Two-dimensional small-radius solitons in magnets

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Two-dimensional topological solitons with precession of the magnetization in ferromagnets and antiferromagnets are considered. An analytic dependence of the precession frequency on the number of bound magnons for small-radius solitons is obtained. It is shown that its character differs substantially for solitons with different values of the topological charge ν: For $\nu = 1$ the dependence is nonanalytic, while for $\nu > 1$ it is analytic. It is proved that the properties of the two-dimensional solitons are unstable with respect to inclusion in the energy of higher powers of the spatial derivatives of the magnetization: Even when the corresponding terms are small their inclusion fundamentally changes the properties of the solitons. Contradictions in the results of previous numerical and analytical calculations of the characteristics of small-radius solitons are explained on the basis of this conclusion. In the generalized model the region of permissible values of the soliton frequency is substantially widened and static stable solitons can exist.

In the modern physics of essentially nonlinear phenomena, based on the theory of solitons, there is increasing interest in the properties of non-one-dimensional solitons. For the analysis of these, constructive analytical methods such as the method of the inverse problem of scattering theory[1] are rarely applicable, and the main thrust is toward the study of isolated soliton solutions[2] on the basis of topological arguments[3] and numerical analysis; analytical results can be obtained in the limiting cases of solitons of large and small radii (see Ref. 2).

Two-dimensional topological solitons in a ferromagnet are interesting from both the purely theoretical and the applied point of view for the description of domains in thin magnetic films.[4] A numerical analysis of such solitons was carried out in Refs. 5–7. It was found that the results of the numerical analysis differ substantially from the analytical asymptotic formulas obtained in Ref. 7 for the dependence of the soliton energy E and precession frequency ω on the number N of magnons in the soliton. In the previously considered problems concerning magnetic solitons without topological charge there was no such disagreement.

In the present paper[1)] we consider the structure of topological solitons of small radius in ferromagnets and antiferromagnets. It is shown that the properties of the solitons of the standard model of a magnet,[8,9] in which the energy is quadratic in the magnetization gradients, are unstable with respect to inclusion of higher powers of the gradients. The properties of the solitons in the standard model and the generalization model are investigated.

I. THE FERROMAGNET

In the phenomenological description of a ferromagnet on the basis of the Landau-Lifshitz equations (see Refs. 8 and 9) one usually starts from the following form of the energy as a functional of the normalized magnetization vector $\mathbf{m}$ ($\mathbf{m}^2 = 1$):

$$W = \frac{1}{2} a M_0^2 \int \{\alpha(\nabla\mathbf{m})^2 + \beta(m_x^2 + m_y^2)\} d^2x. \qquad (1)$$

In the form (1) the magnet is assumed to be uniaxial, $\beta > 0$ is the anisotropy constant, $M_0 = 2\mu_0 s/a^3$ is the saturation magnetization, μ_0 is the Bohr magneton, $\alpha \sim (J/\mu_0 M_0)a^2$, J is the exchange integral, a is the lattice constant, and the factor a in front of the integral has been added to make the dimensions correct.

In a uniaxial magnet not only the energy (1) but also the total z-component I_z of the magnetization is an integral of the motion. This integral of the motion determines the number N of magnons bound in the soliton:

$$N = I_z/2\mu_0 = (aM_0/2\mu_0) \int (1 - m_z) d^2x. \qquad (2)$$

The unit vector $\mathbf{m}$ is conveniently described in angular variables:

$$m_z = \cos\theta, \qquad m_x + im_y = e^{i\varphi} \sin\theta. \qquad (3)$$

In these variables a two-dimensional topological soliton corresponds to (Ref. 5; see also Ref. 2)

$$\varphi = \omega t + \nu\chi, \quad \theta = \theta(r), \quad \theta(0) = \pi, \quad \theta(\infty) = 0, \qquad (4)$$

where ω is the frequency of precession of the magnetization, the integer ν defines the topological charge of the soliton, and r, χ are the polar coordinates of the points of the two-dimensional magnet. The form of the function $\theta(r)$ is determined by the requirement that the energy (1) with allowance for (4) has a minimum at the given value of N. This corresponds to the condition $\delta[W - \hbar\omega N] = 0$. Writing the energy W and the number N of magnons in angular variables, taking into account the concrete form of φ (4), and varying $W - \hbar\omega N$ with respect to $\theta(r)$, we obtain an ordinary differential equation for $\theta(r)$. We write it in the form

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{\nu^2}{r^2}\sin\theta\cos\theta = F(\theta)$$

$$= \frac{1}{l_0^2}\sin\theta\left(\cos\theta - \frac{\omega}{\omega_0}\right). \qquad (5)$$

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Here $l_0 = (\alpha/\beta)^{1/2}$ is the characteristic length in the problem, $l_0 \cong (J/\hbar\omega_0)^{1/2} a \gg a$, and $\omega_0 = 2\mu_0 \beta M_0/\hbar$ is the frequency of the linear ferromagnetic resonance. In the right-hand side we have separated out the terms that arise from the anisotropy energy.

