Final-state interaction of β electrons and related phenomena

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The final-state interaction of a β electron with atomic electrons has been calculated to accuracy $(\alpha Z/v)^2$. It is shown that previous studies devoted to the final-state interaction have not taken into account all diagrams contributing in the first nonvanishing approximation. Correct allowance for the final-state interaction makes it impossible to explain the discrepancy between the theory and the experimental results of Simpson by emission of a neutrino. The influence of the final-state interaction on the value of G_V^2 obtained from superallowed $0^+ \to 0^+$ transitions is investigated. The correction for final-state interactions to the decay probabilities of tritium, muonium, muonic atoms, and pionium is calculated.

I. INTRODUCTION

The final-state interaction of a β electron with the electron of an atomic shell is a characteristic phenomenon which must be taken into account in a precise analysis of a large group of phenomena studied in weak decay processes: in search for a neutrino mass, determination of the cosine of the Cabibbo angle of the basis of superallowed $0^+ \to 0^+$ transitions, the lifetimes of the muon and pion in matter, and so forth. As was shown in our brief note,[1] previous analyses (see the citations in Ref. 2, for example) took into account correctly the energy dependence of the β-electron final-state interaction,[3] but not its absolute magnitude. This was due to neglect of the contribution of diagrams with double exchange of a photon, which give a contribution of the same order as diagrams with single exchange.

In this article we shall give a detailed derivation of the equations describing the final-state interaction of a β electron with accuracy ξ^2, where $\xi = \alpha E/p = \alpha/v$ is the parameter of the Coulomb interaction between the β electron and a bound electron, and v is the velocity of the β electron (Sec. II). In particular, we shall show that the final-state interaction changes the energy distribution of the β electrons by an amount

$$\frac{dW/dE}{dW^{(0)}/dE} = 1-\xi^2\langle\chi|\sum_j^z \frac{r_0}{|r_j|}|\chi\rangle, \tag{1}$$

where $|\chi\rangle$ is the wave function of the initial state of the atom, z is the number of electrons, and $r_0 = 1/m\alpha$ is the Bohr radius. We also obtain formulas describing the final-state interaction in transition of an atom to a given final state. In the subsequent sections we shall discuss a number of applications of the relations obtained.

In Section III we have considered corrections for experiments on the mass of the ν_e in the β decay of tritium, which turn out to be smaller by a factor of about 8 than was stated in Ref. 4. In addition, we have shown that the data of the experiment of Ref. 5 on the search for a heavy neutrino, when the final-state interaction is taken into account correctly, are inconsistent with the existing limitations for mixing of ν_e.

In Section IV we discuss corrections to the probability of superallowed $0^+ \to 0^+$ transitions, which is proportional to G_V^2 and is used for determination of G_V^2. It is shown that the systematic inclusion of the final-state interaction leads to the result that G_V^2 extracted from experiment on the basis of the existing theory (see for example Ref. 6) rises monotonically with atomic number A. This indicates internal difficulties of the existing calculations and the unsatisfactory nature of the procedure for determining G_V^2 which was used in Refs. 7 and 8.

In Section V we consider corrections to the lifetime of bound states. It is shown that the correction to the ratio of the lifetime of a negative muon in a bound state to the lifetime in the free state is about 20% of the entire correction for heavy nuclei. The correction to the lifetime of a μ^+ in matter due to formation of muonium is $\sim\alpha^2$ and in comparable with the measurement accuracy achieved. In the case of decay of a π^+ meson in matter, the corrections to the total decay probability are suppressed by a factor m_e/m_μ and as a result the corrections to the π^+ lifetime in matter turn out to be small with the existing experimental accuracy.[9]

II. DERIVATION OF THE GENERAL FORMULAS

In this section we obtain the basic formulas for the energy distribution of β electrons and the total probability of β decay with inclusion of the final-state interaction in the first nonvanishing approximation.

With neglect of the final-state interaction the β-decay amplitude of a nucleus is

$$A_n^{(0)} = gF_n^{(0)}, \tag{2}$$

where g is the weak amplitude and

$$F_n^{(0)} = \langle\psi_n|\chi\rangle\Phi. \tag{3}$$

In Eq. (3) ψ_n and χ are the wave functions of the initial and final atoms, and the function Φ describes the interaction of the β electron with the nucleus. Here

$$\Phi = \left[\frac{2\pi\xi_Z}{1-\exp(-2\pi\xi_Z)}\right]^{1/2} \exp(i\xi_Z\varphi), \tag{4}$$

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where $\xi_Z = \alpha Z/v$, is the Coulomb parameter of the interaction of the β electron with the nucleus and φ is a phase factor. The function

$$|\Phi|^2 = \frac{2\pi\xi_Z}{1-\exp(-2\pi\xi_Z)} \tag{5}$$

is the standard Fermi function. As is well known,[10] the right-hand side of Eq. (5) is the square of the wave function of the β electron at the center of the nucleus ($r=0$).

