Relativistic electromagnetic gas dynamics

V. Ts. Gurovich and L. S. Solov'ev

Physics Institute, Kirgiz Academy of Sciences
(Submitted 19 March 1986)
Zh. Eksp. Teor. Fiz. **91**, 1144–1159 (October 1986)

Relativistic electromagnetic gas dynamics provides the best mathematical formalism for the solution of a large class of physical problems involving the description of objects that contain large ensembles of particles interacting with strong gravitational and electromagnetic fields. Usually, the equations of gas dynamics are derived from kinetics, and are restricted to special problems. In the present paper we consider the derivation of a general system of equations of relativistic electromagnetic gas dynamics in the framework of the general and special theories of relativity. Like classical gas dynamics, the equations of relativistic electromagnetic gas dynamics contain transport coefficients, which are assumed to be known functions of the thermodynamic variables. Dissipative processes and processes associated with interaction of gases are described by appropriate energy-momentum tensors. The Lagrangian and Hamiltonian forms of the equations of dissipationless relativistic electromagnetic gas dynamics are obtained, and a system of equations for symmetric steady flows is derived.

INTRODUCTION

For the solution of a large class of physical problems involving the description of objects that contain large ensembles of particles interacting with strong gravitational and electromagnetic fields, relativistic electromagnetic gas dynamics provides the best mathematical formalism. Currently topical are above all the equations of two-fluid relativistic electromagnetic gas dynamics, which describe the dynamics of a plasma and relativistic beams of charged particles in the framework of the special theory of relativity. The formalism of relativistic electromagnetic gas dynamics in the general theory of relativity[1] is of independent interest for the investigation of electromagnetic processes in compact astrophysical objects, in particular, for pulsar physics.

The equations of relativistic electromagnetic gas dynamics are usually derived from kinetic equations (see, for example, Refs. 2–4), but in the general case such an approach is rather complicated and leads to perspicuous results only for special problems and when a number of simplifying assumptions is made. At the same time, the required phenomenological equations are essentially contained in the equations of Einstein's general relativity,[1] which establish a general connection between the metric tensor and the matter energy-momentum tensor, and in the energy-momentum conservation equations that follow from them. The explicit expressions for the covariant energy-momentum tensors of the matter and the electromagnetic field introduced in Ref. 1 lead to a system of gas-dynamic equations for ideal noninteracting gases. The presence of equilibrium radiation, and also the influence of dissipative processes and processes associated with the interaction of gases and radiative energy losses in elastic particle collisions can be taken into account by adding to the right-hand side of the equations of general relativity the energy-momentum tensors that describe these processes. The superposition principle employed here is due to the fact that any energy contributes to the space-time curvature, and this curvature is manifested as a gravitational field.

Thus, to obtain the complete system of equations of relativistic electromagnetic gas dynamics it is sufficient for all the processes that are to be taken into account to be represented by symmetric covariant energy-momentum tensors. One then obtains simultaneously both the equations of motion of the gas (momentum conservation) as well as the equations for the heat release (energy conservation). To write down the complete system of gas-dynamic equations in the framework of the special theory of relativity it is obviously sufficient to know the expression for the 4-vector divergence of the total energy-momentum tensor.

The system of equations of phenomenological gas dynamics obtained in this manner is a direct generalization of classical gas dynamics and, like it, contains transport coefficients that are assumed to be known functions of the thermodynamic variables. Similarly, the closed system of equations of relativistic electromagnetic gas dynamics is a generalization of classical magnetohydrodynamics to the two-fluid (electrons and ions) relativistic case. It should be noted that a consistent relativistic description of plasma dynamics is possible only in two-fluid relativistic electromagnetic gas dynamics.

In the present paper, we consider the dissipative tensors of the viscosity, thermal conductivity, radiative thermal conductivity, electrical conductivity, and heat transfer between the ion and electron gases. The description of the last two processes of interaction of the gases by the introduction of the corresponding energy-momentum tensors is given for the first time. The radiative energy losses in inelastic collisions can, like the release of energy due to a change of the rest mass, be taken into account by introducing as a source a given scalar radiation power in the equation of rest mass conservation, whereas to take into account radiation in elastic collisions it is necessary to add to the energy-momentum conservation law the covariant 4-vector of the force that then arises.

In Sec. 1, we consider the gas-dynamic equations that follow from general relativity. We derive the total energy-momentum tensor of matter in conjunction with equilibrium isotropic radiation and the thermodynamic functions it contains for ideal and degenerate gases.

In Sec. 2, we consider the gas-dynamic equations for a neutral gas. We derive the mass-entropy conservation equation, which decomposes, for known power of the energy release due to change in the rest mass, into two equations, these generalizing the conservation laws for the matter and entropy in Newtonian gas dynamics. We obtain generally covariant expressions for the tensors of the viscosity, thermal conductivity, and radiative thermal conductivity. The equations of dissipationless gas dynamics are given in Lagrangian and Hamiltonian forms for an arbitrary metric.

In Sec. 3, we consider symmetric steady flows in the dissipationless gas dynamics of a neutral gas. In this case, the system of gas-dynamic equations in a given gravitational field reduces to a single equation for the flow function Ψ, this equation containing three arbitrary functions of Ψ: the entropy, the energy, and a generalized momentum along a cyclic coordinate.

In Sec. 4, we consider the equations of two-fluid relativistic electromagnetic gas dynamics that describe the motion of electron and ion gases in the framework of general relativity. To obtain a complete system, we must augment these equations by Einstein's equations for the gravitational field and the general relativistic generalizations of Maxwell's equations for the electromagnetic field. The influence of the finite electrical conductivity and heat transfer between the two components of the charged gases is taken into account by the introduction of the 4-vectors of the divergences of the corresponding energy-momentum tensors. As in Sec. 2, the equations of dissipationless relativistic electromagnetic gas dynamics can also be represented in Lagrangian and Hamiltonian forms.

In Sec. 5, we consider generally relativistic symmetric flows in dissipationless two-fluid relativistic electromagnetic gas dynamics.

Section 6 is devoted to a more detailed exposition of two-fluid relativistic electromagnetic gas dynamics in the framework of the special theory of relativity.

In the Appendix we consider a steady cylindrical flow of plasma, taking into account ohmic resistivity and bremsstrahlung. This problem generalizes the Pease-Braginskiĭ problem[5,6] to the case of two-fluid relativistic electromagnetic gas dynamics.

1. THE EQUATIONS OF THE GENERAL THEORY OF RELATIVITY

The equations of relativistic gas dynamics in an arbitrary metric defined by $x^j = (ct, x^\alpha)$, $ds^2 = g_{ik} dx^i dx^k$ are contained in Einstein's equations[1]:

$$R_i^k - {}^1/_2 \delta_i^k R_l^l = \varkappa \hat{T}_i^k,$$

$$\hat{T}^k_{i;k} \equiv (-g)^{-1/2} \frac{\partial}{\partial x^k} (-g)^{1/2} \hat{T}_i^k - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} \hat{T}^{kl} = 0. \qquad (1)$$

Here, R_i^k is the curvature tensor, and $\hat{T}_i^k$ is the total energy-momentum tensor, since any energy contributes to the space-time curvature, which itself is manifested as a gravitational field. The second equation in (1) is a consequence of the first and expresses the energy-momentum conservation law. To describe processes in the framework of relativistic electromagnetic gas dynamics, the tensor $\hat{T}_i^k$ is represented as a sum:

$$\hat{T}_i^k = T_i^k + \mathcal{T}_i^k + \hat{t}_i^k, \qquad (2)$$

where T_i^k is the material energy-momentum tensor of the matter in conjunction with isotropic radiation, $\mathcal{T}_i^k$ is the energy-momentum tensor of the electromagnetic field, and by $\hat{t}_i^k$ we have denoted the sum of the energy-momentum tensors that describe the various dissipative processes.

The gas-dynamic matter energy-momentum tensor has the form[1,7]

$$T_i^k = \rho W u_i u^k - p \delta_i^k, \qquad T_{ik} = \rho W u_i u_k - p g_{ik}, \qquad (3)$$

where $u^i = dx^i/ds$, and ρ and p are the rest mass density and pressure in the comoving frame. The enthalpy W and entropy S per unit rest mass are determined by

$$W = c^2 + \mathscr{E} + p/\rho, \qquad T dS = dW - dp/\rho, \qquad (4)$$

where T is the temperature and $\mathscr{E}$ is the internal energy per unit rest mass of the matter; T_{ik} is a symmetric tensor constructed from the metric tensor g_{ik}, the 4-vector u^i, and the scalar functions ρW and p.