A small-radius soliton corresponds to localization of $\theta(r)$ in a region of order R, where $R \ll l_0$. In this case, $l_0(d\theta/dr) \sim l_0/R$, and in the region $r \ll l_0$ we can assume that the right-hand side of (5) is small and solve this equation by iterations in $F(\theta)$ (more precisely, in $R^2F(\theta)$). The zeroth approximation corresponds to the static scale-invariant soliton solution $\theta_0(r)$ in an isotropic ferromagnet, found by Belavin and Polyakov[10]:

$$\mathrm{tg}(\theta_0/2) = (R/r)^{|\nu|} \tag{6}$$

(below we assume that $\nu > 0$, and omit the modulus sign).

We shall consider corrections to the asymptotic solution (6). For this we write $\theta(r) = \theta_0 + \psi$ and consider the form of the correction $\psi(r)$. For the latter, on the basis of (5), it is easy to obtain a linear equation with a right-hand side:

$$\hat{L}\psi = F(\theta_0), \quad \hat{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\nu^2}{r^2}(1 - 2\sin^2\theta_0). \tag{7}$$

It is easy to convince onself that the operator $\hat{L}$ has a zero eigenvalue, to which corresponds the eigenfunction $\sin\theta_0$: $\hat{L}\sin\theta_0 = 0$. A localized solution of this equation exists in the case when the right-hand side $F(\theta_0)$ is orthogonal to the $\sin\theta_0$, i.e., when

$$\int_0^\infty r\,dr \sin\theta_0 F(\theta_0) = 0.$$

This condition for $R \to 0$ leads to the limiting frequency value $\omega = \omega_0/\nu$ obtained earlier[7] by another method, namely on the basis of (6) and the identity

$$\frac{1}{2}\int_0^\infty \sin^2\theta\, r\,dr = \frac{\omega}{\omega_0}\int_0^\infty (1 - \cos\theta) r\,dr, \tag{8}$$

which can be obtained by integrating Eq. (5). However, consideration of the problem on the basis of (7) also makes it possible to find the behavor of $\omega(N)$ for small but finite values of N. By writing ω in the form $\omega = \omega_0/\nu + \Delta\omega(N)$, and using (8), we can express $\Delta\omega$ in terms of the value of the correction ψ:

$$\Delta\omega(N)\int_0^\infty (1 - \cos\theta_0) r\,dr = \omega_0 \int_0^\infty \sin\theta_0(\cos\theta_0 - 1/\nu)\psi(r) r\,dr. \tag{9}$$

If the solution $\psi(r)$ is known, we can find $\Delta\omega$ as a function of R and express the soliton frequency in the desired form in terms of N by means of the relation

$$\frac{N}{N_2} = \frac{2\pi}{\nu\sin(\pi/\nu)}\left(\frac{R}{l_0}\right)^2, \quad \nu > 1 \tag{10}$$

($N_2 = 2\pi s(l_0/a)^2 \gg 1$ is the characteristic value of the number of magnons), which is obtained by direct calculation from formula (2) with allowance for (6). This relation is valid for $\nu > 1$; the case $\nu = 1$ is special and will be considered below.

The derivation of the dependence $\Delta\omega(N)$ reduces to solving Eq. (7) and calculating the integrals in (9). The solution that vanishes as $r \to 0$ can be found exactly and has the form

$$\psi = -(r^2/4\nu l_0^2)\sin\theta_0. \tag{11}$$

It is small in the region of localization of the soliton, i.e., for $r \lesssim R$. However, the behavior of the solution at large values of r shows that for $r \sim l_0$ the correction ψ becomes comparable to the solution θ_0 itself, and the use of (6) is not justified. This circumstance can be easily understood by noting that the power-law asymptotic form (6) of θ_0 differs from the exponential asymptotic form of the exact solution of Eq. (5). We can refine the solution (6) by writing

$$\mathrm{tg}\frac{\theta}{2} = \frac{2}{(\nu-1)!}\left[\frac{R(1-\omega/\omega_0)^{1/2}}{2l_0}\right]^\nu K_\nu\left(\frac{r(1-\omega/\omega_0)^{1/2}}{l_0}\right), \tag{12}$$

where K_ν is the Macdonald function. In the actual region $r \lesssim R \ll l_0$ (12) goes over into (6), and for $r \gtrsim l_0$ gives the correct (exponential) asymptotic form (5) (Ref. 7). In the analysis of the soliton it may turn out to be necessary to start from this refined solution and calculate the correction ψ on the basis of this solution. Then in the region $r \lesssim R$ we again obtain for ψ the expression (10), while for $r \gg R$ it is necessary to take into account the difference between (6) and (12).

