Quasiclassical theory of three-dimensional wave collapse

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(Submitted 16 January 1986)
Zh. Eksp. Teor. Fiz. **91**, 1310–1324 (October 1986)

A quasiclassical solution that describes a strong wave-collapse regime with finite energy trapped in the collapse zone is constructed within the framework of the three-dimensional nonlinear Schrödinger equation. The quasiclassical solution is matched to the linear one. The role of radiation in collapse is clarified and the stability of the quasiclassical solution is investigated.

INTRODUCTION

Wave collapse, as a phenomenon in which a singularity is produced within a finite time, plays in the dynamics of nonlinear waves just as fundamental a role as to solitons. Classical examples of wave collapses are well known, such as formation of discontinuities in gasdynamics, whitecaps on a choppy sea, self-focusing of light in non-linear dielectrics, and collapse of plasma waves.

The existence of solitons is due to the equilibrium between two opposing effects—dispersive spreading of waves, and nonlinear increase of their gradients; this equilibrium is stable. In wave collapses, the equilibrium is upset and the nonlinear processes predominate. Whether a given medium is characterized by stable solitons or by wave collapse depends essentially on the dimensionality of the problem. Experience has shown that collapse is more typical of multidimensional systems, whereas stable solitons are typical of lower dimensionalities.

The present paper is devoted to three-dimensional wave collapse in one of the fundamental models of nonlinear physics—the nonlinear Schrödinger equation (NLS):

$$i\psi_t+{}^1/_2\Delta\psi+|\psi|^2\psi=0. \qquad (1)$$

This equation has a broad spectrum of various applications. In particular, it describes the evolution of a quasimonochromatic wave packet in a conservative isotropic medium with positive dispersion ($\omega''>0$) and inertialess nonlinearity. This situation is realized, for example, for sufficiently long ($kr_d<(m_e/m_i)^{1/2}$) Langmuir waves in a plasma or for electromagnetic waves in nonlinear dielectrics of certain types.

Equation (1) is derived by averaging the initial system of equations that describe the nonlinear medium over the fast frequency $\omega(k_0)$ corresponding to the center of the wave packet. Compared with the initial system, Eq. (1) contains an additional integral of motion—the wave action or the "number of particles"

$$N=\int|\psi|^2\,d\mathbf{r},$$

which corresponds, apart from small terms, to the wave-packet energy.

Equation (1) is a Schrödinger equation with a potential $u=-|\psi|^2$. From the quantum-mechanical viewpoints this equation describes, in the self-consistent-field approximation, a Bose gas with attraction. It is this attraction which causes the collapse. In the one-dimensional case, however, the compression of the wave packet by attraction is halted by dispersion, and a stable soliton results. In the two- and three-dimensional cases, collapse is already possible, and a sufficient condition for it is that a negative Hamiltonian of the system (1):

$$H=\frac{1}{2}\int[|\nabla\psi|^2-|\psi|^4]\,d\mathbf{r}.$$

This conclusion, rigorously proven theoretically,[1,2] is corroborated by numerical calculations.[3–5]

Of great fundamental and practical interest is the structure of the field $\psi(\mathbf{r},t)$ in the vicinity of the singularity point in a collapse. This question is quite difficult at $d=2$. It was suggested in Ref. 6, on the basis of an analysis of numerical experiments that were quite flawed in their time, that the amplitude of the field ψ has near the collapse point a self-similar character:

$$|\psi|\to\frac{1}{f(t_0-t)}R\left[\frac{r}{f(t_0-t)}\right],\quad f(0)=0, \qquad (2)$$

with the function $R(\xi)$ well defined and of the same form as a stationary two-dimensional soliton (with the so-called Townes mode). Equation (2) means that at as $t\to t_0$ there flows into the collapse point $r=0$ a finite amount of energy

$$N=2\pi\int_0^\infty rR^2(r)\,dr. \qquad (3)$$

Such a collapse was called strong in Ref. 2. Equation (2) was tested later with more exact experiments and is not subject to doubt at present. However, the expression $f=(t_0-t)^{2/3}$ obtained in Ref. 5 on the basis of simple variational estimates is too crude and is not confirmed by an accurate numerical experiment. It agrees better with the expression $f\approx(t_0-t)^{1/2}|\ln(t_0-t)|^{-1/2}$ proposed in Ref. 3. The now most accurate numerical experiment, reported in Ref. 4, offers the best corroboration of the relation $f\approx(t_0-t)^{1/2}$. We shall show that this relation follows from quite general considerations based on the virial theorem. Nonetheless, this relation has not yet been successfully deduced consistently from Eq. (1), the acknowledged best approach being that of Fraĭman.[7] On the whole, the problem of the character of two-dimensional collapse cannot be regarded as solved,

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although there is no doubt that this collapse is strong.

In the three-dimensional case ($d = 3$) the situation is at first glance simpler. Equation (1) permits the self-similar substitution[2]

$$\psi=\frac{1}{(t_0-t)^{1/2+i\alpha}}\chi\left[\frac{r}{(t_0-t)^{1/2}}\right]. \tag{4}$$

Analysis of the corresponding self-similar solution shows that as $t \to t_0$ singularities are formed, of the type

$$|\psi|^2 \to C^2/r^2. \tag{5}$$

At $d = 1$ and 2 this singularity is not integrable and a self-similar solution has no physical meaning. At $d = 3$, however, the singularity is integrable and a physical meaning can be give to the self-similar solution. This solution, which we discuss in greater detail below, corresponds to a "weak" collapse in which, formally speaking, zero energy enters the singularity point. It was previously suggested that this weak collapse is all that can be deduced within the framework of Eq. (1). The actual amount of energy absorbed in a weak collapse is determined by that level of the amplitude ψ at which Eq. (1) becomes meaningless and dissipation sets in.