If the velocity of the β electron is considerably greater than the average velocity of the ground-state electrons

$$v^2 \gg (\alpha Z)^2, \quad \xi_Z^2 \ll 1, \tag{6}$$

then the final-state interaction actually occurs in two stages: interaction of the β electron with the nucleus, which occurs at small distances $r \sim 1/p \ll 1/m\alpha Z$ and which leads to appearance of the factor (4), and the subsequent interaction of the β electron with the bound electrons, which occurs at distances $r \sim 1/m\alpha Z$ (see Appendix). As the result, with accuracy to terms $\xi_Z^2 \ll 1$ we have

$$A_n = g\Phi T_n, \tag{7}$$

where T_n is the amplitude describing the interaction of the β electron with the atomic electrons. The latter can be expanded in the parameters

$$\xi = \alpha/v \ll 1, \tag{8}$$

$$T_n = \sum_i T_n^{(i)}. \tag{9}$$

The amplitude $T_n^{(i)}$ describes the interaction of a free β electron with the bound electrons in the ith order of perturbation theory (Fig. 1). The amplitude

$$T_n^{(0)} = \langle\psi_n|\chi\rangle, \tag{10}$$

obviously, is real, while the first-order amplitude is mainly imaginary: $T_n^{(1)} \sim iC_1\xi + C_2\xi\xi_Z$. Therefore the interelectron interaction gives a contribution $\sim\xi^2$ to the probability of β decay. A contribution of the same order is given by the real part $T_n^{(2)}$.[1] Therefore in calculation of the first nonvanishing correction $\sim\xi^2$ due to the final-state interaction it is

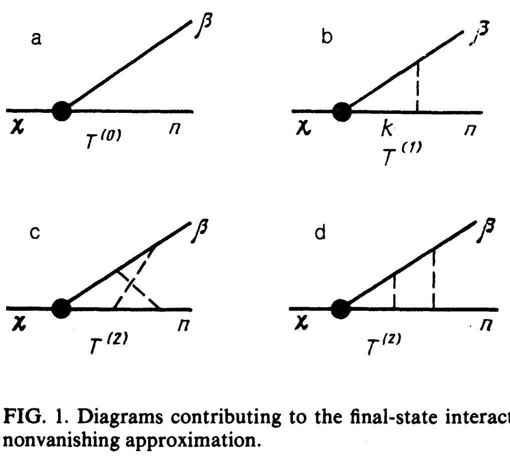

FIG. 1. Diagrams contributing to the final-state interaction in the first nonvanishing approximation.

necessary to take into account diagrams of both first and second orders of perturbation theory:

$$T_n = T_n^{(0)} + T_n^{(1)} + T_n^{(2)}. \tag{11}$$

For the square of the amplitude with inclusion of the final-state interaction we have

$$|T_n|^2 = |T_n^{(0)}|^2 + 2\,\mathrm{Re}\,T_n^{(1)}T_n^{(0)} + |\mathrm{Im}\,T^{(1)}|^2 + 2\,\mathrm{Re}\,T_n^{(2)}T_n^{(0)}, \tag{12}$$

and the last term was not taken into account in the studies which were made before Ref. 1.

Here and below we have used the notation $\langle n| \equiv \langle\psi_n|$. The amplitude of first order is

$$T_n^{(1)} = 4\pi\alpha\sum_k\int\frac{d^3f}{(2\pi)^3}\frac{2m}{-2(\mathbf{pf})+2m\omega_{nk}-f^2+i\varepsilon}\frac{1}{f^2} \times\langle n|\sum_j\exp[-i(\mathbf{fr}_j)]|\mathbf{k}\rangle\langle k|\chi\rangle, \tag{13}$$

where $\omega_{nk} = E_n - E_k$, $E_{n,k}$ are the energies of the states of the daughter atom.

The main contribution to (13) is from the pole of the Green function ($f_z = (\omega_{nk}/v) - i\varepsilon$ (z is the direction of the β-electron momentum). Therefore

$$i\,\mathrm{Im}\,T_n^{(1)} = i\frac{2\pi\alpha}{v}\sum_k\int\frac{d^2\mathbf{f}_t}{(2\pi)^2}\frac{1}{f_t^2+(\omega_{nk}/v)^2} \times\langle n|\sum_j\exp[-i(\mathbf{f}_t\mathbf{r}_{tj})]|k\rangle\langle k|\chi\rangle. \tag{14}$$

The term with $k=n$ in the right-hand side of Eq. (14) contains an infrared divergence corresponding to the Coulomb phase shift of scattering of the β electron by a bound electron in a state $|n\rangle$, which of course cancels in the expression for $|T_n|^2$.