For an ideal gas of particles with effective rest mass m that is in equilibrium with isotropic radiation (a photon gas), the thermodynamic functions in T_{ik} can be expressed by the formulas

$$p = nT + {}^1/_3 aT^4, \quad \rho\mathscr{E} = nT/(\gamma-1) + aT^4 \quad (a = \pi^2 k^4 / 15 c^3 \hbar^3), \qquad (5)$$

where $n = \rho/m$ is the concentration in the rest frame, and γ is the specific-heat ratio. In accordance with (5) the material tensor T_{ik}, defined in (3), is the sum of the tensors of the matter and the equilibrium radiation.

For a degenerate Fermi gas with $nT/p \ll 1$, we have in accordance with Ref. 8

$$\rho = \lambda x^3, \quad p = \lambda c^2 f(x), \; \rho\mathscr{E} = \lambda c^2 \{x^3[(1+x^2)^{1/2} - 1] - f(x)\},$$
$$W = c^2 (1+x^2)^{1/2}, \qquad (6)$$

where

$$8 f(x) = 3 \operatorname{Arsh} x - x(1+x^2)^{1/2}(3 - 2x^2), \qquad \lambda = m^4 c^3 / 3\pi^2 \hbar^3,$$

where $mcx = p_F$ is the limiting Fermi momentum. The resulting equation of state $p = p(\rho)$ is a subsidiary condition in the system of gas-dynamic equations contained in (1).

2. GAS DYNAMICS OF A NEUTRAL GAS

In the absence of electric charges and currents, the energy-momentum tensor $\hat{T}_i^k$ is determined by (2), in which $\mathcal{T}_i^k = 0$. By means of identity transformations the expression for $\hat{T}^k_{i;k}$ can be reduced to the form

$$\hat{T}_{i;k}^{k}=\rho\frac{d}{ds}Wu_i-\frac{\partial p}{\partial x^i}+u_iW(\rho u^k)_{;k}-\frac{1}{2}\rho Wu^ku^l\frac{\partial g_{kl}}{\partial x^i}+\hat{t}_{i;k}^{k}, \quad (7)$$

where

$$\frac{d}{ds}=u^i\frac{\partial}{\partial x^i},\quad (\rho u^k)_{;k}=(-g)^{-1/2}\frac{\partial}{\partial x^k}(-g)^{1/2}\rho u^k.$$

Multiplying (7) by u^i, summing over i, and taking into account $u_iu^i=1$, we obtain

$$u^i\hat{T}_{i;k}^{k}=W(\rho u^k)_{;k}+\rho T\frac{dS}{ds}+\rho W\left(u^i\frac{du_i}{ds}-\frac{u^ku^l}{2}\frac{dg_{kl}}{ds}\right)+u^i\hat{t}_{i;k}^{k}. \quad (8)$$

It is easy to show that the expression in the round brackets is zero, and, therefore, in accordance with the second equation of (1), we obtain the mass-entropy conservation law:

$$W(\rho u^k)_{;k}+\rho T\,dS/ds+u^i\hat{t}_{i;k}^{k}=0. \quad (9)$$

This equation was obtained in the framework of special relativity without allowance for dissipative terms in Ref. 9.

Since the physical meaning of the expression $c(\rho u^k)_{;k}$, which represents the change in the rest mass, is clear, Eq. (9) expresses the general law of conservation of thermal energy with allowance for the transformation into heat of the energy released in nuclear or chemical reactions. For known energy production $\rho\varepsilon(n,T)$ in a cubic centimeter per second Eq. (9) decomposes into two equations[10]:

$$(\rho u^k)_{;k}=-\frac{\rho\varepsilon}{c^3},\quad \rho T\frac{dS}{ds}=\frac{\rho\varepsilon W}{c^3}-u^i\hat{t}_{i;k}^{k}, \quad (10)$$

which generalize, respectively, the laws of matter and entropy conservation in Newtonian gas dynamics. It is also obviously necessary to include in $\rho\varepsilon$ the changes in the energy in arbitrary inelastic collisions of the particles. With allowance for (7) and (10), the energy-momentum conservation equation takes the form

$$\rho\frac{d}{ds}Wu_i=\frac{\partial p}{\partial x^i}+\frac{\rho\varepsilon}{c^3}Wu_i+\frac{1}{2}\rho Wu^ku^l\frac{\partial g_{kl}}{\partial x^i}-\hat{t}_{i;k}^{k}. \quad (11)$$

Equations (10) and (11) in conjunction with Eqs. (1), which determine the gravitational field g_{ik}, form the complete system of equations of the relativistic gas dynamics of a neutral gas in the general theory of relativity. The energy conservation law, which is expressed by the time component of Eq. (11), is a consequence of the remaining equations of the system (10)–(11). Use of the identity

$$u^ku_{i;k}=du_i/ds-{}^1/_2u^ku^l\partial g_{kl}/\partial x^i$$

makes it possible to represent Eq. (11) in the form[7]

$$\rho Wu^ku_{i;k}=\partial p/\partial x^i-u_i\,dp/ds+t_{i;k}^{k}-u_iu^kt_{k;l}^{l}, \quad (12)$$

which expresses the change in the 4-velocity u_i and does not contain the term $\rho\varepsilon$.

It is interesting to note that for a degenerate gas (6) and in the absence of dissipative processes the mass-entropy conservation equation yields the equation

$$(nu^k)_{;k}=(1-4p/\rho W)\,ndm/mds, \quad (13)$$

from which it follows that when energy release is due to the "burning" of rest mass, $dm/ds<0$, the particle number n increases, this process being saturated in the ultrarelativistic limit, when $\rho W\to 4p$.

To take into account the dissipative processes in Eqs. (1), (10), and (11), it is necessary to substitute the energy-momentum tensors that describe the corresponding processes. We give explicit expressions for the energy-momentum tensors for the thermal conductivity, radiative thermal conductivity, and viscosity.

The thermal conductivity tensor obtained in Refs. 11 and 12 in the framework of the special theory of relativity, which can be represented in the form

$$t_{ik}=\chi c\left(u_i\frac{\partial T}{\partial x^k}+u_k\frac{\partial T}{\partial x^i}-\frac{1}{T}\frac{d}{ds}u_iu_kT^2+T\frac{dg_{ik}}{ds}\right), \quad (14)$$

where χ is the coefficient of thermal conductivity, has the form of a covariant symmetric 4-tensor and, therefore, is an energy-momentum tensor in general relativity. At the same time, $u^iu^kt_{ik}=0$ as[11,12] in the special theory of relativity.

For the radiative thermal conductivity in the framework of the special theory the tensor has the form[13]

$$t_{ik}=\frac{4a}{15\nu\rho T}\left\{\frac{\partial u_iT^5}{\partial x^k}+\frac{\partial u_kT^5}{\partial x^i}+\frac{\partial}{\partial x^l}[(g_{ik}-6u_iu_k)u^lT^5]\right\},$$

where ν is the absorption coefficient and

$$u^iu^kT_{ik}=-(4aT^4/3\nu\rho)(\partial u^l/\partial x^l+3dT/Tds).$$

The generalization to general relativity is achieved by replacing the ordinary derivatives by the covariant derivatives. Using then the formula

$$u_{i;k}=\partial u_i/\partial x^k-\Gamma_{ik}^{l}u_l$$

to transform the obtained expression, we find the covariant tensor

$$t_{ik}=\frac{4a}{15\nu\rho T}\left\{\left(g_{il}\frac{\partial}{\partial x^k}+g_{kl}\frac{\partial}{\partial x^i}\right)T^5u^l+[(g_{ik}-6u_iu_k)T^5u^l]_{;l}+6T^5\frac{dg_{ik}}{ds}\right\} \quad (15)$$

which satisfies the condition

$$u^iu^kT_{ik}=-(4aT^4/3\nu\rho)(u_{;l}^{l}+3dT/Tds).$$

In accordance with Refs. 7, 11, and 12, the viscosity tensor can be represented in the framework of the special theory in the form

$$t_{ik}=\eta c\left(\frac{\partial u_i}{\partial x^k}+\frac{\partial u_k}{\partial x^i}-\frac{d}{ds}u_iu_k\right)+c\left(\xi-\frac{2}{3}\eta\right)(g_{ik}-u_iu_k)\frac{\partial u^l}{\partial x^l},$$

where η and ξ are the coefficients of shear and bulk viscosity. The generalization to general relativity is done in the same way as in (15) and for $u^iu^kt_{ik}=0$ gives

$$t_{ik}=\eta c\left\{\left(g_{il}\frac{\partial}{\partial x^k}+g_{kl}\frac{\partial}{\partial x^i}\right)u^l+\frac{d}{ds}(2g_{ik}-u_iu_k)+c\left(\xi-\frac{2}{3}\eta\right)(g_{ik}-u_iu_k)u_{;l}^{l}\right\}. \quad (16)$$

In dissipationless gas dynamics the equations of motion can be represented in the Lagrangian form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha}-\frac{\partial L}{\partial x^\alpha}=\frac{mT}{u^0}\frac{\partial S}{\partial x^\alpha},\qquad L=\frac{mW}{u^0},\tag{17}$$

which can be readily verified by direct substitution in Eq. (11), setting $\rho\varepsilon=0$, $\hat{t}_i^k=0$.