However, the conclusion that a collapse is weak in the three-dimensional case contradicts to some degree our physical intuition. Let a caviton be compressed and its total number of particles N be preserved. Then the intensity at center of the caviton will increase more rapidly than in the two-dimensional case, and the role of the nonlinear effects should at any rate be not weaker. Thus, we are justified in expecting a strong collapse to be possible in the three-dimensional case. The same conclusion should hold also for the equation

$$\Delta(i\psi_t+{}^1/_2\Delta\psi)+\operatorname{div}(|\nabla\psi|^2\nabla\psi)=0, \tag{6}$$

which describes in the "static approximation" the collapse of Langmuir waves in a plasma.[2]

The question of whether a collapse is strong or weak within the framework of Eqs. (1) and (6) is important also because its solution determines whether the collapse is effective as a nonlinear energy-dissipation mechanism.

We show in the present paper that in the three-dimensional case in the nonlinear Schrödinger equation there exists a strong wave-collapse regime in which a finite amount of energy is trapped in the singularity. This collapse regime sets in under quasiclassical initial conditions, with the quasiclassicism criterion improving in the course of compression. This collapse regime is therefore called quasiclassical. It was reported by us, jointly with Musher, in a preliminary communication[8] that contains also the results of a numerical calculation. We confine ourselves here to theoretical question, viz., construction of the solution itself and matching it to the linear solution, investigation of the stability of the quasiclassical solution, and assessment of the role of radiation in the collapse. We do not touch upon numerical simulation of a wave collapse and the associated questions, which we hope to cover in a separate paper.

§1. INSTABILITY OF THREE-DIMENSIONAL SOLITONS

We begin with a discussion of the role of three-dimensional solitons in the dynamics of the system (1). We recall a few known facts.

We represent Eq. (1), which is related to Hamilton's equations, in the form

$$i\psi_t=\delta H/\delta\psi^* \tag{1.1}$$

with the Hamiltonian H of Eq. (3). With this we construct in standard fashion the Lagrangian

$$L=\int\frac{i}{2}(\psi^*\psi_t-\psi\psi_t^*)\,d\mathbf{r}-H$$

and formulate a variational principle for the action

$$S=\int L\,dt. \tag{1.2}$$

The soliton solutions $\psi_0(r,t) = \exp(i\lambda^2 t)g(\mathbf{r})$ are stationary points H at a fixed number N of quanta

$$\delta(H+\lambda^2N)=0$$

or

$$-\lambda^2g+{}^1/_2\Delta g+|g|^2g=0. \tag{1.3}$$

It can be easily seen that in this solution the number of particles decreases in inverse proportion to λ:

$$N=\lambda^{-1}\int|f(\xi)|^2\,d\xi, \tag{1.4}$$

where $f(\xi)$ is determined from the equation

$$-f+{}^1/_2\Delta f+|f|^2f=0. \tag{1.5}$$

Among the stationary solutions described by Eq. (1.5) there is one solution with a minimum value of H (at a fixed N). This is the ground state, with respect to which the eigenfunction is real, is spherically symmetric, and has no zeros.

It is known (see, e.g., Ref. 9) that the soliton solution corresponding to the ground state is unstable. This result follows directly from an analysis of equations linearized with respect to the soliton solution. According to Ref. 9, the instability condition is of the form $\partial N_s/\partial\lambda^2 < 0$, so that according to (1.4) the instability is exponential. The existence of instability is implied also indirectly by the following arguments. Consider a scale transformation that preserves the number of particles:

$$\psi_0(r)\to a^{-3/2}\psi_0(r/a). \tag{1.6}$$

Under these transformations, the Hamiltonian becomes a function of the parameter a:

$$H(a)={}^1/_2(I_1/a^2-I_2a^3), \tag{1.7}$$

where

$$I_1=\int|\nabla\psi_0|^2\,dr,\qquad I_2=\int|\psi_0|^4\,dr.$$

(A plot of this function is shown in Fig. 1.). As $a \to 0$ this function is unbounded from below, and as $a \to \infty$ it tends to

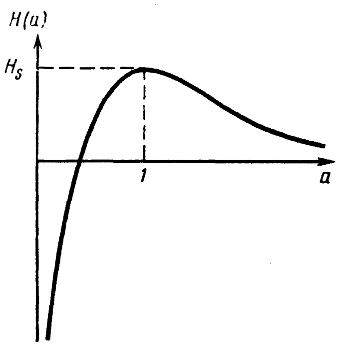

FIG. 1.

zero. The maximum of this function is reached precisely on the soliton: $a = 1$ and $H_s = \lambda^2 N_s$ (Refs. 10 and 11).

If we consider another very simple transformation, a gauge transformation $\psi_0(r) \to \psi_0(r)\exp[i\chi(r)]$, which also preserves N, then the soliton, conversely, realizes a minimum:

$$H = H_s + \int (\nabla\chi)^2 \psi_0^2\, dr.$$

Thus, in the three-dimensional case the soliton is a saddle point of an energy functional. In systems with a large number of degrees of freedom, this fact usually leads to instability of the equlibrium state.

Note that in the one-dimensional case the soliton minimizes the Hamiltonian, and is therefore stable.