The calculation of $\Delta\omega(N)$ has shown that the behavior of $\omega(N)$ can be fundamentally different for solitons with different values of the topological charge ν. Only for $\nu > 2$ can we start from the form (6) for the solution and (11) for the correction; then, to within terms of order $(R/l_0)^4$ or $(N/N_2)^2$,

$$\frac{\omega(N)}{\omega_0} = \frac{1}{\nu} - \eta_\nu \frac{N}{N_2},$$

where

$$\eta_\nu = \sin^2(\pi/\nu)/2\pi\nu^2\sin(2\pi/\nu) \tag{13}$$

and the derivative $d\omega/dN$ is finite as $N/N_2 \to 0$. In the case $\nu = 2$ the use of formula (6) and (11) leads to divergent integrals in (9) and it is necessary to refine these formulas on the basis of (12). In this case it turns out that the correction ψ to the refined solution (12) is described by formula (11) for $r < R$ and decreases as $1/r^2$ for $R < r \ll l_0$. The calculation of the quantity $\omega(N)$ led to formula (13) with $\eta_2 = 0.027$. thus, η_ν decreases monotonically with increase of ν; by virtue of (13), $\eta_3 = 0.015$ and $\eta_4 = 0.005$. This behavior agrees well with the result of the numerical calculation of Ref. 7, if we confine ourselves to the region of not very small N ($N/N_2 > 10^{-2}$).

For a soliton with $\nu = 1$ the situation is fundamentally different: to obtain the asymptotic dependence $\omega(N)$ it is sufficient to use formula (12) directly. In this case we can make use of formula (8), rewriting it in the form

$$\left(1-\frac{\omega}{\omega_0}\right)\int_0^\infty \sin^2\frac{\theta}{2}r\,dr=\int_0^\infty \sin^4\frac{\theta}{2}r\,dr. \qquad (14)$$

Calculating these integrals with the aid of (12) and using the formula for N, we obtain

$$N=(8N_2/e\gamma^2)(1-\omega/\omega_0)^{-2}\exp\{-1/(1-\omega/\omega_0)\},\qquad \nu=1, \quad (15)$$

where $\gamma=1.78$ is the Euler constant. By virtue of (15), $\omega\to\omega_0$ as $N/N_2\to 0$, but the dependence $\omega(N)$ for $\nu=1$ is nonanalytic and $d\omega/dN\to\infty$ as $N/N_2\to 0$. This does not contradict the result of the numerical calculation of Ref. 5; on the graph of the dependence $\omega(N)$ in the latter paper the sharp increase of the derivative $d\omega/dN$ upon decrease of N/N_2 is clearly visible.

Thus, the analysis performed has demonstrated the fundamentally different behavior of the dependence $\omega(N)$ in the model (1) for different values of the topological charge of the soliton: For $\nu=1$ this dependence is nonanalytic, while for $\nu>1$ the value of the derivative $d\omega/dN$ for $N/N_2\to 0$ is finite and decreases with increase of ν: $(N_2/\omega_0)(d\omega/dN)\cong 1/4\nu^3$ for $\nu\gg 1$. This result sharply contradicts the data of numerical calculations, according to which the limiting value $\omega=\omega_0/\nu$ for $\nu=2, 3, 4$ is not reached for any value of N in the interval of N/N_2 from 10^2 to 10^{-5} (Refs. 6, 7). For $N/N_2<10^{-3}$, instead of reaching a constant value the quantity $\omega(N)$ decreases sharply to values less than $0.1\omega_0$ with decrease of N. The case $\nu=1$ was analyzed numerically in Ref. 5, in which such a steep drop of $\omega(N)$ at small N was not observed. It seems to us that an explanation of this anomalous behavior is possible on the basis of an analysis of a certain, more general model of a ferromagnet.

2. THE GENERALIZED MODEL

We shall consider a generalization of the model (1) of a ferromagnet, consisting in the inclusion of terms containing higher powers of the spatial derivatives in the nonuniform-exchange energy. We shall take the energy of the ferromagnet in the form $W=W_0+\Delta W$, where W_0 is defined by the standard formula (1), and

$$\Delta W=\frac{1}{4}a^3M_0^2\int\{\alpha_1(\Delta\mathbf{m})^2+\alpha_2[(\nabla\mathbf{m})^2]^2\}d^2x. \qquad (16)$$

Besides these terms there are two further exchange invariants

$$\frac{\partial^2 m_\alpha}{\partial x_i\,\partial x_k}\frac{\partial^2 m_\alpha}{\partial x_i\,\partial x_k},\qquad \left(\frac{\partial m_\alpha}{\partial x_i}\frac{\partial m_\alpha}{\partial x_k}\right)\left(\frac{\partial m_\beta}{\partial x_i}\frac{\partial m_\beta}{\partial x_k}\right),$$

but their inclusion does not lead to new effects since the corresponding constants enter additively to $(\alpha_1+\alpha_2)$; therefore, for the analysis we shall confine ourselves to (16). The invariants of fourth order in the derivatives arise naturally in the expression for the exchange energy of a discrete spin system in the form of an expansion in derivatives of $\mathbf{m}(\mathbf{r})$ (see Ref. 8). The signs of α_1 and α_2 are determined by the relative values of the exchange integrals between nearest-neighbor and between next-nearest-neighbor spins, and we shall assume that $\alpha_1\sim\alpha_2\sim\alpha$.