Calculations carried out in the same way as those performed previously in Ref. 11 give

$$i\,\mathrm{Im}\,T_n^{(1)} = -\frac{i}{2}\xi\sum_k\langle n|\sum_j\ln\frac{r_{tj}^2}{a^2}|k\rangle\langle k|\chi\rangle, \tag{15}$$

where $a^2 = (\omega_{nk}/v)^2 + \lambda^2$ and λ is infrared cutoff of the integral over f. Since $\langle n|k\rangle = \delta_{nk}$, we have

$$i\,\mathrm{Im}\,T_n^{(1)} = -\frac{i}{2}\xi\langle n|\sum_j\ln r_{tj}^2/\lambda^2|\chi\rangle. \tag{16}$$

In the calculation of $\mathrm{Re}T_n^{(1)}$ we note that states with $\omega_k \gg m(\alpha Z)^2$ give a smallness due to the smallness of the corresponding matrix elements. Therefore, after integrating over angles and then expanding the right-hand side of powers of $(m\omega_{nk} + f^2)/pf \ll 1$, we obtain

$$\mathrm{Re}\,T_n^{(1)} = -\frac{\xi^2}{2}\left[\langle n|\sum_j\frac{r_0}{r}|\chi\rangle - \omega_{nk}\langle n|\sum_j|r_j||k\rangle\langle k|\chi\rangle\right]. \tag{17}$$

Equation (12) is valid for states with $E_n < m(\alpha Z)^2 v/\alpha Z$. For states with higher energy the final-state interaction contains an additional smallness for the same reasons.

The right-hand side of (17) transforms into

$$\operatorname{Re} T_n^{(1)} = -\frac{\xi^2}{2}\Big[\langle n|\sum_j \frac{r_0}{|r_j|}|\chi\rangle - \langle n|[H|r_j|]|\chi\rangle\Big], \qquad (18)$$

where H is the complete Hamiltonian of the daughter atom, or

$$\operatorname{Re} T_n^{(1)} = \frac{\xi^2}{2} r_0 \langle n|\sum_j \frac{\partial}{\partial r_j}|\chi\rangle. \qquad (19)$$

It remains to calculate $T_n^{(2)}$. Note that the amplitude $T_n^{(2c)}$ (Fig. 1c) does not contribute, since after integration over f_0 all poles of the Green functions in f_{10} turn out to be in a single half-plane. Therefore

$$T_n^{(2)} = T_n^{(2d)} = \left(\frac{4\pi\alpha}{v}\right)^2 \times \sum_{l,k}\int \frac{d^3f}{(2\pi)^3}\frac{d^3f_1}{(2\pi)^3}\frac{1}{(-f_z - f_{1z} + \omega_{nk}/v + i\varepsilon)} \times \frac{1}{(-f_z + \omega_{nl}/v + i\varepsilon)}\frac{1}{f^2}\frac{1}{f_1^2}\langle n|e^{-i(\mathbf{f}\mathbf{r})}|l\rangle\langle l|e^{-i(\mathbf{f}_1\mathbf{r})}|k\rangle\langle k|\chi\rangle. \qquad (20)$$

Integrating over f_Z and f_{1Z}, we obtain

$$\operatorname{Re} T_n^{(2)} = -\frac{1}{2}\sum_l T_{nl}^{(1)} T_l^{(1)} \qquad (21)$$

with accuracy to terms $\sim \xi^2$. Here T_{nl} is the amplitude of the transition between states $|n\rangle$ and $|l\rangle$. The main contribution to $\operatorname{Re} T_n^{(2)}$ is from the poles of the electron propagators.

Using Eq. (15), we obtain

$$\operatorname{Re} T_n^{(2)} = -\frac{\xi^2}{4}\langle n|\sum_{ij}\ln\frac{r_{ti}^2}{\lambda^2}\ln\frac{r_{tj}^2}{\lambda^2}|\chi\rangle. \qquad (22)$$

Combining Eqs. (10), (16), (19), and (22), we finally have for the change of the probability of β decay accompanied by transition of an electron to a state $|n\rangle$:

$$\frac{(dW_n/dE) - (dW_n^{(0)}/dE)}{(dW_n^{(0)}/dE)} = \frac{\xi^2}{2}\Big\{ r_0\langle\chi|n\rangle\langle n| \times \sum_j \frac{\partial}{\partial r_j}|\chi\rangle + \frac{1}{2}\Big[\,|\langle\chi|\sum_j \ln r_{tj}^2/\lambda^2|n\rangle|^2 - \langle\chi|n\rangle\langle n|\sum_{i,j}\ln(r_{ti}^2/\lambda^2)\ln(r_{tj}^2/\lambda^2)|\chi\rangle\Big]\Big\}. \qquad (23)$$

It is easy to see that the right-hand side of Eq. (23) actually does not depend on χ.