To identify qualitatively the nonlinear stage of development of this instability, we use a variational principle for the nonlinear Schrödinger equation, stipulating minimization of the action (1.2) on trial functions of the form

$$\psi(r,\ t) = a^{-3/2} g(r/a) \exp(i\lambda^2 t + i\mu r^2), \tag{1.8}$$

where a and μ are real functions of the time.

Substitution of (1.8) in the action (1.2) and integration with respect to r leads to the following expression for S:

$$S = \int [-\lambda^2 N - \mu_t a^2 c - 2\mu^2 a^2 c - H(a)]\, dt, \tag{1.9}$$

where $H(a)$ [Eq. (1.7) is the value of the Hamiltonian on the functions $a^{-3/2}\psi_0(r/a)$ (and coincides with that in Ref. 7), and $c = \int \xi^2 g^2 d\xi$.

Variation of (1.9) with resepct to μ and a yields

$$\mu = a_t/2a, \tag{1.10}$$

$$2ca(\mu_t + 2\mu^2) = -\partial H(a)/\partial a. \tag{1.11}$$

Substituting (1.10) in (1.11) we obtain for a the Newton equation

$$c\ddot{a} = -\partial H(a)/\partial a.$$

In this equation the soliton corresponds to the unstable equilibrium position $a = 1$. On moving to the left from $a = 1$, the particle falls after a finite time to the center [and a corresponding singularity appears in the distribution (1.8)]. As the point $a = 0$ is approached, the influence of the dispersive term proportional to I_1 becomes negligible, and we have thus

$$c\ddot{a} \approx -3I_2/2a^4. \tag{1.12}$$

Integration of this equation specifies the law of singularity formation in (1.8):

$$a(t) \propto (t_0 - t)^{2/5}. \tag{1.13}$$

It will be shown later that this behavior near a singularity corresponds to a quasiclassical wave-collapse regime, while a substitution of the type (1.8) corresponds to a quasiclassical wave function for Eq. (1).

§2. VIRIAL THEOREM

The results of the preceding section are qualitative in character and cannot be regarded as exact. Exact results for a three-dimensional collapse were obtained by one of us.[2] They comprise a generalization of the known result of Vlasov, Petrishev, and Talanov,[1] obtained for the case $d = 2$.

We present below these results, since in our opinion their consequences have not been fully utilized. Equation (1) leads to the relation

$$\frac{d^2}{dt^2}\int r^2|\psi|^2\, d\mathbf{r} = 4H - (d-2)\int |\psi|^4\, d\mathbf{r}, \tag{2.1}$$

which is usually called the virial theorem.[2)]

At $d = 2$ the relation (2.1) is integrated twice:

$$\langle r^2\rangle = \int r^2|\psi|^2\, d\mathbf{r} \Big/ \int |\psi|^2\, d\mathbf{r} = 2Ht^2/N + C_1 t + C_2, \tag{2.2}$$

with the constants C_1 and C_2 determined from the initial data:

$$C_2 = \langle r^2\rangle|_{t=0}, \qquad C_1 = \frac{\partial}{\partial t}\langle r^2\rangle|_{t=0}.$$

Obviously, $C_2 > 0$. A sufficient condition for collapse is the vanishing of the right-hand side at certain $t = t_0 > 0$. At $H > 0$ it is necessary to satisfy the conditions

$$C_1 < 0, \quad C_1^2 \geqslant (8H/N)C_2, \tag{2.4}$$

which mean a certain "additional focusing" of the initial distribution. In the case of the general condition $C_1^2 > 8HC_2/N$ the quadratic trinomial in (2.2) crosses the real axis, and in the special case $C_1^2 = 8HC_2/N$ tangency takes place.

Assume that the general condition is satisfied and the asymptote of the amplitude ψ is given by Eq. (2). More accurately speaking, as $t \to t_0$ we have

$$|\psi|^2 = \frac{1}{f^2}\left\{R^2\left(\frac{r}{f}\right) + O\left(f, \frac{r}{f}\right)\right\}, \tag{2.5}$$

where $O(f) \to 0$ as $f \to 0$. Then

$$\langle r^2\rangle = f^2[A + O(f)], \quad A = \int r^2 R^2\, d\mathbf{r} \Big/ \int R^2\, d\mathbf{r}. \tag{2.6}$$

Substituting (2.6) in (2.2), we verify that

$$f \to \tilde{C}(t_0 - t)^{1/2}[1 + O(t_0 - t)] \quad \text{as} \quad t \to t_0.$$

Here $\tilde{C}$ is a certain constant.

At $d = 3$ the equality in (2.1) is replaced by an inequality, from which follow the previous sufficient collapse criteria (2.3) and (2.4), where the constants C_1 and C_2 have the same meaning. In this case, however, it is possible to ascertain from (2.1) when $\langle r^2\rangle$ vanishes. This calls for finding a solution of Eq. (1).

§3. WEAK SELF-SIMILAR COLLAPSE

The simplest of these solutions is the self-similar one mentioned above, determined by the substitution (4). We consider only spherically symmetric solutions, for which the function χ satisfies the equation

$$i\left[\left(\frac{1}{2}+i\alpha\right)\chi+\frac{1}{2}\xi\chi_\xi\right]+\frac{1}{2}\chi_{\xi\xi}+\frac{1}{\xi}\chi_\xi+|\chi|^2\chi=0. \quad (3.1)$$

Here $\xi = r(t_0-t)^{-1/2}$ is the self-similar variable.