If the characteristic length scale of the variation of $\mathbf{m}$, determined by the soliton radius R, is much greater than the lattice constant a (otherwise, a macroscopic treatment of the soliton makes no sense), then

$$\alpha_1a^2(\Delta\mathbf{m})^2\sim\alpha_2a^2(\nabla\mathbf{m})^4\sim\alpha(a^2/R^4)\ll\alpha/R^2$$

and the value of (16) is small in comparison with the usual magnitude of the exchange energy. It turns out, however, that the inclusion of (16), even for $R\gg a$, substantially alters the form of $\omega(N)$.

With the inclusion of the (16) the soliton solution can be sought, as before, in the form (4), and the structure of the soliton is determined by the minimum of the energy $W=W_0+\Delta W$ with allowance for the explicit form of $\varphi(\chi)$ (4). This is an extremely cumbersome equation of fourth order in $\theta(r)$, but it can be analyzed in the case of interest to us, viz., the case of a soliton of small radius $(a\ll R\ll l_0)$. In this case

$$\Delta W\ll aM_0^2(\nabla\mathbf{m})^2R^2,\qquad \beta\sin^2\theta\ll\alpha(\nabla\mathbf{m})^2$$

and the contribution (16), like the contribution of the anistopy energy, is small. In this case we can again assume that the solution is close to $\theta_0(r)$ [see (6) or (12)], and as a result the analysis of the problem reduces to the solution of a linear equation of the type (7) but with a different right-hand side: $\hat{L}\psi=F(\theta_0)+\Delta F(\theta_0)$.

The form of F is given by (5), and the function of ΔF arises from the additional term ΔW (16) in the energy of the magnet. The rather cumbersome expression for ΔF includes the function θ_0 and its derivatives up to the fourth.

For brevity, we give the form of this function after simplification with the use of the relation $d\theta_0/dr=-(\nu/r)\sin\theta_0$ [see (6)]:

$$\Delta F=-(2\nu^3a^2/\alpha r^4)\sin^3\theta_0(\alpha_1+2\alpha_2+2\nu\alpha_2\cos\theta_0). \qquad (17)$$

The condition of orthogonality of $\sin\theta_0$ and $F(\theta_0)+\Delta F(\theta_0)$ leads for $\nu>1$ to the relation

$$\frac{1}{\nu}-\frac{\omega}{\omega_0}=\frac{4(\alpha_1+\alpha_2)}{3\alpha}\nu(\nu^2-1)\left(\frac{al_0}{R^2}\right)^2.$$

Hence, using the relation (10) between N and R, we obtain the desired dependence $\omega(N)$ for the generalized model for $\nu>1$:

$$\frac{\omega}{\omega_0}=\frac{1}{\nu}-\frac{16\pi^2(\nu^2-1)}{3\nu\sin^2(\pi/\nu)}\frac{\alpha_1+\alpha_2}{\alpha}\left(\frac{a}{l_0}\right)^2\left(\frac{N_2}{N}\right)^2. \qquad (18)$$

If in these formulas we assume that $R\gg(al_0)^{1/2}$ or $N\gg aN_2/l_0$, then $\omega\cong\omega_0/\nu$. But upon decreases of R or N the values of the function $\omega(N)$ changes sharply in comparison with these characteristic values. If $\alpha_1+\alpha_2<0$, then $\omega(N)$ increases with decrease of N. In the more interesting case when $\alpha_1+\alpha_2>0$, which we shall discuss below, $\omega(N)$ decreases sharply with decrease of N. At $N=N_0$ or $R=R_0$, where

$$N_0=N_2\left(\frac{a}{l_0}\right)\left(\frac{\alpha_1+\alpha_2}{3\alpha\nu}\right)^{1/2}\frac{4\pi(\nu^2-1)^{1/2}}{\sin(\pi/\nu)},$$
$$R_0^2=2al_0\left[\frac{\nu(\nu^2-1)(\alpha_1+\alpha_2)}{3\alpha}\right]^{1/2}, \qquad (19)$$

the value of the frequency vanishes and the soliton is static.

For $N<N_0$ or $R<R_0$ the value of the frequency is negative, which can never be the case in the standard model (1) (Ref. 7).