We shall now calculate the change of the energy distribution summed over final states $|n\rangle$. Using Eqs. (10), (16), and (22), we obtain

$$\sum_n |\operatorname{Im} T_n^{(1)}|^2 + \sum_n 2\operatorname{Re} T_n^{(0)} T_n^{(2)} = 0. \qquad (24)$$

We note that Eq. (24) is easily obtained from consideration of the corresponding graphs (Fig. 2), taking into account that the main contribution to $\operatorname{Re} T^{(2)}$ is from the poles of the electron Green functions. Indeed, we can see from Fig. 2 that $\operatorname{Re}\sum_n T_n^{(0)} T_n^{(2)}$ and $\sum_n \operatorname{Im} T_n^{(1)} \operatorname{Im} T_n^{(1)}$ correspond to identical diagrams. The relative sign is obvious from the condition of cancellation of the infrared divergences.

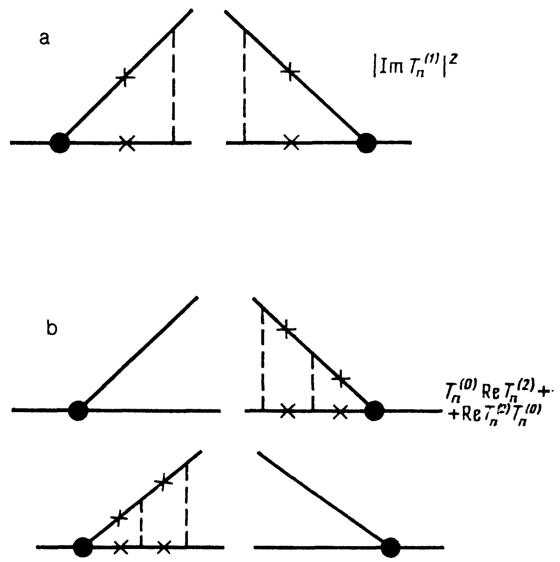

FIG. 2.

When Eq. (24) is taken into account, Eq. (12) takes the form

$$\sum_n |T_n|^2 - \sum_n |T_n^{(0)}|^2 = 2\operatorname{Re}\sum_n T_n^{(0)} T_n^{(1)}. \qquad (25)$$

Noting that in the right-hand side of Eq. (17)

$$\sum_{k,n} \omega_{nk}\langle\chi|n\rangle\langle n|\,|r_j|\,|k\rangle\langle k|\chi\rangle = 0 \qquad (26)$$

as the result of antisymmetry of ω_{nk} with respect to n and k, we obtain Eq. (1).

The formulas obtained permit us to find also the correction to the total probability of β decay. In fact, for $v^2 \lesssim (\alpha Z)^2$ the final-state interaction cannot be described by perturbation theory and it is necessary to construct the complete wave function of the final state. However, the region $v^2 \lesssim (\alpha Z)^2$ gives a small contribution to the spectrum, which can be written in the form

$$dW = g^2|\Phi|^2\sum_n |T_n|^2(\varepsilon_0^{(n)} - \varepsilon)^2(\varepsilon + m)(\varepsilon^2 + 2m\varepsilon)^{1/2}\,d\varepsilon, \qquad (27)$$

where $\varepsilon = E - m$ is the kinetic energy of the β electron and $\varepsilon_0^{(n)}$ is its maximum value. Then the integral over the region $\varepsilon \sim m(\alpha Z)^2$ gives a small contribution (much less than ξ_Z^2) to the spectrum, and the correction due to the final-state interaction is as before much smaller than unity.