The solution of Eq. (3.1) is accurate to within a phase factor $\exp(i\theta)$. It can have at some $\xi=\xi_0$ a singularity of the form

$$\chi\approx\frac{aq}{\sqrt{2}}\exp\left[i\frac{a}{(\xi-\xi_0)q}\right]\left\{\frac{1}{(\xi-\xi_0)^{1+q}}+\cdots\right\} \quad (3.2)$$

(the terms discarded are singular to a lesser degree). Here a and q are certain constants. We are interested only in a regular solution of (3.1) that decreases as $\xi\to\infty$ and satisfies the equation

$$(^1/_2+i\alpha)\chi+{}^1/_2\xi\chi_\xi=0. \quad (3.3)$$

Consequently χ has an asymptote

$$\chi\to C/\xi^{1+2i\alpha}, \quad (3.4)$$

where C is a certain constant that can be assumed, without loss of generality, to be positive and real.

The requirement that the solution be regular eliminates the ambiguity in the choice of the constants α and C. We have in fact a nonlinear eigenvalue problem for Eq. (3.1). This problem can be solved numerically by the "shooting" method, which yields

$$\alpha=0.545, \quad C=1.01. \quad (3.5)$$

Plots of the functions $|\chi(\xi)|$ and $V(\xi)=\partial\Phi/\partial\xi$ $(\Phi=\arg\chi)$ are shown in Fig. 2.

We proceed to interpret the self-similar solution. We note first that for any fixed point of physical space with coordinate r, the corresponding self-similar coordinate ξ tends to infinity as $t\to t_0$. The self-similar solution goes over then into its asymptote (3.4), which takes in the physical variables r and t the form

$$\psi\to C/r^{1+2i\alpha}, \quad (3.6)$$

i.e., it remains finite as $t\to t_0$. The self-similar solution is realized in physical space in a certain region with coordinate $r<r_0$, where r_0 is the dimension of the region. The integrable singularity (3.6) "grows out" at the center of this region as $t\to t_0$. At first glance, the self-similar substitution (4) leads to nonconservation of the integral

$$N=\int|\psi|^2\,dr \quad (3.7)$$

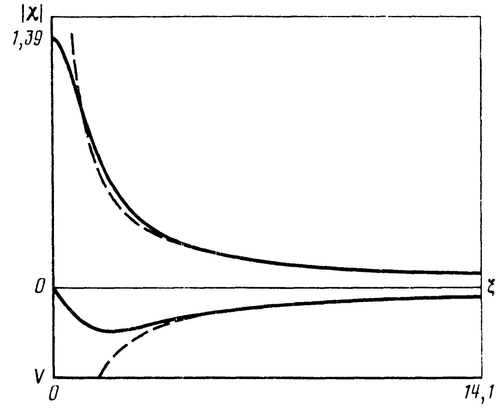

FIG. 2. Plots of $|\chi(\xi)|$ and $V(\xi)$. The dashed lines correspond to the asymptotics $|\chi(\xi)|\to c/\xi$ and $V(\xi)\to -2\alpha/\xi$.

and corresponds by the same token to a value $N=\infty$ for the integral. This is indeed the case if the self-similar solution is considered in all of space. In any finite region $r<r_0$, however, the value of the integral N remains constant.

Indeed, after substituting (4) in (3.7) we have for the region $r<r_0$

$$N=(t_0-t)^{1/2}4\pi\int_0^{r_0(t_0-t)^{-1/2}}\xi^2|\chi(\xi)|^2\,d\xi. \quad (3.8)$$

When account is taken of the asymptote (3.4), the integral in (3.8) diverges on the upper limit and tends to a finite value as $t\to t_0$. Let now r_0 be large enough. The value of the integral N in the region $r<r_0$ should be close to its value in this region at the instant of the collapse $t=t_0$. In other words, the following equation should hold:

$$\int_0^\infty\{|\chi|^2-C^2/\xi^2\}\xi^2\,d\xi=0. \quad (3.9)$$

This equation was verified with high accuracy for the computer-generated $\chi(\xi)$ and C.

Note that a similar situation with formal nonconservation of the integral of motion is a frequency occurrence for self-similar solutions. It was analyzed in detail in Ref. 14, using as the example the integral of the total number of particles for supersonic collapse of Langmuir waves.[14] The solution constructed here corresponds to weak collapse—formally speaking, zero energy enters the singularity at $r=0$. This means in fact that if the characteristic amplitude value at which energy absorption in the collapse sets in and Eq. (1) no longer holds is of the order of ψ_0, then the amount of energy absorbed in one collapse act is

$$\Delta N\sim\psi_0^2\xi_0^3\sim1/\psi_0.$$

Here $\xi_0\sim1/\psi_0$ is the characteristic dimension of the absorption region.

To conclude this section, we note that the weak regime of wave collapse was first demonstrated in numerial experiments[5] and later in Ref. 8.

§5. QUASICLASSICALLY STRONG COLLAPSE

We discuss now the feasibility of realizing a strong collapse within the framework of the nonlinear Schrödinger equation (1). We note first that this collapse must be quasiclassical in the sense that as the collapse point is approached the conditions for applicability of the quasiclassical approximations improve for Eq. (1) regarded as a Schrödinger equation with a potential $U= -|\psi|^2$. In fact, these conditions are of the form

$$Q=Ua^2\gg 1. \tag{4.1}$$

Here $U\sim|\psi_0|^2$ is the characteristic value of the potential, and a is its characteristic scale in space. In the case of strong collapse, the relation $Ua^3\sim|\psi_0|^2a^3\sim\text{const}=N$ is satisfied, so that $Q\sim\text{const}/a$. In a collapse we have $a\to 0$ and the condition (4.1) can be met regardless of the value of the constant N. To obtain the corresponding solution of (1), we separate in it the amplitude from the phase

$$\psi=Ae^{i\Phi}, \tag{4.2}$$

$$\partial A^2/\partial t+\operatorname{div} A^2\nabla\Phi=0, \tag{4.3}$$

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2-A^2=\frac{1}{2}\frac{\Delta A}{A}. \tag{4.4}$$