We note that the above-indicated characteristic value of the soliton radius, although small in comparison with l_0, is nevertheless macroscopic: If we assume that $l_0\sim(J/\hbar\omega_0)^{1/2}a\gg a$, then

$$R_0\sim(al_0)^{1/2}\sim(J/\hbar\omega_0)^{1/4}a\gg a.$$

Analogously,

$$N_0\sim N_2a/l_0\sim(sl_0/a)\gg1$$

and the soliton can be described in the framework of a semiclassical phenomenological approach. This result is entirely specific to the two-dimensional case: In the three-dimensional case the radius of the static soliton in the model (1), (16) is of the order of the lattice constant, and a macroscopic description of it is inadequate. For solitons in superfluid ^{3}He this conclusion was reached in Refs. 3.

In the special case $\nu=1$ it is necessary to use the refined solution (12), and here the relationship between ω and R is more complicated:

$$1+\frac{\alpha_1+\alpha_2}{2\alpha}\left(\frac{al_0}{R}\right)^2$$
$$=\left(1-\frac{\omega}{\omega_0}\right)\ln\left(\frac{4l_0^2}{e\gamma^2R^2(1-\omega/\omega_0)}\right),$$
$$\nu=1. \qquad (20)$$

Analysis of this formula and of the dependence $\omega(N)$ obtained on the basis of it shows that the qualitative behavior of $\omega(N)$ is the same of $\nu=1$ and $\nu\neq1$: In a broad interval of values of $N(N_0\ll N\ll N_2)$ the soliton frequency is close to the limiting value ω_0/ν for $\nu\neq1$ and ω_0 for $\nu=1$, and then decreases sharply and passes through zero at $N=N_0$ (see Fig. 1). The change of value of the small parameter a/l_0 changes only the left boundary of the region in which $\omega\cong\omega_0/\nu$, i.e., changes the position of the point at which the steep drop of the frequency begins. We emphasize that the steep drop occurs for an arbitrarily small value of a/l_0, and the corresponding value $R_0\gg a$. The situation is characteristic only for the two-dimensional case. In the one- and three-dimensional cases the presence of the leading derivatives is manifested only when $R\sim a$, when a macroscopic approximation is simply inapplicable. It can be said that in the two-dimensional case the problem of the investigation of topological solitons is unstable with respect to the inclusion of the higher derivatives.

It may be postulated that the contradiction between the numerical data of Refs. 6 and 7 and the analytical calculation of $\omega(N)$ for the model (1) is a consequence of this instability. The point is that in solving a differential equation numerically one always replaces it by a difference equation, as a result of which uncontrollable higher derivatives appear in the problem. The role of the lattice constant, or, more precisely, of a/l_0, is played in this case by the step in the difference scheme. In any case, the dependence $\omega(N)$ obtained by the numerical methods differs from the result of the analysis of the model (1) but corresponds well to the dependence $\omega(N)$ in the generalized model with higher derivatives, if in the latter we set $a/l_0\cong10^{-7}$. Such instability of the two-dimensional topological solitons describable by second-order equations should certainly be taken into account in the numerical analysis of these solitons.

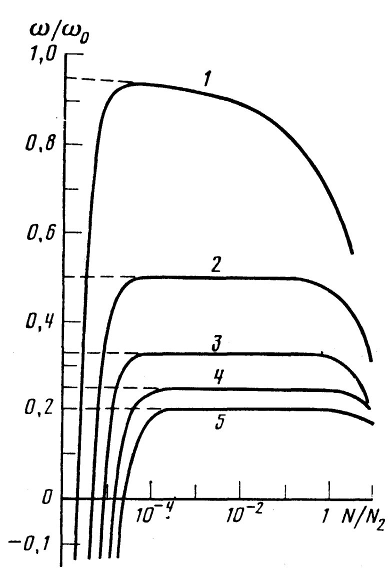

FIG. 1. Plot of $\omega(N)$ in a ferromagnet for different values of ν; the number of the curve corresponds to the value of ν. The solid curves are for $(\alpha_1+\alpha_2)a^2/al_0^2=1.3\cdot10^{-13}$; the dashed curves are for $\alpha_1+\alpha_2=0$.

3. STABILITY

The region of the "steep drop" corresponds to $dN/d\omega>0$, whereas the previously formulated condition for the stability of solitons for a wide class of models[11] has the form of the opposite inequality. We note, however, that the proof that a soliton is stable when $dN/d\omega<0$ and unstable when $dN/d\omega>0$ was carried through only for dynamical solitons without topological charge.[12,13] A one-dimensional topological soliton of the domain-boundary type is stable independently of the sign of $dN/d\omega$ (Ref. 13).