Introducing the 4-vector of the generalized momentum $\mathscr{P}_k=(-\mathscr{H}/c,\ \mathscr{P}_\alpha)=-mWu_k/c$, where $\mathscr{P}_\alpha=\partial L/\partial\dot{x}^\alpha$, $\mathscr{H}=\dot{x}^\alpha\partial L/\partial\dot{x}^\alpha-L$, we obtain in accordance with (10) and (11) the complete system of equations in Hamiltonian form:

$$\frac{\partial}{\partial x^k}(-g)^{1/2}\rho u^k=0,\qquad \frac{dS}{dt}=0,$$
$$\frac{d\mathscr{P}_k}{dt}=-\frac{\partial\mathscr{H}}{\partial x^k}+\frac{mT}{u^0}\frac{dS}{\partial x^k}.\tag{18}$$

It follows from this that in the presence of symmetry ($\partial/\partial x^3=0$) and for steady flows ($\partial/\partial t=0$) not only the entropy S but also $\mathscr{P}_3$ and $\mathscr{H}$, respectively, are integrals of the motion (functions frozen into the matter).

Introduction of the antisymmetric tensor

$$K_{ik}=\partial\mathscr{P}_k/\partial x^i-\partial\mathscr{P}_i/\partial x^k\tag{19}$$

and use of the equation $\dot{x}^\alpha\partial\mathscr{P}_\alpha/\partial x^\beta=0$, which follows from (18), make it possible to represent the equations of motion in the form

$$\frac{\partial\mathscr{P}_\alpha}{\partial t}-\dot{x}^\beta K_{\alpha\beta}=-\frac{\partial\mathscr{H}}{\partial x^\alpha}+\frac{mT}{u^0}\frac{\partial S}{\partial x^\alpha}.\tag{20}$$

In the case of isentropic processes, $S=\text{const}$, it follows from (18)–(20) that

$$(-g)^{1/2}\rho u^0\frac{d}{dt}\frac{K_{\alpha\beta}}{(-g)^{1/2}\rho u^0}$$
$$=\left(K_{\beta\gamma}\frac{\partial}{\partial x^\alpha}+K_{\gamma\alpha}\frac{\partial}{\partial x^\beta}+K_{\alpha\beta}\frac{\partial}{\partial x^\gamma}\right)\dot{x}^\gamma.\tag{21}$$

This equation generalizes the equation, well known in classical hydrodynamics, which expresses the freezing of the streamlines of the vector curl **v**. In the special case of symmetric flow, $\partial/\partial x^3=0$, and if only the one component K_{12} is nonzero, we obtain one further conservation law:

$$\frac{d}{dt}\frac{K_{12}}{(-g)^{1/2}\rho u^0}=0.\tag{22}$$

3. SYMMETRIC STEADY FLOWS

In the case when there is no dependence on one of the spatial coordinates, $\partial/\partial x^3=0$, and on the time, $\partial/\partial t=0$, the existence of the integrals S, $\mathscr{P}_3$ and $\mathscr{H}$ makes it possible to simplify appreciably the system of equations of dissipationless gas dynamics. In this case, using the continuity equation (18), we introduce the flow function Ψ:

$$(-g)^{1/2}nu^1=\partial\Psi/\partial x^2,\qquad (-g)^{1/2}nu^2=-\partial\Psi/\partial x^1,\tag{23}$$

which is also an integral of the motion: $d\Psi/dt=0$. Accordingly, the integrals S, $\mathscr{P}_3$ and $\mathscr{H}$ will be arbitrary functions of the single argument Ψ. In accordance with the definition (19),

$$K_{13}=\partial\mathscr{P}_3/\partial x^1=\mathscr{P}_3'(\Psi)\partial\Psi/\partial x^1,$$
$$K_{23}=\partial\mathscr{P}_3/\partial x^2=\mathscr{P}_3'(\Psi)\partial\Psi/\partial x^2\tag{24}$$

and Eqs. (20) reduce to the single equation

$$K_{12}/(-g)^{1/2}n-u^3\mathscr{P}_3'(\Psi)=-u^0\mathscr{H}'(\Psi)+mTS'(\Psi),\tag{25}$$

which contains three arbitrary functions of Ψ.

We transform Eq. (25) further in such a way that it contains the contravariant component K^{12} and the covariant component u_3. Following Ref. 14, we introduce the three-dimensional tensor $q_{\alpha\beta}$ and three-dimensional vector g^α:

$$q^{\alpha\beta}=-g^{\alpha\beta},\quad g^\alpha=-g^{0\alpha},\quad q_{\alpha\beta}=-g_{\alpha\beta}+g_{00}g_\alpha g_\beta,\quad g=-g_{00}q,\tag{26}$$

where q is the determinant of $q_{\alpha\beta}$. The components K^{12} and u_3 can be represented in the form

$$\bar{K}^{12}=K^{12}-q^{-1/2}[\mathbf{g}\nabla\mathscr{H})_3=q^{-1}(q_{31}K_{23}+q_{32}K_{31}+q_{33}K_{12}),$$
$$\bar{u}_3=-u_3-u_0g_3=q_{31}u^1+q_{32}u^2+q_{33}u^3.\tag{27}$$

Substitution of these expressions in Eq. (25) gives ($h=g_{00}$)

$$\bar{K}^{12}/q_{33}nh^{1/2}-\frac{\bar{u}_3}{q_{33}}\mathscr{P}_3'(\Psi)+u^0\mathscr{H}'(\Psi)-mTS'(\Psi)=0.\tag{28}$$

Calculating $\bar{K}^{12}$ and adding to (28) the conservation equations for the angular momentum and the energy (which generalizes the Bernoulli equation), we obtain the system of equations for dissipationless steady symmetric flows:

$$m\bar{i}_3/q_{33}nh^{1/2}-(\bar{u}_3/q_{33})\mathscr{P}_3'(\Psi)+u^0\mathscr{H}'(\Psi)-mTS'(\Psi)=0,$$
$$mWu_3=-\mathscr{P}_3(\Psi),\qquad mWu_0=\mathscr{H}(\Psi).\tag{29}$$

Here, $\bar{i}_3$ is expressed by a differential operator:

$$\bar{i}_3=-q_{33}\,\mathrm{div}(W\nabla\Psi/q_{33}nh^{1/2})+(W\bar{u}_3/q_{33})\mathbf{e}_3\,\mathrm{rot}\,\mathbf{e}_3+\mathbf{e}_3\,\mathrm{rot}\,Wu_0\mathbf{g},\tag{29a}$$

in which $\mathbf{e}_\alpha$ are the coordinate vectors of a curvilinear coordinate system in which $dl^2=q_{\alpha\beta}dx^\alpha dx^\beta$, $\mathbf{a}=\mathbf{e}_\alpha a^\alpha$. The components u_0 and u^0 are related by

$$u_0=hu^0-\frac{h}{q_{33}}\left(g_3\bar{u}_3-\frac{1}{nh^{1/2}}[\mathbf{g}\mathbf{e}_3]\nabla\Psi\right).\tag{29b}$$

The contravariant components u^α are expressed in terms of the gas-dynamic velocities $v^\alpha=\dot{x}^\alpha$ by the formulas $u^\alpha=v^\alpha/u^0$, $u^0=c(g_{ik}\dot{x}^i\dot{x}^k)^{-1/2}$.