For sufficiently small Z, such that

$$\pi\xi_Z(\varepsilon_0) \ll 1, \qquad (28)$$

it is possible to obtain an analytic form for the effect of the final-state interaction on the total probability of β decay. Indeed, for $\varepsilon_0 \sim \varepsilon$ one can set $|\Phi|^2 = 1$, and the contribution of the region in which $\pi\xi_Z \sim 1$ is negligible with accuracy to (28). Then, integrating Eq. (1), we obtain

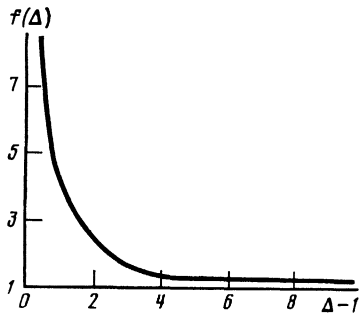

FIG. 3. Plot of the function $f(\Delta)$.

$$W=W^{(0)}\left[1-\alpha^2\langle\chi|\sum_j\frac{r_0}{|r_j|}|\chi\rangle f(\Delta)\right], \tag{29}$$

where $\Delta = E_0/m$, and

$$f(\Delta)=\frac{(\Delta^2-1)^{1/2}[{}^{11}/_{30}\,\Delta^4+{}^{11}/_{60}\,\Delta^2+{}^{8}/_{15}]-{}^{3}/_{4}\,\Delta\ln[\Delta+(\Delta^2-1)^{1/2}]}{(\Delta^2-1)^{1/2}[{}^{1}/_{30}\Delta^4-{}^{3}/_{20}\Delta^2-{}^{2}/_{15}]+{}^{1}/_{4}\,\Delta\ln[\Delta+(\Delta^2-1)^{1/2}]}. \tag{30}$$

A plot of the function $f(\Delta)$ is given in Fig. 3.

In the ultrarelativistic limit we have

$$f(\infty)=1. \tag{31}$$

In the nonrelativistic limit $\Delta^2 - 1 \ll 1$ but $\Delta^2 - 1 \gg (\alpha Z)^2$

$$f(\Delta)=7/(\Delta^2-1). \tag{32}$$

All formulas of this section are given for the case of β^- decay. For β^+ decay it is necessary to change the sign of the amplitude $T^{(1)}$. Here the Fermi function is

$$|\Phi|^2=2\pi\xi_Z/[\exp(2\pi\xi_Z)-1]. \tag{33}$$

Note that in the case of β^+ decay the appearance of a falling exponential for the case $\pi\xi_Z \gtrsim 1$ increases the accuracy of the formulas for the total decay probability.

III. SEARCH FOR NEUTRINO MASS

In the case of β decay of tritium ($\varepsilon_0 = 18.6$ keV) the atomic electrons are described by Coulomb functions. The matrix elements in Eq. (21) are calculated analytically, and in particular

$$\langle\chi|r_0/r|\chi\rangle=1. \tag{34}$$

The decay of tritium is a traditional tool for the search for neutrino mass. We shall consider the application of the results obtained in the previous section.

1. Search for a heavy neutrino

In the search based on the β decay of tritium[5] the deviation of the data in the range 0.75 keV $\leqslant\varepsilon\leqslant 1.5$ keV from the calculations has been interpreted as an indication of the existence of a heavy neutrino with mass $M_\nu \sim 17$ keV having a mixing angle with $\bar{\nu}_e$ corresponding to $\sin^2\theta\leqslant 2.5\cdot 10^{-2}$. The interaction of the β electron with the atomic electron was taken into account by the method of Durand,[12] the principal deficiency of which is that the interaction is taken into account only at the moment of time at which the β electron is at the point $r = 0$. However, from the discussion of the preceding section it follows that the main contribution to the final-state interaction arises when the β electron is at a distance $r\sim 1/m\alpha Z$ from the nucleus. Therefore the method of Ref. 12 underestimates the effect of the final-state interaction.

We have analyzed the results of Simpson[5] on the basis of his method. Analysis of the data on the basis of Eq. (1) shows that for explanation of the results as emission of a heavy neutrino it is necessary that

$$M_\nu=16\ \text{keV},\qquad \sin^2\theta>5.5\cdot 10^{-2} \tag{35}$$

($\sin^2\theta$ increases, for larger M_ν), which is inconsistent with the limit known from other experiments[13]

$$\sin^2\theta<4\cdot 10^{-2}. \tag{36}$$

For tritium implanted in a medium (in the case of Ref. 5 in silicon) the functions $|\chi\rangle$ changed and Eq. (34) is not satisfied. To estimate the value of $\langle\chi|r_0/r|\chi\rangle$ it is possible to use a crude model in which the effect of the medium reduces to replacement of the charge of the tritium nucleus $Z = 1$ by some effective Z. In turn Z_{eff} can be estimated from measurements of the hyperfine splitting in normal muonium (μ^+e^-) implanted in silicon[1)]:

$$|\psi_{Si}(0)|^2/|\psi_{vac}(0)|^2\sim 0.45\ \ [14].$$

Then, since $|\psi(0)|^2\sim Z^3$, $\langle\chi|r_0/r|\chi\rangle\sim Z$,[10] we obtain

$$\langle\chi|r_0/r|\chi\rangle\sim 0.75. \tag{37}$$

Analysis of the results of the experiments of Ref. 5 with inclusion of the influence of the medium (Eq. (37)) gives

$$M_\nu=17\ \text{keV},\qquad \sin^2\theta>5\cdot 10^{-2}; \tag{38}$$

that is, as before it is inconsistent with the limit (36).