The calculations we have performed by the Ritz method have shown that static solitons with $N\cong N_0$ correspond to the minimum of the energy, and these solitons are stable. Evidently, for two-dimensional topological solitons, as for one-dimensional solitons, the condition $dN/d\omega<0$ is not necessary for stability. The possibility of the existence of stable two-dimensional magnetization-field solitons of extremely small radius (almost-singular solitons) is of interest in connection with the proposal to use them as information carriers in magnetic-memory systems[14] [the statement in Ref. 15 concerning their stability pertained only to the standard model (1); in our theory, however, such solitons exist only in the generalized model (16), which has not been discussed before].

4. THE ANTIFERROMAGNET

The method developed is also easily applied to the investigation of topological solitons of small radius in an antiferromagnet (AFM). The dynamics of the magnetization of an AFM is conveniently described in terms of the unit vector **l** of the antiferromagnetism.[16,17] In angular variables for **l** ($l_x + il_y = \sin\theta e^{i\varphi}$), a soliton, as in a ferromagnet, corresponds to a solution of the form (4). The form of $\theta(r)$ is determined by the condition $\delta[W - \hbar\omega N] = 0$, where

$$N=\frac{2\omega a}{\mu_0 g\delta}\int \sin^2\theta\, d^2x. \qquad (21)$$

We shall write the energy of a uniaxial AFM, to within terms quadratic in the gradients of **l**, in the form

$$W_0=aM_0^2\int\left\{\frac{\alpha}{2}(\nabla \mathbf{l})^2+\frac{\alpha}{2c^2}\left(\frac{\partial \mathbf{l}}{\partial t}\right)^2+w_a(l_z^2)\right\}d^2x. \qquad (22)$$

Here the meaning of the constants a and α is the same as for the ferromagnet [see (1)], M_0 is the sublattice magnetization, $c = \mu_0 M_0(\alpha\delta)^{1/2}/\hbar$ is the phase velocity of the AFM spin waves,)$\delta = 4/\chi$ is the exchange constant, χ is the susceptibility of the AFM, $\delta \sim \alpha/a^2$, and w_a is the anistropy energy.

We choose w_a in the form

$$w_a={}^1/_2\beta\sin^2\theta-{}^1/_4 b\sin^4\theta,\quad \beta>0,\quad b>0, \qquad (23)$$

since for the existence of dynamical solitons it is necessary to include in the energy w_a not only terms proportional to l_z^2 but also terms proportional to l_z^4 (for more detail, see Ref. 12). We shall consider also a generalized model of an AFM, corresponding to inclusion of higher powers in the gradients of the vector **l**. In this model the energy $W = W_0 + \Delta W$, where W_0 is given by (22) and ΔW is obtained from the corresponding formula (16) for the ferromagnet by replacing **m** by **l**.

The angle θ in the AFM soliton is determined by an equation of the type (5), but with a different form of right-hand side. As in the ferromagnet, the right-hand side of this equation can be represented in the form $F(\theta) + \Delta F(\theta)$. The function ΔF is determined by the variation ΔW and is described by formula (17): the form of $F(\theta)$ is easily obtained by noting that, by virtue of (4), $(\partial \mathbf{l}/\partial t)^2 = \omega^2\sin^2\theta$, and using the explicit form (23) of the anistropy energy of the AFM:

$$F=\left(\frac{\beta}{\alpha}-\frac{\omega^2}{c^2}\right)\sin\theta\cos\theta-\frac{b}{\alpha}\sin^3\theta\cos\theta.$$

We note that $F(\theta)$ can be rewritten in the form

$$F=\frac{1}{l_a^2}\sin 2\theta(\cos 2\theta-\Omega),$$

$$\Omega=\frac{\omega^2-\omega_t^2}{\omega_1^2-\omega_t^2},\qquad l_a^2=\frac{2\alpha}{b}. \qquad (24)$$

Here $\omega_1 = \mu_0 M_0(\beta\delta)^{1/2}/\hbar$ is the frequency of AFM magnons with $k = 0$, i.e., the lower boundary of the continuous spectrum of the magnons, $\omega_t = 2\mu_0 H_t/\hbar$, and $H_t = \mu_0[\delta(\beta - b/2)]^{1/2}/2$ is the field of the spin-flop transition. The quantity l_a coincides with the thickness of the 90-degree interphase boundary that exists at the spin-flop transition point in a magnetic field H equal to H_t. We note that $F(\theta)$ in the AFM differs from the corresponding function for the ferromagnet by the replacements $\omega\to\Omega$, $l_0\to l_a$, and $\theta\to 2\theta$. For $\nu = 0$, i.e., in the case of a dynamical soliton without topological charge, it follows from this circumstance that the equations for θ and 2θ in the cases of the ferromagnet and antiferromagnet, respectively, are, with the appropriate replacements,[12] identical, and the soliton exists for $0 < \Omega < 1$, i.e., $\omega_t^2 < \omega^2 < \omega_1^2$. Because of the term $(\nu^2/r^2)\sin\theta\cos\theta$ in the equation for θ there is no such simple correspondence for topological solitons, and it is necessary to perform the analysis anew. The investigation of small-radius solitons is carried out using the same method as in the case of the ferromagnet [see formulas (6)–(8)], with allowance for the concrete form (24) of $F(\theta)$ for the AFM.

