathscr{E} \rangle = e^2 E_0^2 T t/m + \mathscr{E}_0.$$

In the case when the Larmor radius becomes comparable to the wavelength of the central mode the diffusion law changes. This occurs, however, in the region in which the particle energy is not too large, i.e., $k_0 r_H \gtrsim 1$. With increase of the particle energy we have $k_0 r_H \gg 1$ and the role of the

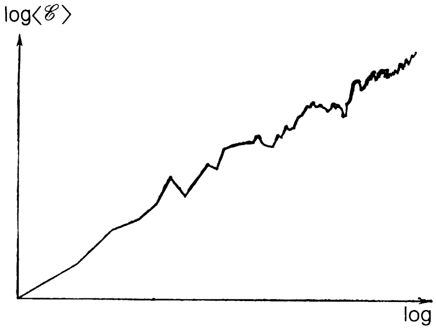

FIG. 10. Increase of the average energy as a function of time.

correction due to the Bessel function in (7.9) becomes negligibly small. The average energy then increases with time in accordance with the law

$$\langle \mathscr{E} \rangle = e^2 E_0^2 Tt/2m. \qquad (7.11)$$

This law is also represented on the graph obtained by the numerical analysis (Fig. 10).

8. CONCLUSION

From the results given we shall select those which seem to us to be fundamentally new and important for applications. In the construction of the so-called quasilinear description of different phenomena in a plasma an important role is played by the condition for the possibility of such a description. This condition reduces to the determination of a criterion for the stochastization of certain phase variables—a criterion that makes it possible to perform the corresponding averaging operation. One serious difficulty in the path to obtaining such a criterion involves the fact that the separatrices of the different waves in the packet are displaced relative to each other because of the dispersion of the waves.[8] This circumstance is manifested especially strongly in an external magnetic field, which leads to twisting of the trajectories of the particles that are interacting resonantly with individual waves of the packet. As a result of this twisting the entire phase space is covered by a stochastic web for arbitrarily small perturbations, i.e., for arbitrary wave amplitudes.

The existence of the stochastic web leads to universal diffusion of particles, analogous to Arnol'd diffusion in the multidimensional case. This diffusion leads to stochastic heating of particles. Although the particles are few because the stochastic web is thin, the unbounded increase of their energy leads to the existence of a universal damping of the wave packet in a transverse magnetic field.

The structure of the stochastic web in the general case has a fractal character, and the sizes of its cells have a weak stochastic spread. This very important circumstance shows how a quasicrystalline structure can be created on the plane. In the given case this structure is formed as the trajectory of a particle moving along the channels of the stochastic web. It contains rotational symmetry determined by the rational ratio of the Larmor frequency to the frequency shift between neighboring waves in the wave packet. In addition, the stochastic destruction of the short-range order in the structure of the web shows how it is possible to cover the plane by weakly deformed polygons while preserving the symmetry of the long-range order (rotational symmetry).

To this it may be worth adding that the stochastic web with twisting angle $\alpha_q = 2\pi/3$ corresponds to a classical fractal with elements of the Koch-curve type. In the problem considered this fractal is generated by the trajectory of a particle, and this is the first example in which a fractal of the Koch-curve type has been generated by the motion of a particle in a real field.

In conclusion the authors express their sincere gratitude to V. I. Arnol'd for an interesting discussion and comments.

[1)] The limit $\alpha \to 0$ has, in itself, no physical meaning. It can be realized either at $T = 0$, which is unnatural, or at $\omega_H = 0$, but in this case it is necessary also to redefine the variables in accordance with (2.11).

[2)] If $p > q$, the integer part must be subtracted from the ratio p/q.

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