We emphasize that our conclusion supplements measurements made on ^{35}S, which do reveal no indications of a heavy neutrino.[15] Therefore our analysis excludes the explanation[16] whereby the contradiction between the experiments of Refs. 5 and 15 is a consequence of the fact that ν_H is not produced in Fermi transitions.

2. Search for the mass of $\bar{\nu}_e$

In this case the experimenters are interested in the probabilities of transitions of bound electrons to specific states of the ^{3}He atom. The calculations are simplified if one notes that for the ground state $\chi = 2\exp(-r/r_0)$, and therefore

$$\frac{\partial}{\partial r}|\chi\rangle=-\frac{1}{r_0}|\chi\rangle$$

and Eq. (23) takes the form

$$|T_n|^2=|T_n^{(0)}|^2(1-\xi^2)+\xi^2 b_n, \tag{39}$$

in which $b_n = [|\langle\chi|\ln r_i^2|n\rangle|^2 - \langle\chi|n\rangle\langle n|\ln^2 r_i^2|\chi\rangle]/4$. The electron-distribution change due to the final-state interaction is obviously given by the the quantity $\varkappa_n = \{|T_n|^2 - |T_n^{(0)}|^2\}/\sum_n |T_n|^2$. Taking into account Eq. (39), we obtain

$$\varkappa_n = b_n. \tag{40}$$

A specific calculation gives

$$b_1 = -0.307, \quad b_2 = 0.141, \quad b_3 = 0.021. \tag{41}$$

In electronic transitions produced by the final-state interaction, the final electron, in contrast to the case of shaking, turns out in 40% of the cases to be in the continuum. Substituting the numerical value of ξ^2, we find that the change of the level populations[2)] $\varkappa_n$ are

$$\varkappa_1 = -2.2\cdot 10^{-4}, \quad \varkappa_2 = 1.0\cdot 10^{-4}, \quad \varkappa_3 = 1.5\cdot 10^{-5}. \tag{42}$$

Therefore the final-state interaction leads to excitation of a state with a probability $2.2\cdot 10^{-4}$, i.e., almost eight times smaller than the prediction of Ref. 4. The effect of the final-state interaction on the measurement of $m_{\nu e}$ consequently is negligible for $m_{\nu e} \gg 10^{-2}$ eV.

IV. SUPERALLOWED TRANSITIONS AND DETERMINATION OF G_V^2

In heavy nuclei the correction due to the final-state interaction can reach tens of percent, but in the general case it cannot be explicitly identified experimentally as the result of the large uncertainty in the nuclear matrix elements. An exception is the case of superallowed transitions $0^+ \to 0^+$, since the latter do not depend on the detailed properties of the nuclear wave functions.[6] As is well known, they are used for determination of the weak constant G_V^{12}.

A recent view[7] gives values of the quantity $ft \sim G_V^{-2}$, obtained by means of calculations[8] in which the final-state interaction was taken into account by the Durand method[12] (see Table I). In the last column of Table I we have given values of ft obtained with use of Eq. (1). As can be seen, the differences in the values of ft (and in G_V^2) obtained from different nuclei reach a magnitude $\sim 5\cdot 10^{-3}$.

Therefore with correct inclusion of the final-state interaction the ft value obtained by us rises monontonically with Z. This indicates a systematic defect of the existing theory of inclusion of corrections which increase with Z. Therefore the values of $\cos^2\theta_C$, where θ_C is the Cabibbo angle, obtained from these measurements obtain a significant systematic error. Extrapolation to small Z in the spirit of Ref. 17 leads to an increase of $\cos^2\theta_C$ and practically removes the difference of the sum $\sin^2\theta_C + \cos^2\theta_C$, from unity which was discussed in Ref. 18. This question will be discussed in detail separately (see *Note added in proof*).