When the quasiclassicism condition (4.1) is met, the right-hand side of (4.4) can be neglected. The resultant system of hydrodynamic equations

$$\begin{aligned}&\partial A^2/\partial t+\operatorname{div}(A^2\nabla\Phi)=0,\\ &\partial\Phi/\partial t+{}^1/_2(\nabla\Phi)^2-A^2=0\end{aligned} \tag{4.5}$$

describes a gas with negative pressure and with an adiabatic exponent $\gamma=2$. We seek a solution of these equations in self-similar form, assuming this solution to be spherically symmetric and to conserve the total number of particles

$$A_0^2=a^{-3}n(r/a). \tag{4.6}$$

Substitution of (4.6) in (4.3) leads to the equation

$$a_ta\frac{1}{\xi^2}\frac{\partial}{\partial\xi}(\xi^2n)=\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2n\frac{\partial\Phi}{\partial\xi}\right), \tag{4.7}$$

which has an integral of the form

$$\frac{\partial\Phi}{\partial\xi}=a_ta\xi \quad \text{или} \quad \Phi=\frac{a_ta}{2}\xi^2+\Phi_0(t). \tag{4.8}$$

Substituting (4.8) in (4.5) we get

$$n(\xi)=\lambda^2(1-\xi^2),$$

$$\Phi=\int_0^t\frac{\lambda^2\,dt}{a^3}+\frac{a_ta}{2}\xi^2=\int_0^t\frac{\lambda^2\,dt}{a^3}+\frac{a_tr^2}{2a}. \tag{4.9}$$

Here λ^2 is an arbitrary constant, and $a(t)$ obeys the Newton equation

$$\ddot a=-\partial V/\partial a=-2\lambda^2/a^4, \tag{4.10}$$

that describes the falling of a classical particle to the center in a potential $V=-2\lambda^2/3a^3$. As $t\to t_0$ we get

$$a\to({}^{25}/_3\lambda^2)^{1/5}(t_0-t)^{2/5},\quad A_0^2(0,t)\propto(t_0-t)^{-6/5}.$$

The solution (4.9), (4.10) is meaningful only at $\xi\leqslant 1$, and if $\xi>1$ we must set A^2 equal to zero.

The solution constructed describes a strong collapse of a wave packet as a whole. It is interesting that this collapse regime is qualitatively described correctly with the aid of the rather crude estimates given in §1. This can be easily verfied by comparing Eqs. (1.12) and (4.10). Qualitatively fair agreement is obtained also for the concluding stage of the collapse.

The solutlon (4.9), (4.10) is not valid, of course, in a certain region near the poinst $\xi=1$. The size of this region can be estimated by comparing the quantities

$$\frac{A_{rr}}{A}\approx\frac{1}{a^2(\Delta\xi)^2},\quad A^2\approx\frac{\lambda^2}{a^3}\Delta\xi,$$

from which we have

$$\Delta\xi\approx a^{1/3}\lambda^{-2/3}. \tag{4.11}$$

It can be seen from (4.11) that as $a\to 0$ the relative width of the region in which the self-similar solution (4.9) is not valid becomes narrower as the collapse point is approached, and the problem of finding the solution in this narrow region (spherical layer) becomes esentially one-dimensional.

To solve this problem, we note that the phase Φ obtained from Eq. (4.3) is determined by integration over the entire region of ξ and that near the point $\xi=1$ it should be to the quasiclassically specified expression (4.9).

The phase enters in Eq. (4.4) for the amplitude only in the combination

$$\partial\Phi/\partial t+{}^1/_2(\nabla\Phi)^2-A_0^2=0. \tag{4.12}$$

Here $A_0^2(r,t)$ is the quasiclassical value of A^2 specified by Eq. (4.9).

Equation 4.4 has thus taken the form

$$A(A_0^2-A^2)={}^1/_2\Delta A. \tag{4.13}$$

We change over in this equation to a new coordinate

$$r'=r-a(t) \tag{4.14}$$

and expand $A_0^2(r,t)$ in a Taylor series in the vicinity of the point $r'=0$. Recalling that $A_0^2[a(t),t]=0$ we retain only the term linear in r'. Next, replacing the spherical Laplacian by a planar one, we get

$$\left[2\frac{\lambda^2r'}{a^4}+A^2\right]A+\frac{1}{2}\frac{\partial^2A}{\partial r'^2}=0. \tag{4.15}$$

By a simple change to nondimensional variables

$$r'=ax(a/2\lambda^2)^{1/3},\quad A=(2\lambda^2/a^4)^{1/3}\varphi \tag{4.16}$$

we rewrite (4.15) in the form

$$[x+\varphi^2]\varphi+{}^1/_2\varphi_{xx}=0. \tag{4.17}$$

This equation has a solution expressed in terms of Painleve transcendental functions.