We shall give the final formulas for the dependence $\omega(N)$. In the generalized model without allowance for corrections in N of the form (18) we find that for $\nu > 1$

$$\left(\frac{\omega}{\omega_1}\right)^2=\frac{3\nu^2-2(\nu^2-1)(1-\omega_t^2/\omega_1^2)}{3\nu^2+(\nu^2-1)[16\pi N_*\lambda/N\sin(\pi/\nu)]^2}, \qquad (25)$$

where

$$\lambda^2=4(\alpha_1+\alpha_2)a^2\beta/\alpha^2,\quad N_*=4\pi s a^2\beta^{1/2}/\alpha\delta^{1/2}.$$

For $\nu = 1$ it is necessary to start from the refined equations (12); we shall not give the corresponding formulas, but represent the result of the analysis in graphical form (see Fig. 2).

Formula (25) describes a dependence $\omega(N)$ that is fundamentally different from that in the ferromagnet [see (18)]: With decrease of N the value of the frequency vanishes only at $N = 0$. A static soliton with $\omega = 0$ corresponds to $N = 0$ but finite values of the radius R_0 and the energy; R_0 coincides in order of magnitude with the radius of the static soliton in the ferromagnet. The vanishing of N at $\omega = 0$ is connected with the fact that, by virtue of (21), $N \propto \omega$. In the

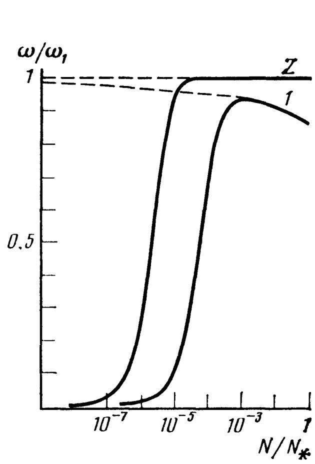

FIG. 2. Plot of $\omega(N)$ for an antiferromagnet for $\nu = 1$. Curve 1 corresponds to a value $\omega_t \ll \omega_1$, and curve 2 to a value $\omega_t \cong \omega_1$. The solid curves are plotted for $\lambda^2 = 1.3\cdot 10^{-13}$, and the dashed curves for $\alpha_1 + \alpha_2 = 0$.

presence of an external magnetic field H parallel to the z axis the value of N in the static soliton is finite: $N \propto H$.

Upon increase of N to values greater than λN_*, where $\lambda N_* \gg 1$ for $\lambda \gg (\beta/\delta)^{1/2}$, ω becomes independent of N and the soliton frequency reaches a limiting value:

$$\omega^2 \to (1/3\nu^2)[(\nu^2+2)\omega_1^2 + 2(\nu^2-1)\omega_t^2].$$

For $N > \lambda N_*$ the effect α_1 and α_2, i.e., allowance for higher powers of the gradients of **l**, is not important, and the structure of the soliton is determined by variation of the energy W_0 (22).

The results of the analysis of the model with $\alpha_1, \alpha_2 = 0$ are more conveniently described in terms of the quality Ω, connected with the soliton frequency by the relation (24).

As in the case of the ferromagnet, we can find the dependence $\Omega(N)$ for large (see Ref. 7) and small values of N. In the latter case it is sufficient to calculate the correction $\psi(r)$ to the solution $\theta_0(r)$ (6) with allowance for the concrete form of $F(\theta)$ for the AFM. To calculate Ω for a known form of ψ it is sufficient to make use of the relation

$$4\Omega \int_0^\infty \sin^2\theta r\, dr = \int_0^\infty \sin^2 2\theta r\, dr$$

[see formula (47) of Ref. 12]. Finally, we find that for $\nu = 1$ the value of $d\Omega/dN$ for $N \to 0$ is infinite:

$$N/N_a = [e\gamma(1-\Omega)]^{-2}\exp\{-4/3(1-\Omega)\}, \qquad (26)$$

and for $\nu > 1$ the function $\Omega(N)$ is analytic:

$$\Omega = (2+\nu^2)/3\nu^2 - \eta_\nu(\omega/\omega_1)(N/N_a). \qquad (26')$$

Here N_a is the characteristic magnon number that is convenient for the description of solitons in an antiferromagnet with $\alpha_1, \alpha_2 = 0$:

$$N_a = (\pi s/2)(\alpha\beta^{1/2}/b\delta a^2).$$

The limiting value of Ω as $N \to 0$ has been obtained by Voronov and Kosevich.[18] The factor η_ν for $\nu > 2$ is determined by the formula

$$\eta_\nu = \frac{1}{64\pi}\left(\frac{\nu^2-1}{3\nu^2}\right)^2 \frac{\sin^2(\pi/\nu)}{\sin(2\pi/\nu)}$$

and for $\nu = 2$ is equal to $7\cdot 10^{-3}$. The quantity η_ν decreases with increase of ν, but more slowly than in the ferromagnet: $9\cdot 10^{-4}/\nu$ for $\nu \gg 1$. This is connected with the fact that the limiting value of the soliton frequency in the AFM for $\nu \gg 1$ and $\alpha_1, \alpha_2 = 0$ remains finite ($\Omega \to 1/3$), and does not vanish as in the ferromagnet ($\omega \to \omega_0/\nu$).