Equations (29) and (29a) simplify appreciably in the case of an orthogonal metric with $g_{0\alpha}=0$; then $u_0=hu^0$, and the first equation in (29) for the flow function Ψ takes the form

$$\frac{m}{nh^{1/2}}\,\mathrm{div}\frac{W\nabla\Psi}{g_{33}nh^{1/2}}-\frac{u_3}{g_{33}}\mathscr{P}_3'(\Psi)+u^0\mathscr{H}'(\Psi)-mTS'(\Psi)=0.\tag{30}$$

4. ELECTROMAGNETIC GAS DYNAMICS

Whereas the motion of a neutral gas can usually be described effectively in the framework of single-fluid gas dynamics, the adequate macroscopic description of the motion of charged gases in an electromagnetic field requires use of two-fluid gas dynamics.

The electrodynamic equations are generalized in general relativity[1] by introducing the 4-vectors $j^i = (\rho_e, j^\alpha)$, $A^i = (\varphi, A^\alpha)$ and the antisymmetric electromagnetic field tensor F_{ik}:

$$j^i = c\sum enu^i, \quad F_{ik} = \partial A_k/\partial x^i - \partial A_i/\partial x^k, \tag{31}$$

where φ and A^α are the scalar and vector potentials, ρ_e and j^α are the charge and electric current densities, and there is summation over the species of gases, the individual particles of which have charges $e_\pm$. The electromagnetic field is described by the Maxwell equations

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0, \quad (-g)^{-1/2}\frac{\partial}{\partial x^k}(-g)^{1/2}F^{ik} = -\frac{4\pi}{c}j^i, \tag{32}$$

the first of which is a consequence of (31). As can be seen from (31) and (32), the electromagnetic field is determined by the total current in the same way as the gravitational field in (1) is determined by the sum of all energies. The conservation law

$$j^k_{;k} \equiv (-g)^{-1/2}\frac{\partial}{\partial x^k}(-g)^{1/2}j^k = 0 \tag{33}$$

for the total charge is a consequence of the second equation of (32).

The electromagnetic field energy-momentum tensor and its covariant derivative are, respectively,

$$\mathcal{T}_i^k = -\frac{1}{4\pi}\left(F^{kl}F_{il} - \frac{1}{4}\delta_i^k F^{lm}F_{lm}\right),$$

$$\mathcal{T}^k_{i;k} = -\frac{1}{c}j^k F_{ik}. \tag{34}$$

The energy-momentum conservation law (1) requires the vanishing of the covariant derivative of the total tensor $\hat{T}_i^k$. For example, for a mixture of neutral and positively and negatively charged dissipationless noninteracting gases we obtain with allowance for (34)

$$^0T^k_{i;k} + {}^+T^k_{i;k} + {}^-T^k_{i;k} = c^{-1}j^k F_{ik} \equiv c^{-1}(j_+{}^k + j_-{}^k)F_{ik}, \tag{35}$$

where $j^k_\pm = ce_\pm n_\pm u^k_\pm$ are the 4-currents of the positive and negative charges. To obtain the equations of motion of each individual gas species, we must obviously require fulfillment of the partial equations

$$^0T^k_{i;k} = 0, \quad {}^+T^k_{i;k} = c^{-1}j_+{}^k F_{ik}, \quad {}^-T^k_{i;k} = c^{-1}j_-{}^k F_{ik}. \tag{36}$$

Adding to Eqs. (10) and (11) the electromagnetic forces $\mathcal{T}^k_{i;k}$ and bearing in mind that, the tensor F_{ik} being antisymmetric, the electromagnetic field does not contribute to the mass-entropy conservation law (9), since; $u^i\mathcal{T}^k_{i;k} = 0$, we arrive at the system of equations of electromagnetic gas dynamics for each gas species separately:

$$(\rho u^k)_{;k} = -\frac{\rho\varepsilon}{c^3}, \quad \rho T\frac{dS}{ds} = \frac{\rho\varepsilon}{c^3}W - u^i\hat{t}_i,$$

$$\rho\frac{d}{ds}Wu_i = \frac{\partial p}{\partial x^i} + \frac{\rho\varepsilon}{c^3}Wu_i + \frac{1}{2}\rho Wu^k u^l\frac{\partial g_{kl}}{\partial x^i} + enu^k F_{ik} - \hat{t}_i. \tag{37}$$

Here, the 4-vector $\hat{t}_i = \hat{t}^k_{i;k}$ describes the total effect of the dissipative processes on the given gas. In general relativity, the system (37) must be augmented by Eqs. (1), which determine the gravitational field g_{ik}, and Eqs. (32), which determine the electromagnetic field F_{ik}.

The equations of motion can also be represented in the form (12)

$$\rho Wu^k u_{i;k} = \partial p/\partial x^i - u_i dp/ds + enu^k F_{ik} - \hat{t}_i + u_i u^k \hat{t}_k. \tag{38}$$

In the second equation of (37), the right-hand side can be interpreted as the release of heat in each gas due to the change in the rest mass and the effect of the dissipative forces. If the rest mass does not change, $\rho\varepsilon = 0$, then the continuity equations in (37) express also charge conservation in each gas, and, accordingly, the conservation equation (33) for the total charge is a consequence of (37). In the case when the rest mass does change, we obtain from (33) and (37) for two charged gases the relation

$$(\rho\varepsilon)_+ e_+/m_+ + (\rho\varepsilon)_- e_-/m_- = 0.$$

From this in particular there follows the natural result that when charged particles of equal masses and opposite charges are annihilated the same amount of energy is released in both gases.

The influence of the finite electrical conductivity σ of the plasma due to the friction of the electrons on the ions is described by the conductivity tensor t_i^k, whose covariant derivative $t_i = t^k_{i;k}$ has the form[15]

$$t_i = \mp ce_+e_-n_+n_-\sigma^{-1}[u_i{}^+ - u_i{}^- - \alpha(1 - u_k{}^+u_-{}^k)u_i],$$

$$\alpha \equiv (m_+ - m_-)/(m_+ + m_-). \tag{39}$$

The expression (39) has covariant form, and in the nonrelativistic limit and for $n_+e_+ = -n_-e_-$ it leads to the well-known expressions for the forces that act on gases of positively and negatively charged particles and for the heat released in them.[16] The expression for the friction force on the right-hand side of Eq. (38),

$$t_i^* = t_i - u_i u^k t_k = \mp ce_+e_-n_+n_-\sigma^{-1}[u_i{}^+ - u_i{}^- - (u_k{}^+ - u_k{}^-)u^k u_i], \tag{40}$$

does not depend on the masses of the ions and electrons and is symmetric with respect to transposition of the indices $\pm$.

The amount of heat released in the ion and electron gases (in 1 cm^3/sec) due to Joule heating is determined in accordance with (37) and (39) by the formulas

$$Q_\pm = -cu^i t_i = \frac{|e_+e_-|}{\sigma}\frac{n_+n_-m_\mp c^2}{m_+ + m_-}(u_+{}^i u_i{}^- - 1) \geq 0. \tag{41}$$

The total heat release, $Q = Q_+ + Q_-$, does not depend on the masses $m\pm$.

In the special case $m_+ = m_-$, $n_+ = n_-$, $e_+ = -e_- = e$, the 4-vector t_i can be expressed in terms of j_i:

$$t_i = \pm ce^2n^2\sigma^{-1}(u_i{}^+ - u_i{}^-) = \pm en\sigma^{-1}j_i. \tag{42}$$

In the framework of special relativity, the 4-vector j_i can be represented as the derivative of a mixed tensor, $j_i = \Pi^k_{i;k}$, where the symmetric tensor Π_{ik} can be expressed in terms of the potentials of the electromagnetic field:

$$\Pi_{ik}=\partial A_k/\partial x^i+\partial A_i/\partial x^k-2g_{ik}\partial A^l/\partial x^l.$$

Writing Π_{ik} in covariant form, we obtain

$$\Pi_{ik}=A_{k;i}+A_{i;k}-2g_{ik}A^l_{;l}=\left(g_{kl}\frac{\partial}{\partial x^i}+g_{il}\frac{\partial}{\partial x^k}-2g_{ik}\frac{\partial}{\partial x^l}\right)\times A^l+gA^l\frac{\partial}{\partial x^l}\frac{g_{ik}}{g}. \qquad (43)$$

Accordingly, the covariant tensor of the electrical conductivity can be represented in the case (42) in the form

$$t_{ik}=\pm en\sigma^{-1}\Pi_{ik}. \qquad (44)$$

A second effect associated with the finite conductivity is the heat transfer between the two components of the plasma; this leads to an equalization of their temperatures $T_\pm$. The corresponding heat transfer process can be described by the covariant derivative

$$t_i=\mp\frac{32e_+e_-n_+n_-}{\pi c\sigma}\frac{T_+-T_-}{m_++m_-}u_i. \qquad (45)$$

In accordance with (37), the heat releases in the ion and electron gases are determined by the expressions

$$Q_\pm=-cu^it_i=\pm\frac{32e_+e_-n_+n_-}{\pi\sigma}\frac{T_+-T_-}{m_++m_-}, \qquad (46)$$

which are identical to the nonrelativistic expressions given in Ref. 17. The expression for σ at nonrelativistic temperatures has the form[17]

$$\sigma=\frac{2^{5/2}}{\pi^{3/2}L}\frac{\mu}{|e_+e_-|}\left(\frac{T_+}{m_+}+\frac{T_-}{m_-}\right)^{3/2},\quad \mu=\frac{m_+m_-}{m_++m_-}, \qquad (47)$$

where L is the Coulomb logarithm. For the generalization to the case of relativistic temperatures, see Ref. 18.