We shall not consider here the theory of these functions (see, e.g., Ref. 15). We indicate only that Eq. (4.17) has a solution $\varphi(x)$ with an asymptotic $\varphi_- = (-x)^{1/2}$ as $x \to -\infty$. The general solution of (4.17) has for $x \to -\infty$ an asymptotic

$$\varphi \to (-x)^{1/2} + \frac{c_1}{(-x)^{1/4}} \exp[4i(-x)^{3/2}] + \frac{c_2}{(-x)^{1/4}} \exp[-4i(-x)^{3/2}], \tag{4.18}$$

where $c_1 = c_2^*$ is a certain complex constant to be determined. A more detailed analysis shows that in the next approximation in the parameter $\varepsilon = (a/2\lambda^2)^{1/2}$ the constants c_1 and c_2 become slow functions of x, and moreover $c_1 \neq c_2^*$, so that Eq. (4.17) must be replaced by a more general complex equation. This difficult task is outside the scope of the present article. We note only that Eq. (4.17) becomes a linear Airy equation as $x \to \infty$, and its solution must be matched to the solution of the linearized Schrödinger equation. The character of this solution can be understood from qualitative considerations.

We consider now the behavior of the oscillation energy N_c (of the integral N) contained in this layer. Assuming a layer of width $x \sim 1$, we have

$$N_c \approx (2\lambda^2/a^4)^{2/3} a^2 a (a/2\lambda^2)^{1/2} \approx (2\lambda^2)^{1/6} a^{2/3}.$$

$N_c \to 0$ as $a \to 0$, so that the boundary layer loses energy as the collapse point is approached. This energy is emitted as low-amplitude waves through the transition zone. The characteristic emitted wavelength is of the order of the boundary-layer thickness and tends as $t \to t_0$ to zero like $\lambda \sim a^{4/3} \sim (t_0 - t)^{8/15}$.

A quasiclassically collapsing caviton is thus an emitter of low-amplitude waves.

Besides the considered self-similar solution (4.9), (4.10), which describes a strong collapse, the quasiclassical equations (4.5) admit of large class of self-similar solutions of the form

$$A^2 = (t_0 - t)^{-\alpha} n(\xi), \quad \Phi = \lambda^2 \int^t \frac{dt}{(t_0 - t)^\alpha} + (t_0 - t)^{1-\alpha} \varphi(\xi),$$
$$\xi = r(t_0 - t)^{\alpha/2 - 1}, \tag{4.19}$$

where n and φ are determined from the solution of the set of equations

$$\alpha n + \frac{2-\alpha}{2} \xi \frac{\partial n}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^2 n \frac{\partial \varphi}{\partial \xi} = 0,$$
$$\lambda^2 + \left[(1-\alpha)\varphi + \frac{2-\alpha}{2} \xi \frac{\partial \varphi}{\partial \xi} \right] + \frac{1}{2} \left(\frac{\partial \varphi}{\partial \xi} \right)^2 - n = 0. \tag{4.20}$$

The quasiclassicism criterion (4.1) is satisfied for the solution (4.19) if $\alpha > 1$. All these solutions have a power-law time-independent asymptotic form $A^2 \to C^2 r^{2\alpha/(\alpha-2)}$. This means that as $t \to t_0$ a singularity of the form $Cr^{2\alpha/(\alpha-2)}$ is produced at the point $r = 0$. This singularity is integrable if $\alpha < 6/5$ (only such solutions have any meaning). Thus, the solutions (4.19) describe an intermediate collapse regime, ranging from the fastest with scale $r_0 \sim (t_0 - t)^{1/2}$ to the slowest with $r_0 \sim (t_0 - t)^{2/5}$. All are weak wave collapses. It should be added that the quasiclassical approach is not valid also for these self-similar solutions if r exceeds a certain value R determined from a comparison of A^2 and $\Delta A/A$.

Note that the existence, within the framework of (4.20), of regular quasiclassical solutions that describe a weak self similar collapse is still a moot question.

§5. STABILITY OF QUASICLASSICAL SELF-SIMILAR COLLAPSE

Let us examine the stability of the obtained solution (4.9), (4.10). Since $|\psi|^2 \gg 1/a^2$ for a strong collapse, the most dangerous from the standpoint of stability are short-wave perturbations with $k \gg 1/a$. Recall that the growth rate of the modulation instability of a monochromatic wave with $k = 0$, $\psi = A \exp(i|A|^2 t)$, called the condensate, is defined as

$$\Gamma = (|A|^2 k^2 - {}^1/_4 k^4)^{1/2}. \tag{5.1}$$

It is a maximum at $k_m^2 = 2|A|^2$ and is equal to $\Gamma_{\max} = |A|^2$, i.e., the dispersion terms in the region of the maximum growth rate are of the order of the nonlinear ones. At $k^2 \ll |A^2|$ the dispersion terms, conversely, are insignificant and the instability is quasiclassical in this case.

According to (5.1), the maximum growth rate for the investigated solution should be located at $k^2 \sim |\psi|^2 \gg 1/a^2$ in the short-wave region. We can therefore use this expression for estimates, replacing $|A|^2$ by λ^2/a^2 and assuming a t-dependence of the perturbations in the form $\exp \int \Gamma dt'$. In addition, we must take into account the spatial shrinking of the wave packet. This means that the wave number k also depends on the time: $k = p/a(t)$, where p is a conserved quantity.