The analysis performed has shown that for small values of N the behavior of the soliton in the AFM displays the same nonanalyticity as in the ferromagnet. The behavior of the soliton parameters is fundamentally different for a finite but small value of $\alpha_1 + \alpha_2$ and for $\alpha_1 + \alpha_2 = 0$. The range of the soliton frequencies for $\alpha_1 + \alpha_2 \neq 0$ extends from $\omega = 0$ to $\omega = \omega_1$, and is much wider than in the case $\alpha_1 + \alpha_2 = 0$ (the inequality $0 < \Omega < 1$ corresponds to $\omega_t^2 < \omega_2 < \omega_1^2$, and usually ω_t is close to ω_1).

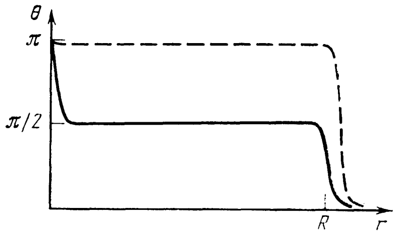

FIG. 3. Form of the function $\theta(r)$ for topological solitons of large radii. The dashed curve corresponds to the case of the ferromagnet, and the solid curve to the case of the antiferromagnet.

5. DISCUSSION

We shall discuss the properties of a large-radius topological soliton in an AFM. As in ferromagnets, these solitons correspond to a large value of N ($N \gg N_2$ and $N \gg N_a$, respectively). In Ref. 7 it was shown that in a ferromagnet for $N \gg N_2$ the soliton corresponds to a rather large region of radius R (in which $\theta \cong \pi$), separated from the rest of the magnet by a domain boundary of thickness l_0 (the region of the singularity near the center of the soliton is small).

A qualitative analysis of the problem has shown that the form of the solution for the AFM is essentially different: In a large part of the inner region of the soliton the value of θ is close to $\pi/2$, and at the center of the soliton there is a vortex of finite amplitude. The inner part of the soliton is separated from the rest of the AFM by a 90-degree interphase boundary in which $\tan\theta = \exp[(R-r)/l_a]$ (see Fig. 3).

On the basis of this model we have calculated the dependence of the energy and the number of magnons in the soliton on R, and, on the basis of the relation $dW/dN = \hbar\omega$, have found the dependence of $\Omega(N)$:

$$\Omega = \left(\frac{N_a}{N}\right)^{1/2}\left[1 + 16\nu^2\left(\frac{N_a}{N}\right)^{1/2}\right], \quad N \gg N_a. \qquad (27)$$

The main term in this relation ($\Omega \propto N^{-1/2}$) is the same as in the ferromagnet[7]; the interesting term is the ν-dependent correction. This correction arises from the term $(\nabla\varphi)^2\sin^2\theta$ in the energy density of the AFM; in the central

FIG. 4. "Joining" of the asymptotic dependences $\Omega(N)$ for $N < N_a$ and $N > N_a$; the number of the curve corresponds to the value of ν.

part of the soliton ($\theta \cong \pi/2$) it is close to ν^2/r^2 and gives a logarithmic term of the form $\nu^2 \ln(R/l_a)$ in the soliton energy.

By virtue of the different dependence of $\Omega(N)$ on the topological charge ν at large and small values of N, $\Omega(N)$ increases with increase of ν for $N \gg N_a$ and decreases for $N \ll N_a$ [see (26) and (27)]; these curves should intersect at finite values of N/N_a (Fig. 4). The fact that curves of $\Omega(N)$ with different ν intersect has been established independently by Voronov and Kosevich[18] in a numerical analysis of the soliton in the AFM. We note that at the center of a ferromagnetic soliton we have $\theta \cong \pi$ and there is no logarithmic term of the type found in the antiferromagnetic case. Therefore, the dependence of the energy E and frequency ω on the topological charge in a ferromagnet is weaker than in an antiferromagnet:

$$dW/d\nu^2 \sim \text{const}, \quad d\omega/d\nu^2 \sim \omega_0 (N_2/N)^{1/2}. \tag{28}$$

It is clear that, because of this, intersection of the $\omega(N)$ curves for a ferromagnet for different values of ν was not observed in the numerical experiment of Ref. 7; it follows from (28) that the intersection can occur at large values of N/N_2.

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