The ratio of the transfer heat release Q_2 to the Joule heat Q_1,

$$\frac{Q_2}{Q_1}=\mp\frac{16}{\pi m_\mp c^2}\frac{T_+-T_-}{u_+{}^iu_i{}^--1}, \qquad (48)$$

does not depend on σ and is determined by the difference between the temperatures and the relative velocity of the charged gases.

The energy and momentum losses due to radiation in elastic collisions of the particles can, like the heat transfer, be taken into account by introducing the 4-vector $t_i=Gu_i/c$, where $G_\pm$ is the power of the emission from 1 cm^3 of each gas. For example, in the case of untrapped bremsstrahlung at nonrelativistic temperatures

$$G_\pm=\frac{m_\mp}{m_++m_-}\frac{8\pi n_+n_-e^6}{3c^3\mu\hbar}\left(\frac{T_+}{m_+}+\frac{T_-}{m_-}\right)^{1/2}, \qquad (49)$$

where $e_+=-e_-=e$, and $\hbar$ is Planck's constant.

The generalized Maxwell equations can be represented in vector form[14] by using the metric (26). Following Ref. 14, we introduce the 3-vectors $\mathbf{E}$ and $\mathbf{D}$ and the 3-tensors $B_{\alpha\beta}$ and $H^{\alpha\beta}$:

$$E_\alpha=E_{0\alpha},\quad B_{\alpha\beta}=F_{\alpha\beta},\quad D^\alpha=-h^{1/2}F^{0\alpha},\quad H^{\alpha\beta}=h^{1/2}F^{\alpha\beta}.$$

The vectors $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{g}$ are related by the equations

$$\mathbf{D}=\mathbf{E}h^{-1/2}-[\mathbf{gH}],\quad \mathbf{B}=\mathbf{H}h^{-1/2}+[\mathbf{gE}]. \qquad (50)$$

Maxwell's equations (32) take the form

$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{div}\mathbf{D}=4\pi h^{1/2}\rho_e, \qquad (51)$$

$$\frac{1}{cq^{1/2}}\frac{\partial}{\partial t}q^{1/2}\mathbf{B}=-\operatorname{rot}\mathbf{E},\qquad \frac{1}{cq^{1/2}}\frac{\partial}{\partial t}q^{1/2}\mathbf{D}=\operatorname{rot}\mathbf{H}-\frac{4\pi}{c}h^{1/2}\mathbf{j}.$$

Here

$$h=g_{00},\quad g^\alpha=g^{0\alpha},\quad \rho_e=j^0=\sum enu^0,\quad j^\alpha=\sum cenu^\alpha,$$

and the vector operators div a and curl a are determined in curvilinear coordinates by

$$\operatorname{div}\mathbf{a}=q^{-1/2}\frac{\partial}{\partial x^\alpha}q^{1/2}a^\alpha,\qquad \operatorname{rot}\mathbf{a}=q^{-1/2}\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3}\\ a_1 & a_2 & a_3\end{vmatrix},$$

where $q=-g/h$ is the determinant of the third tensor $q_{\alpha\beta}$. From (51) we obtain a conservation equation for the total charge in vector form equivalent to (33):

$$c^{-1}q^{-1/2}\partial[(qh)^{1/2}\rho_e]/\partial t+\operatorname{div}h^{1/2}\mathbf{j}=0.$$

The equations of motion of dissipationless two-fluid electromagnetic gas dynamics can also be represented in Lagrangian and Hamiltonian forms:

$$\frac{d}{dt}\frac{\partial L}{\partial v^\alpha}-\frac{\partial L}{\partial x^\alpha}=\frac{mT}{u^0}\frac{\partial S}{\partial x^\alpha},\quad L=-\frac{mW}{u^0}-ev^kA_k, \qquad (52)$$

$$\frac{d\mathscr{P}_k}{dt}=-\frac{\partial\mathscr{H}}{\partial x^k}+\frac{mT}{u^0}\frac{\partial S}{\partial x^k},\quad v^\alpha=\frac{\partial\mathscr{H}}{\partial\mathscr{P}_\alpha}, \qquad (53)$$

where the 4-vector $\mathscr{P}_k$ of the generalized momentum and the Hamiltonian $\mathscr{H}$ are

$$\mathscr{P}_k=-mWu_k-eA_k,\quad \mathscr{H}=mWu_0+eA_0=-\mathscr{P}_0. \qquad (54)$$

Equations (52) and (53) are double sets of equations for the positively and negatively charged gases, and accordingly the indices $\pm$ are understood for all quantities. For dissipationless flows, the continuity equations and entropy conservation equation (18) must also be satisfied for both gases separately. When $\partial/\partial x^3=0$ and $\partial/\partial t=0$, there are two pairs of frozen functions $S_\pm$, $\mathscr{P}_3^\pm$ and $\mathscr{H}_\pm$ for each case. Further, since the definition of the antisymmetric tensor K_{ik} in (19) and the form of the individual equations (52) and (53) are the same as for the single-fluid equations (17) and (18) and do not depend on the explicit expressions for $\mathscr{P}_k$, it follows that for both gases Eqs. (20) are still valid and with them the equations (21) and (22) that follow from them for $S=\text{const}$, where $\mathscr{P}_k$ are now determined by the expressions (54), which contain the electromagnetic potentials A_k.

5. SYMMETRIC STEADY FLOWS IN TWO-FLUID ELECTROMAGNETIC GAS DYNAMICS

In the presence of symmetry, $\partial/\partial x^3=\partial/\partial t=0$, the 3-vectors $\mathbf{B}$, $h^{1/2}\mathbf{j}$, $h^{1/2}\mathbf{u}$, and $\mathbf{i}=\operatorname{curl}Wu^0\mathbf{v}$, whose divergences vanish, can be represented by the formulas

$$q_{33}\mathbf{B}=[\nabla\psi\mathbf{e}_3]+B_3\mathbf{e}_3,\qquad q_{33}\mathbf{j}=h^{-1/2}(\nabla\bar{H}_3\mathbf{e}_3)+\bar{j}_3\mathbf{e}_3. \qquad (55)$$

$$q_{33}\mathbf{u}=h^{-1/2}[\nabla\Psi\mathbf{e}_3]+\bar{u}_3\mathbf{e}_3,\qquad q_{33}\mathbf{i}=[\nabla W\bar{u}_3\mathbf{e}_3]+\bar{i}_3\mathbf{e}_3,$$

where in accordance with (50) and (51)

$$\bar{B}_3=\bar{H}_3 h^{-1/2}-[\mathbf{g}\nabla\varphi]_3, \quad \mathbf{E}=-\nabla\varphi, \tag{56}$$

and $\psi=\bar{A}_3$ is the flow function of the magnetic field **B**. The bar above a symbol denotes the covariant components of the 3-vectors in the curvilinear coordinate system, $dl^2=q_{\alpha\beta}dx^\alpha dx^\beta$, these being related to the components of the 4-vectors in the metric g_{ik} by $\bar{a}_\alpha=-a_0 g_\alpha-a_\alpha$.