V. LIFETIME OF BOUND STATES

1. Tritium

The influence of the final-state interaction on the probability of decay of tritium can be calculated by means of Eq. (27),

$$\delta_1 = (W - W_0)/W_0 = -5.1\cdot 10^{-3}. \tag{43}$$

The value (43) is significantly greater than the accuracy $\sim 3\cdot 10^{-4}$ in measurement of the tritium lifetime.[19] Direct comparison of theory and experiment is impossible as the result of uncertainty in the nuclear matrix element. In an investigation[19] of the influence of atomic effects on the tritium lifetime, and the final-state interaction was taken into account using the results of Ref. 4, which give $\delta_1 = -2\cdot 10^{-3}$. This led to an inaccuracy in calculation of the total lifetime change due to electromagnetic effects. The value $(\Delta f/f)_{tot} = 0.82\cdot 10^{-2}$ given in Table II of Ref. 19 should be replaced by

$$(\Delta f/f)_{tot} = 0.53\cdot 10^{-2}. \tag{44}$$

2. Muonic atoms

It is well known (see for example Ref. 20) that the probability of decay of a negative muon decreases by about 15% if it is in a bound state near a heavy nucleus. For high Z, as follows from Eq. (27), the effects of the final-state interaction can reach several percent, i.e., may amount to about 20% of the total change of the probability of μ^- decay. In Table II we give the results, taken from the review of Ref. 20, of measurements and of calculations, without taking into account the final-state interaction, of the ratio of the decay probability of a bound and a free negative muon, and the change of this ratio on taking into account the final-state interaction.

3. Muonium

The difference in the lifetimes of a free positive muon and the system (μ^+e^-) is determined mainly by the final-state interaction, since the effects of change of phase space give an additional smallness $\sim m_e/m_\mu$. Here the muonium lifetime should be smaller by an amount $\sim 5\cdot 10^{-5}$.

TABLE I.

nucleus	ft from Ref. 7	ft from our result	nucleus	ft from Ref. 7	ft from our result
^{14}O	3075.5±3.9	3075.8	^{42}Se	3089.3±7.5	3097.8
^{26}Al	3072.9±3.7	3075.9	^{46}V	3088.6±4.3	3098.4
^{34}Cl	3076.9±4.7	3082.4	^{50}Mn	3085.9±5.7	3097.2
^{38}K	3076.6±4.6	3083.4	^{54}Co	3087.5±4.4	3101.4

TABLE II.

Z	Calculation without allowance for final-state interaction	Calculation with final-state interaction	Experiment
30	0.96	0.95	0.93±0.04
50	0.92	0.90	0.87±0.04
74	0.85	0.82	0.78±0.04
82	0.84	0.81	0.86±0.04

At the present time measurements of the muon lifetime in vacuum and in matter are known, which give respectively[21,22]

$$\tau_1=(2{,}197078\pm8\cdot10^{-5})\mu\text{sec},\quad \tau_2=(2.197110\pm7.3\cdot10^{-5})\mu\text{sec}. \tag{45}$$

As can be seen, the error in Refs. 21 and 22 is comparable with correction for the final-state interaction if the muonium if formed in matter (in Ref. 22 water with small additions or organic materials was used). Obviously, a small increase of the measurement accuracy can provide an answer to the question of the sign of the difference of τ_1 and τ_2 and the possibility of formation of muonium in various media.

4. Pionium

The calculation of the final-state interaction in the decay of pionium $(\pi^+e^-)\to\mu^+\bar{v}_\mu e^-$ is carried out in the same way as was done in Section II. Here ε_μ is fixed: $\varepsilon_\mu=(m_\pi-m_\mu)^2/2m_\pi$; effects due to the change of phase space give an additional smallness $(m_\pi-m_\mu)^2/m_\pi^2\sim0.1$. The change of lifetime is determined by the final-state interaction:

$$\frac{\tau_{(\pi^+e^-)}-\tau_{\pi^+}}{\tau_{\pi^+}}=-\frac{\alpha^2}{v_\mu{}^2}\frac{m_e}{m_\mu}\langle\chi|\frac{r_0}{r}|\chi\rangle\sim-3\cdot10^{-6}. \tag{46}$$

Therefore the rescattering of π^+-decay muons by the electrons of the medium is negligible as long as the measurement accuracy does not reach $\sim10^{-5}$. At the present time the experimental error[9] is $\sim2\cdot10^{-4}$, that is, almost two orders of magnitude larger than the final-state interaction effect.

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APPENDIX

Here we shall prove Eq. (7). For simplicity we shall restrict the discussion to nonrelativistic β electrons, $v^2\ll1$, but the generalization to the relativistic case is quite obvious.