As a result we have for

$$\Gamma_p(t) = [p^2\lambda^2/a^5 - p^4/4a^4]^{1/2}. \tag{5.2}$$

It can be seen from this equation that as $a \to 0$ the principal term under the square-root sign is the first. This means that as the perturbations increase they become quasiclassical. Recalling that as $t \to t_0$ we have

$$a(t) \to (25\lambda^2/3)^{1/5} (t_0 - t)^{2/5},$$

we obtain $\Gamma_p(t) \to (3/25)^{1/2} p/(t_0 - t)$, and

$$\exp\left(\int \Gamma dt' \right) \to (t_0 - t)^\gamma, \quad \gamma = -3^{1/2} p/5. \tag{5.3}$$

Since $p \gg 1$, Eq. (5.2) demonstrates the instability of the strong-collapse regime to short-wave perturbations; this instability is of the disruptive type. This result follows also directly from Eqs. (4.3) and (4.4) if they are linearized against the background of the quasiclassical solution (4.9), (4.10):

$$\frac{\partial n'}{\partial t} + (\mathbf{v}\nabla) n' + \frac{3a_t}{a} n' + \operatorname{div}(A_0^2 \nabla \Phi') = 0,$$
$$\frac{\partial \Phi'}{\partial t} + (\mathbf{v}\nabla)\Phi' - n' = \frac{1}{4} \frac{\Delta n'}{A_0^2}.$$

Here n' and Φ' are the perturbations of the intensity A^2 and of the phase Φ, while $\mathbf{v} = \nabla\Phi = a_t \mathbf{r}/a$. If we change in these equations to the self-similar variables $\xi = \mathbf{r}/a(t)$ and $t' + t$,

$$\partial n'/\partial t + 3a_t n'/a + a^{-2}\,\mathrm{div}_\xi (A^2 \nabla\Phi') = 0, \tag{5.4}$$

$$\partial\Phi'/\partial t - n' = \Delta_\xi n'/4A_0^2 a^2, \tag{5.5}$$

it can be seen by comparing the last term of (5.5) with n' that as $a \to 0$ the contribution of this term tends to zero. It follows thus directly from the equations that as $a \to 0$ the perturbations become quasiclassical. If we neglect the spatial dependence of A_0^2 on ξ by putting $A_0^2 = \lambda^2/a^3$, then Eqs. (5.3) and (5.4) have as $t \to t_0$ solutions of the form

$$\Phi' \sim (t_0 - t)^\gamma \exp(i\mathbf{p}\xi),$$

where we obtain the earlier result (5.3) for $\gamma = -3^{1/2}p/5(p \gg 1)$.

For long-wave perturbations with wavelengths comparable with $a(t)$, account must be taken of the dependence of A_0^2 on ξ. In this case, as $t \to t_0$, we obtain for the perturbations

$$\Phi' = (t_0 - t)^\gamma \chi(\xi) Y_{lm}(\theta, \varphi)$$

the following spectral problem:

$$\frac{25}{3}\gamma\left(\gamma + \frac{1}{5}\right)\chi + \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^2(1-x^2)\frac{\partial\chi}{\partial x}\right] + (1-x^2)\frac{l(l+1)}{x^2}\chi = 0 \tag{5.6}$$

with the condition that χ be regular at the points $x = 0$ and $x = 1$. Replacing $\chi = x^l W(x^2)$ by W we obtain the differential equation for the hypergeometric function. The solution, regular at $x^2 = 0$ and $x^2 = 1$, is expressed in terms of Jacobi polynomials $P_n^{(l+1/2,0)}(1 - 2x^2)$; then

$$^{25}/_3\gamma(\gamma + {}^1/_5) = 4n(n + l + {}^3/_2) + 2l, \quad n = 0, 1, \ldots. \tag{5.7}$$

At $l = 0$ the solution of (5.6) is expressed in terms of a Legendre polynomial: $\chi(x) = x^{-1}P_{2n+1}(x)$.

The solution (4.9) is stable if $h'/A_0^2 \to 0$ as $t \to t_0$, i.e., at $\gamma > -1/5$. According to (5.7), all the perturbations are unstable, and the instability is stronger the shorter the perturbation wavelength. For smooth initial conditions, the instability will come into play later than for jagged ones.

It can be seen, in analogy with (5.1) and (5.2), that all other self-similar quasiclassical regimes will likewise be unstable, since as $t \to t_0$ we have $A^2 \gg 1/a^2 (A \sim (t_0 - t)^{-\alpha}$, $a \sim (t_0 - t)^{1-\alpha/2}$, $6/5 > \alpha > 1)$. This is a modulation instability. Nothing can be said concerning the regime with $\alpha = 1$, for in this regime $A^2 \sim 1/a^2$. It is possibly stable, but this cannot be rigorously proved.

Thus, the considered quasiclassical self-similar solutions that describe three-dimensional collapse have all turned out to be unstable. This raises the natural questions of their significance and of the role that they can play in the dynamics of the system, in view of their instability, and moreover since the only regime that can be regarded as stable is already known.