The system of equations that describe two-fluid symmetric flows in an electromagnetic field can be derived in the same way as the single-fluid case (29) and represented in the form

$$\frac{m\bar{i}_3+e\bar{B}_3}{q_{33}nh^{1/2}}-\frac{\bar{u}_3}{q_{33}}\mathscr{P}_3'(\Psi)+u^0\mathscr{H}'(\Psi)=mTS'(\Psi),$$

$$j_3=\sum cen\bar{u}_3, \quad j^0=\sum enu^0, \quad H_3=\sum e\Psi, \tag{57}$$

$$\mathscr{P}_3(\Psi)=-mWu_3-e\psi, \quad \mathscr{H}(\Psi)=mWu_0+e\varphi.$$

Here, the first, fifth, and sixth equations are complexes of two equations for positively and negatively charged particles with masses $m_\pm$ and charges $e_\pm$. The first three equations form a system of differential equations for the functions $\Psi_\pm$, ψ, and φ, in which $\bar{i}_3$, $\bar{j}_3$, and j^0 are expressed by the differential operators

$$\bar{i}_3=-q_{33}\operatorname{div}\frac{W\nabla\Psi}{q_{33}nh^{1/2}}+\frac{W\bar{u}_3}{q_{33}}\mathbf{e}_3\operatorname{rot}\mathbf{e}_3+\mathbf{e}_3\operatorname{rot}Wu_0\mathbf{g},$$

$$h^{1/2}\bar{j}_3=-q_{33}\operatorname{div}\frac{h^{1/2}}{q_{33}}(\nabla\psi-g_3\nabla\varphi)+\frac{H_3}{q_{33}}\mathbf{e}_3\operatorname{rot}\mathbf{e}_3, \tag{57a}$$

$$h^{1/2}j^0=-q_{33}\operatorname{div}\left\{\frac{\nabla\varphi}{h^{1/2}}+\frac{h^{1/2}g_3}{g_{33}}(\nabla\psi-g_3\nabla\varphi)+\frac{H_3}{q_{33}}[\mathbf{g}\mathbf{e}_3]\right\},$$

and the components u_0 and u^0 are related by (29b).

The expressions (57a) simplify appreciably for an orthogonal metric g_{ik}, when $g^{0\alpha}=0$. In this case $q_{\alpha\beta}=-g_{\alpha\beta}$, $u_0=hu^0$, $\bar{a}_\alpha=-a_\alpha$, $\bar{B}_3=-H_3$

$$\bar{i}_3=-q_{33}\operatorname{div}(W\nabla\Psi/q_{33}nh^{1/2}), \quad h^{1/2}\bar{j}_3=-q_{33}\operatorname{div}(h^{1/2}\nabla\psi/q_{33}),$$

$$h^{1/2}j^0=-q_{33}\operatorname{div}(\nabla\varphi/h^{1/2}), \tag{57b}$$

so that $\bar{i}_3$, $\bar{j}_3$, and j^0 are represented by differential operators of second order acting, respectively, on the functions Ψ, ψ, and φ

6. TWO-FLUID ELECTROMAGNETIC GAS DYNAMICS IN SPECIAL RELATIVITY

In the framework of the special theory of relativity, the system of equations of two-fluid gas dynamics (for ion and electron gases) contains Maxwell's equations and the conservation equations for the rest mass, entropy, and momentum of each gas:

$$\operatorname{div}\mathbf{E}=4\pi\rho_e, \quad \operatorname{div}\mathbf{B}=0, \tag{58}$$

$$\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\operatorname{rot}\mathbf{B}-\frac{4\pi}{c}\mathbf{j}, \quad \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\operatorname{rot}\mathbf{E}, \tag{59}$$

$$\frac{\partial}{\partial t}\frac{\rho}{\Gamma}+\operatorname{div}\frac{\rho\mathbf{v}}{\Gamma}=-\frac{\rho\varepsilon}{c^2}, \quad \frac{\rho T}{\Gamma}\frac{dS}{dt}=\frac{\rho\varepsilon}{c^2}W-cu^i\hat{t}_i, \tag{60}$$

$$\frac{\rho}{\Gamma c^2}\frac{d}{dt}\frac{W\mathbf{v}}{\Gamma}=-\nabla p+\frac{en}{\Gamma}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}\mathbf{B}]\right)+\frac{\rho\varepsilon W}{\Gamma c^4}\mathbf{v}-\hat{\mathbf{t}}. \tag{61}$$

Here, $\hat{t}=t_{i;k}^{\ k}$ is the 4-divergence of the total dissipative tensor,

$$u^i=(1/\Gamma, \mathbf{v}/\Gamma c), \quad u_i=(1/\Gamma, -\mathbf{v}/\Gamma c), \quad \Gamma=(1-v^2/c^2)^{1/2},$$

$$W=c^2+\mathscr{E}+p/\rho, \quad TdS=dW-dp/\rho,$$

$$\rho_e=\sum(en/\Gamma), \quad \mathbf{j}=\sum(en\mathbf{v}/\Gamma).$$

Equations (58)–(61) constitute a complete system of equations of two-fluid relativistic electromagnetic gas dynamics provided (60) and (61) are the double set of equations for the ion and electron gases.

Maxwell's equations (58) and (59) contain the conservation laws for the total charge and energy of the electromagnetic field:

$$\frac{\partial\rho_e}{\partial t}+\operatorname{div}\mathbf{j}=0, \quad \frac{1}{2c}\frac{\partial}{\partial t}(E^2+B^2)+\operatorname{div}[\mathbf{E}\mathbf{B}]=-\frac{4\pi}{c}\mathbf{j}\mathbf{E}. \tag{62}$$

From Eqs. (61) there follow the energy conservation laws

$$\frac{\rho}{\Gamma}\frac{d}{dt}\frac{W}{\Gamma}=\frac{\partial p}{\partial t}+\frac{en}{\Gamma}\mathbf{v}\mathbf{E}+\frac{\rho\varepsilon}{c^2\Gamma}W-c\hat{t}_{0;k}^{\ k} \tag{63}$$

for each gas species. Summing (63) over the gas species and taking into account the mass-entropy conservation law (60) and the conservation law for the energy of the electromagnetic field (62), we obtain the conservation law of the total energy in the form

$$\frac{\partial}{\partial t}\left\{\sum\left(\frac{\rho W}{\Gamma^2}-p\right)+\frac{E^2+B^2}{8\pi}\right\}$$

$$+\operatorname{div}\left(\sum\frac{\rho W}{\Gamma^2}\mathbf{v}+\frac{c}{4\pi}[\mathbf{E}\mathbf{B}]\right)+c\sum\hat{t}_{0;k}=0. \tag{64}$$

The equation of motion (61), represented in the form (12), gives

$$\frac{\rho W}{c^2\Gamma}\frac{d}{dt}\frac{\mathbf{v}}{\Gamma}=-\nabla p-\frac{\mathbf{v}}{\Gamma^2c^2}\frac{dp}{dt}$$

$$+\frac{en}{\Gamma}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}\mathbf{B}]\right)+\hat{t}-(\hat{t}_iu^i)\mathbf{u}. \tag{65}$$

When allowance is made for the electrical conductivity (39) and the energy loss due to radiation (G, $\mathrm{erg\cdot cm^{-3}\cdot sec^{-1}}$), the relativistic electromagnetic gas-dynamic equations (60) and (61) can be written in the form

$$\frac{\partial}{\partial t}\frac{\rho}{\Gamma}+\operatorname{div}\frac{\rho\mathbf{v}}{\Gamma}=-\frac{\rho\varepsilon}{c^2}, \quad \frac{\rho T}{\Gamma}\frac{dS}{dt}=\frac{W}{c^2}(\rho\varepsilon-G)$$

$$+\frac{2e^2}{\sigma}\frac{n_+n_-m_\mp c^2}{m_++m_-}\left(\frac{1}{\Gamma'}-1\right), \tag{66}$$

$$\frac{\rho}{\Gamma c^2}\frac{d}{dt}\frac{W\mathbf{v}}{\Gamma}=-\nabla p+\frac{W}{\Gamma c^2}(\rho\varepsilon-G)\mathbf{v}\pm\frac{en}{\Gamma}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}\mathbf{B}]\right)$$

$$\mp\frac{e^2n_+n_-}{\sigma}\left\{\frac{\mathbf{v}_+}{\Gamma_+}-\frac{\mathbf{v}_-}{\Gamma_-}+\alpha\left(\frac{1}{\Gamma'}-1\right)\frac{\mathbf{v}}{\Gamma}\right\}, \tag{67}$$

where we have assumed $e_+=-e_-=e$,

$$\Gamma'=\Gamma_+\Gamma_-(1-\mathbf{v}_+\mathbf{v}_-/c^2]^{-1}=(1-v'^2/c^2)^{1/2},$$

$$v'=[(\mathbf{v}_+-\mathbf{v}_-)^2-[\mathbf{v}_+\mathbf{v}_-]^2/c^2]^{1/2}/(1-\mathbf{v}_+\mathbf{v}_-/c^2),$$

and Γ' is an invariant quantity that depends only on the relative velocity v' of the electron and ion gases, $\alpha=(m_+-m_-)/(m_++m_-)$. The right-hand side of the second equation of (66) determines the heat release in 1 cm^3/sec in each gas. In accordance with (41), the Joule heat release in the ion and electron gases is given by

$$Q^{\pm}=\frac{2c^2e^2}{\sigma}\frac{n_+n_-m_\mp}{m_++m_-}\left(\frac{1}{\Gamma'}-1\right). \qquad (68)$$

The total Joule heat $Q=Q^++Q^-$ does not depend on the masses of the ions and the electrons and in the nonrelativistic limit tends for $n_+=n_-$ to the well-known[19] expression $Q=j^2/\sigma$.