We shall consider first the interaction of the β electron with the nucleus. In the first order of perturbation theory

$$\Phi_1=4\pi\alpha Z\int\frac{d^3f}{(2\pi)^3}\frac{1}{f^2}\frac{2m}{+2(\mathbf{pf})+f^2-i\varepsilon}$$
$$=\frac{\alpha Z}{v}\int\frac{df}{f}\ln\frac{f+2p}{f-2p}=\frac{\pi\xi_Z}{2}+i\xi_Z\int^{2p}\frac{df}{f}. \tag{A.1}$$

The main contribution of $\mathrm{Re}\Phi_1$ is from the region $f\sim p$, i.e., a distance $r\sim1/p$ from the nucleus. It is well known that the sum of the series in $\pi\xi_Z$ gives

$$\Phi=\left(\frac{2\pi\xi_Z}{1-\exp(-2\pi\xi_Z)}\right)^{1/2}. \tag{A.2}$$

The second term in Eq. (A.1) contains an infrared divergence. These terms are grouped into the phase factor $\exp[i\xi_Z\ln(2p/\lambda)]$,[23] where λ is the infrared cutoff. In calculation of the probability these factors cancel.

Let us consider now the interaction of a free β electron with the bound electrons of an atomic shell (see Fig. 1c and Eq. (13)). Large f values of the order of p are suppressed by a factor $\langle n|e^{-i(\mathbf{fr})}|\chi\rangle$. The main contribution of Eq. (13) is from momenta which are comparable with the characteristic momenta of the bound electrons $r\sim m\alpha Z\ll p$. The process occurs at distances of the order of the radius of the atomic shell $r\sim r_Z=1/m\alpha Z$.

Let us now take into account the interaction both with nuclei and with electrons. We have seen that the interaction with the nucleus occurs at a distance $r\sim1/p$, after which, going away to a distance $r\sim r_Z$, the β electron interacts with a bound electron.

This mechanism corresponds to the diagram of Fig. 4a. Neglecting the momentum f_1 in comparison with $p+f$, which as was said above corresponds to neglect of terms ξ_Z^2, we obtain

$$F_{1a}=T^{(1)}\Phi\exp(i\xi_Z\varphi). \tag{A.3}$$

In the amplitude shown in Fig. 4c, the β electron returns to the nucleus after interaction with bound electrons at distances $r\sim r_Z$. It is supressed only if the electron with momentum $|\mathbf{p}+\mathbf{f}_1|$ does not turn out to be on the mass shell. In

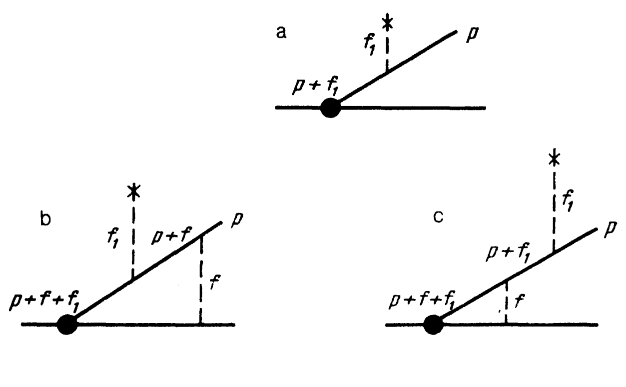

FIG. 4.

the latter case, as was shown above, we acquire a phase factor. Direct calculation of the integral

$$F_{4b}=T^{(1)}\int\frac{p^2}{|\mathbf{p}+\mathbf{f}|^2(|\mathbf{p}+\mathbf{f}+\mathbf{f}_1|^2-p^2+i\varepsilon)f_1^2}\frac{d^3f_1}{(2\pi)^3} \qquad (A.4)$$

proves the statement.

A similar analysis of the second-order amplitude leads us to Eq. (7).

Note added in proof (22 August 1986). Our recent analysis (Leningrad Institute of Nuclear Physics preprint No. 1221, 1986; to be published in Sov. J. Nucl. Phys.) shows that in the analysis of Ref. 7 the values of $ft_{1/2}$ for $0^+\to0^+$ transitions, in addition to the underestimation pointed out in the present work for the final-state interaction of the positron with atomic electrons, use was made of a calculation which greatly exaggerates the interaction of the positron with the residual nucleus. On removing these two defects in Ref. 7 we obtained $ft_{1/2}$ values which are practically independent of Z and which correspond to $\cos^2\theta_C + \sin^2\theta_C = 1.0003 \pm 0.0025$, which completely removes the indications of existence of mixing of first and fourth generations of quarks which were discussed in Ref. 16.

[1)] During the times characteristic of tritium β decay the transition from the anomalous state to the normal state occurs.

[2)] These values of $\varkappa_n$ are an improvement on the values obtained by us in Ref. 1.

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