These questions have several answers. First, collapsing solutions have meaning only so long as the nonlinear equation itself is applicable, i.e., up to a certain finite but quite large amplitude. If the initial condition is close to one solution, albeit unstable, the system can nonetheless exist (evolve) for some time in accordance with the described scenario, until the instability alters greatly the given solution. It may also turn out that the field amplitude had reached its applicability limit before the instability had time to develop. This is precisely the situation observed in Musher's numerical calculations that were published in our joint article.[8] An approach to the strong-collapse regime was observed under initial conditions close to the solution (4.9). Second, in the same experiments with initial conditions $\partial\Phi/\partial r|_{t=0} = 0$ the intensity of the collapse was observed to increase like $(t_0 - t)^{-1}$ at $r = 0$ (see also Ref. 5), with simultaneous trapping of finite energy in the collapse zone. The two observed tendencies, which are related simultaneously to weak and strong collapses, find a natural explanation within the framework of the given theory. Trapping of finite energy into the collapse zone should have led to a quasiclassical strong, but unstable, collapse. A reflection of this instability and of its nonlinear stage is the formation, near $r = 0$, of a zone of the fastest of the weak collapses, with the intensity at the center of the zone varying as $(t_0 - t)^{-1}$. The energy stored in this zone is low and is proportional to the zone dimension: $r_0 \propto (t_0 - t)^{1/2}$. This zone is produced as a result of development of instability of the shortest perturbations. Next, this zone should be surrounded by a transition region with smoothly varying scales $R \sim (t_0 - t)^\beta$, with β ranging from 1/2 to 2/5. The transition region contains information on the unstable perturbations with intermediate scales—from the shortest to the longest. This region terminates in an energy-containing region that should evolve quasiclassically and should vary in size as $(T_0 - t)^{2/5}$. The energy-containing region should also determine the collapse time of the entire collapsing region, estimated at $\tau \sim a_0^{5/2}\varepsilon_{\mathrm{cav}}^{-2}$, where $\varepsilon_{\mathrm{cav}}$ is the energy fed to the collapse zone and a_0 is the initial scale.

These arguments apply to the spherically symmetric case. In general, when no symmetry is imposed by the numerical-experiment conditions, another behavior is possible. It follows from (5.2) that as the collapse point is approached all the short-wave perturbations, which are stable at the initial instant of time, become unstable, and near the instant of collapse they increase faster the shorter their wavelength. By the same token, as the instant of collapse is approached, latent perturbations are, figuratively speaking, awakened in the caviton. As a result, a three-dimensional collapse described by the nonlinear Schrödinger equation can be accompanied by a stochastic fragmentation of the scales, where the collapsing caviton breaks up into a "spray" of smaller ones. Each of the resultant droplets will subsequently contract self-similarly and will also be stochastically fragmented. All this can be regarded as a new form of stochastic behavior in dynamic system. The final answer to the question of the character of the three-dimensional collapse can be provided by a numerical experiment. As shown by the preceding reasoning, such an experiment must be essentially three-dimensional. This is as yet a difficult task even for the better modern computers.

CONCLUSION

Let us discuss briefly the possible application of these results in the theory of a Langmuir collapse described by Eq. (6) that is similar in structure to the nonlinear Schrödinger equation (1). Equation (6) is also Hamiltonian with

$$H=\frac{1}{2}\int (|\Delta\psi|^2-|\nabla\psi|^4)\,d\mathbf{r},$$

and descibes the total number of waves (energy) $N=\int|\nabla\psi|^2d\mathbf{r}$. It is clear therefore that for Eq. (6) there exists the same classification of collapses as for the NSE. Indeed, we can make in (6), just as in (1) the self-similar substitution

$$\nabla\psi=\frac{1}{(t_0-t)^{-1/2+i\alpha}}\frac{\partial}{\partial\xi}f(\xi),\qquad \xi=\frac{\mathbf{r}}{(t_0-t)^{1/2}},$$

which describes a weak collapse regime with $r_0\sim(t_0-t)^{1/2}$. As $t\to t_0$ singularities $|\nabla\psi|^2\to C^2(\theta,\varphi)/r^2$ are formed also in the intensity distribution, and the distribution cannot be normalized—the total number of quanta is infinite.

To describe a strong-collapse regime, it is necessary to go in (6) to the quasiclassical limit. Putting $\psi=A\exp(i\Phi)$, we find, after straightforward but rather laborious calculations, that the quasiclassical equations for (1) and (6) coincide:

$$\partial n/\partial t+\operatorname{div}(n\nabla\Phi)=0,\quad \partial\Phi/\partial t+(\nabla\Phi)^2/2-n=0. \tag{6.1}$$

Here $n=A_2(\nabla\Phi)^2$ is equal, with quasiclassical accuracy, to the wave intensity. It might seem therefore that all the results of §§3 and 4 can be applied to a Langmuir collapse. Equation (6), however, which has a vector structure, has in the case of spherical symmetry a solution with a field $\mathbf{E}=\nabla\psi$ equal to zero at the center. Consequently, the intensity $|\mathbf{E}|^2$ is also zero at the center. Our solution (4.9), obviously, does not meet this condition. This means that for a strong Langmuir collapse the solution must be asymmetric, and should most likely have a dipole structure. As for the self-similarity of the quasiclassical Langmuir collapse, the simplest estimates, and the conclusions concerning their stability, all remain of course the same as before.

It must be added to the foregoing that the quasiclassical equation (6.1), in contrast to (6), admits formulation of a virial theorem [cf. (2.1)]:

$$\frac{\partial}{\partial t^2}\int r^2n\,d\mathbf{r}=4H-(d-2)\int n^2\,d\mathbf{r},$$

$$H=\frac{1}{2}\int [n(\nabla\Phi)^2-n^2]d\mathbf{r},$$

from which we get for the quasiclassical equations the sufficient condition for collapse, $H<0$.

In conclusion, we are grateful above all to S. L. Musher for performing the numerical experiments that contributed to the writing of the present paper, and to L. N. Shchur for computer calculation of the self-similar solution.

[1)] It appears that this approach was first used in Ref. 12 for the self-focusing problem.

[2)] This is, strictly speaking, not quite accurate. The virial theorem is customarily used for quasiperiodic motion and is assumed to incorporate time averaging. The analogy is based on the fact that the simplest method of obtaining the virial theorem in mechanics is to calculate the second derivative of the moment of inertia with respect to time (cf. Ref. 13); the role of this moment is played in our case by the quantity $\int r^2|\psi|^2dr$.

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Translated by J. G. Adashko