Substituting Eq. (67) in the form (65), we obtain

$$\frac{\rho W}{\Gamma c^2}\frac{d}{dt}\frac{\mathbf{v}}{\Gamma}=-\nabla p-\frac{\mathbf{v}}{c^2\Gamma^2}\frac{dp}{dt}\pm\frac{en}{\Gamma}\left(\mathbf{E}+\frac{1}{c}[\mathbf{vB}]\right)$$
$$\mp\frac{e^2n_+n_-}{\sigma\Gamma\Gamma_+\Gamma_-}\left(\mathbf{v}_+-\mathbf{v}_--\frac{1}{c^2}[\mathbf{v}[\mathbf{v}_+\mathbf{v}_-]]\right). \qquad (69)$$

In the case $n_+=n_-$, the condition of compensation of the friction force between the ion and electron gases by the acceleration in the electric field $\mathbf{E}$ leads, as follows from both Eqs. (69), to the relativistic generalization of Ohm's law:

$$\sigma\mathbf{E}=(en/\Gamma_+\Gamma_-)(\mathbf{v}_+-\mathbf{v}_-). \qquad (70)$$

If $n_+\neq n_-$, then the presence of an electric field gives rise to acceleration of the plasma if the conductivity σ is finite. Since this effect also exists in the nonrelativistic limit, we consider the simplest nonrelativistic case of homogeneous streams of charged particles with $e_+=-e_-=e$ in a constant homogeneous electric field $\mathbf{E}$. Then for $v_\pm^2/c^2\ll 1$ and at nonrelativistic temperatures Eqs. (69) can be written in the form

$$m_\pm dv_\pm/dt=\pm eE\mp(e^2n_\mp/\sigma)(v_+-v_-).$$

The solution of these equations

$$v_++v_-=v_+^0+v_-^0+\frac{\rho_+-\rho_-}{\rho_++\rho_-}V_0(1-e^{-\nu t})+2\frac{n_+-n_-}{\rho_++\rho_-}eEt,$$
$$v_+-v_-=\frac{m_++m_-}{\rho_++\rho_-}\frac{\sigma E}{e}$$
$$+V_0e^{-\nu t}\left(\rho_\pm=m_\pm n_\pm,\ \nu=\frac{\rho_++\rho_-}{m_+m_-}\frac{e^2}{\sigma}\right)$$

shows that in the limit $t\to\infty$ the relative velocity tends to a finite limit $e(\rho_++\rho_-)(v_+-v_-)=(m_++m_-)\sigma E$ (analog of Ohm's law), though the mean velocity increases unboundedly if $(n_+-n_-)E\neq 0$.

The equations of dissipationless relativistic electromagnetic gas dynamics can be conveniently represented in an arbitrary curvilinear spatial coordinate system, $dl^2=q_{\alpha\beta}dx^\alpha dx^\beta$, in the Lagrangian form (52):

$$\frac{d}{dt}\frac{\partial L}{\partial v^\alpha}-\frac{\partial L}{\partial x^\alpha}=mT\Gamma\frac{\partial S}{\partial x^\alpha},$$
$$L(x^\alpha,v^\alpha)=-\frac{m\Gamma}{c^2}W-e\left(\varphi-\frac{1}{c}\mathbf{vA}\right), \qquad (71)$$

and also in the form of the Hamilton equations (53):

$$\frac{d\mathcal{P}_\alpha}{dt}=-\frac{\partial\mathcal{H}}{\partial x^\alpha}+mT\Gamma\frac{\partial S}{\partial x^\alpha},\qquad v^\alpha=-\frac{\partial\mathcal{H}}{\partial\mathcal{P}_\alpha},$$
$$\mathcal{H}(x^\alpha,\mathcal{P}_\alpha)=\frac{mW}{\Gamma c^2}+e\varphi. \qquad (71a)$$

The electromagnetic field and the generalized momentum can be expressed in terms of the scalar and vector potentials in accordance with (54) as

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\qquad \mathbf{B}=\mathrm{rot}\,\mathbf{A},\qquad \vec{\mathcal{P}}=\frac{mW}{\Gamma c^2}\mathbf{v}+\frac{e}{c}\mathbf{A}.$$

Use of the vector $\mathbf{K}=\mathrm{curl}\,\vec{\mathcal{P}}$ makes it possible to express the equations of motion in the form

$$\partial\vec{\mathcal{P}}/\partial t-[\mathbf{vK}]=-\nabla\mathcal{H}+m\Gamma T\nabla S. \qquad (72)$$

In particular, for isentropic processes $S=\mathrm{const}$ it follows from this that

$$\partial\mathbf{K}/\partial t=\mathrm{rot}[\mathbf{vK}]. \qquad (73)$$

This equation is identical to the equation for curl $\mathbf{v}$ in classical hydrodynamics and for the magnetic field $\mathbf{B}$ in single-fluid magnetohydrodynamics and is their generalization for two-fluid relativistic electromagnetic gas dynamics. When allowance is made for the continuity equation, (73) can be represented in the form

$$\frac{d}{dt}\frac{\Gamma\mathbf{K}}{n}=\left(\frac{\Gamma\mathbf{K}}{n}\nabla\right)\mathbf{v}, \qquad (74)$$

and this can be interpreted as the equation of the freezing of the field lines of the vectors $\mathbf{K}_\pm$ into the corresponding gases of the charged particles. The nonrelativistic analog of (74) was obtained in particular in Ref. 20.

For symmetric steady flows ($\partial/\partial x^3=0, \partial/\partial t=0$), the first and second contravariant components of the vectors

$$\mathbf{a}_N=\mathbf{B},\ \mathbf{j},\ n\mathbf{v}/\Gamma,\ \mathbf{i}=\mathrm{rot}\,W\mathbf{v}/\Gamma,\ \mathbf{K}=m\mathbf{i}+(e/c)\mathbf{B}$$

can be expressed in terms of the derivatives of the corresponding flow functions $\psi_N=\psi, cB_3/4\pi, \Psi, Wv_3/\Gamma, \mathcal{P}_3$ by

$$q^{1/2}a_N^{\ 1}=\partial\psi_N/\partial x^2,\ q^{1/2}a_N^{\ 2}=-\partial\psi_N/\partial x^1,$$

where q is the determinant of $q_{\alpha\beta}$. Using further the freezing into their flows of the entropies $S_\pm(\Psi_\pm)$, the covariant component of the generalized momentum $\mathcal{P}_3^\pm(\Psi_\pm)$, and the energy $\mathcal{H}_\pm(\Psi_\pm)$, we obtain the following system of equations of two-fluid relativistic electromagnetic gas dynamics for symmetric steady flows in the framework of the special theory of relativity:

$$\frac{\Gamma}{q_{33}n}\left(mi_3+\frac{e}{c}B_3\right)-\frac{v_3}{q_{33}}\mathcal{P}_3'(\Psi)+\mathcal{H}'(\Psi)=m\Gamma TS'(\Psi),$$
$$j_3=e_+n_+v_3^+/\Gamma_++e_-n_-v_3^-/\Gamma_-,$$
$$\frac{c}{4\pi}B_3=e_+\Psi_++e_-\Psi_-,\qquad \frac{mW}{\Gamma c^2}v_3+\frac{e}{c}\psi=\mathcal{P}_3(\Psi),$$
$$\frac{mW}{\Gamma}+e\varphi=\mathcal{H}(\Psi),\qquad \Delta\varphi=-4\pi\left(\frac{e_+n_+}{\Gamma_+}+\frac{e_-n_-}{\Gamma_-}\right). \qquad (75)$$

Here, all the quantities without indices except j_3, B_3, ψ, φ, and c are understood to have the indices $\pm$, i.e., to relate to the ion or electron gases. The covariant components i_3 and j_3 are expressed by the differential operators

$$\frac{i_3}{q_{33}} = -\operatorname{div}\frac{W\nabla\Psi}{q_{33}nc^2} + \frac{Wv_3}{\Gamma}\operatorname{div}\frac{[\mathbf{e}_3\mathbf{e}^3]}{q_{33}},$$

$$\frac{j_3}{q_{33}} = -\operatorname{div}\frac{\nabla\psi}{nq_{33}} + \frac{B_3}{\Gamma}\operatorname{div}\frac{[\mathbf{e}_3\mathbf{e}^3]}{q_{33}}. \qquad (75a)$$

Thus, the first two equations of (75) represent four differential equations containing six arbitrary functions of the flows $S_\pm$, $\mathcal{P}_3^\pm$ and $\mathcal{H}_\pm$. Equations analogous to (75) for nonrelativistic two-fluid gas dynamics were derived in Ref. 21.

APPENDIX: STEADY CYLINDRICAL FLOW

In the framework of two-fluid relativistic electromagnetic gas dynamics, the steady cylindrical flow of a plasma, with allowance for ohmic resistivity and bremsstrahlung, can be described in accordance with (66) and (67) by the system of equations (see also Ref. 22)

$$\frac{1}{r}\frac{d}{dr}rB = \frac{4\pi e}{c}\left(\frac{n_+v_+}{\Gamma_+} - \frac{n_-v_-}{\Gamma_-}\right),$$

$$\frac{1}{r}\frac{d}{dr}rE_r = 4\pi e\left(\frac{n_+}{\Gamma_+} - \frac{n_-}{\Gamma_-}\right), \qquad (A.1)$$

$$E_z = \frac{env'}{\sigma\Gamma'}, \quad \frac{dp}{dr} = \pm\frac{en}{\Gamma}\left(E_r - \frac{v}{c}B\right), \quad G_\pm - Q_\pm = 0.$$

Here, the charges of the ions and electrons are assumed to be the same: $e_\pm = \pm e$,

$$v = v_z, \quad B = B_\varphi, \quad v' = (v_+ - v_-)(1 - v_+v_-/c^2)^{-1},$$

where Q and G are the powers of the Joule heat release and the energy loss due to radiation as determined in (68) and (49).

From the system (A.1) we obtain a balance equation for the radial forces,

$$\frac{d}{dr}(p_+ + p_-) = -\frac{1}{8\pi r^2}\frac{d}{dr}r^2(B^2 - E_r^2), \qquad (A.2)$$

which shows that $B^2 - E_r^2$ is a relativistic invariant. The quantity $B^2 - E^2$ is also invariant since in accordance with (A.1) the component E_z is invariant. From (A2) under the condition $p_\pm(R) = 0$ we obtain

$$2\pi R^2c^2\langle p_+ + p_-\rangle = J^2 - c^2q^2, \qquad (A.3)$$

where the angular brackets denote averaging over the volume, and J and q are the total current and charge per unit length of the cylinder.

In the case $v_\pm = \text{const}$, $n_+ = n_- = n$ we have

$$J^2 - c^2q^2 = 2e^2N^2c^2(1/\Gamma' - 1), \quad N \equiv \pi R^2\langle n\rangle, \quad \Gamma' \equiv (1 - v'^2/c^2)^{1/2}. \qquad (A.4)$$

Setting $T_+ = T_- = T = \text{const}$ and substituting $\sigma = 3T^{3/2}/4(2\pi\mu)^{1/2}e^2L$,[16] we obtain from the energy balance equations $G_\pm = Q_\pm$ for the ions and electrons (for $p = nT$)

$$J^2 - c^2q^2 = 4\cdot 3^{1/2}\hbar\mu^2c^7L/e^4 \equiv J_{\rm PB}^2, \quad \mu \equiv \frac{m_+m_-}{m_+ + m_-}, \qquad (A.5)$$

where $J_{\rm PB}$ is the Pease-Braginskiĭ current.[5,6] In the considered case with $n_+ = n_-$, $T_+ = T_- = \text{const}$, $v_\pm = \text{const}$ we have $q = eN[1 - (1 + J^2/e^2N^2c^2)^{1/2}]$, and it follows from (A.5) that a steady state is possible for a current

$$J = J_{\rm PB}[1 + (J_{\rm PB}/2eNc)^2]^{1/2}. \qquad (A.6)$$

If $J_{\rm PB} \ll 2eNc$, the equilibrium current $J \approx J_{\rm PB}$ does not depend on N as in the single-fluid nonrelativistic case considered in Refs. 5 and 6. It should however be noted that the Pease-Braginskiĭ equilibrium is realized only for very stringent restrictions on the geometry and the distribution of the parameters within the flow.

We thank A. A. Vedenov, V. I. Kogan, and A. P. Pitaevskiĭ for their interest in the work and for valuable comments.

[1] A. Einstein, "On a stationary system with spherical symmetry consisting of many gravitating masses," Ann. Math. **40**, 922 (1939).

[2] Yu. P. Ochelkov, O. F. Prilutskiĭ, I. L. Rozental', and V. V. Usov, Relyativistskaya kinetika i gridrodinamika (Relativistic Kinetics and Hydrodynamics), Atomizdat, Moscow (1979).

[3] D. I. Dzhavakhishvili and N. L. Tsintsadze, Zh. Eksp. Teor. Fiz. **64**, 1314 (1973) [Sov. Phys. JETP **37**, 666 (1973)].

[4] B. É. Meerovich and S. T. Sukhorukov, Zh. Eksp. Teor. Fiz. **68**, 1783 (1975) [Sov. Phys. JETP **41**, 895 (1975)].

[5] R. S. Pease, Proc. Phys. Soc. **70B**, 11 (1957).

[6] S. P. Braginskiĭ, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 1, Izd. AN SSSR, Moscow (1958), p. 115.

[7] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, Moscow (1954) [Pergamon, 1959].

[8] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Chicago (1939) [Russian translation published by IIL, Moscow (1950)].

[9] K. P. Stanyukovich, Neustanovivshiesya dvizheniya sploshnoĭ sredy (Unsteady Motions of a Continuum), GITTL, Moscow-Leningrad (1955) [Pergamon, 1960].

[10] L. S. Solov'ev, K relyativistskoĭ gazodinamike ("On relativistic gas dynamics"), Preprint IAÉ-3362/1, I. V. Kurchalov Institute of Atomic Energy, Moscow (1980).

[11] C. Eckart, Phys. Rev. **58**, 919 (1940).

[12] S. Weinberg, Gravitation and Cosmology, Wiley, New York (1972) [Russian translation published by Mir, Moscow (1975)].

[13] Shih-I Pai, Dinamika izluchayushego gaza (Dynamics of Radiating Gases; Russian translation), Mir, Moscow (1968).

[14] L. D. Landau and E. M. Lifshitz [Teoriya polya, Nauka, Moscow (1973)], *The Classical Theory of Fields*, 4th ed., Pergamon Press, Oxford (1975).

[15] L. S. Solov'ev, in: Problemy teorii gravitatsii i élementarnykh chastits (Problems of the Theory of Gravitation and Elementary Particles), Vol. 13 (ed. K. P. Stanyukovich), Énergoizdat, Moscow (1982), p. 157.

[16] S. P. Braginskiĭ, in: Voprosy teorii plazmy, Vol. 1 (ed. M. A. Leontovich), Atomizdat, Moscow (1963); English translation: Reviews of Plasma Physics, Vol. 1, Consultants Bureau, New York (1965).

[17] V. P. Kogan, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 1, Izd. AN SSSR, Moscow (1958), p. 130.

[18] A. B. Kukushkin and V. I. Kogan, Fiz. Plazmy **5**, 1264 (1979).

[19] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred, Nauka, Moscow (1982); English translation of earlier edition: Electrodynamics of Continuous Media, Pergamon Press, Oxford (1960).

[20] S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, and V. V. Yanovskiĭ, Dokl. Akad. Nauk SSSR **258**, 318 (1981) [Sov. Phys. Dokl. **26**, 479 (1981)].

[21] A. I. Morozov and L. S. Solov'ev, in: Voprosy teorii plazmy (Reviews of Plasma Physics), No. 8 (ed. M. A. Leontovich), Atomizdat, Moscow (1974).

[22] L. S. Solov'ev, Fiz. Plazmy **10**, 1045 (1984) [Sov. J. Plasma Phys. **10**, 602 (1984)].

Translated by Julian B. Barbour