Effect of collapse of sound waves on the structure of collisionless shock waves in a magnetized plasma

E. A. Kuznetsov and S. L. Musher

Institute of Automation and Electrometry, Siberian Branch, Academy of Sciences of the USSR
(Submitted 18 February 1986)
Zh. Eksp. Teor. Fiz. **91**, 1605–1619 (November 1986)

The effect of acoustic collapse on the structure of collisionless weak shock waves propagating through a low-pressure plasma at an angle to the magnetic field is studied. In a first step, it is shown on the basis of the three-dimensional Kadomtsev-Petviashvili equation that acoustic collapse occurs under certain conditions in media with a positive dispersion. A study is made of how the emission of small-amplitude waves affect this process. The results show that, for oblique weak shock waves, the collapse of fast magnetic sound provides a new dissipation mechanism, which results in a transfer of energy to ions. The widths of collisionless shock waves and the effective rate of energy transfer to ions due to the collapse are derived. A comparison is made with other nonlinear three-wave processes. The results show that all of these processes are strongly suppressed as a result of the removal of the excited oscillations from the shock wave front.

INTRODUCTION

Wave collapse, i.e., an event in which a singularity forms in a wave field over a finite time, is one of the fundamental processes in the physics of continuous media. It is of particular importance as a mechanism for the generation of fast particles in plasma physics. The efficiency of many collective plasma-heating methods depends on it. It was originally thought that this phenomenon is peculiar to Langmuir turbulence,[1] but several closely related phenomena have subsequently been discovered for other plasma waves.[2–6] It is now clear that all these cases have much in common and are described by equations of similar structure. Another physical situation arises in systems with a hydrodynamic nonlinearity: sound waves in weakly dispersive media, described by the Kadomtsev-Petviashvili equation. Significantly, in media with a positive dispersion the nonlinear stage of the instability of the initial distribution is wave collapse, as was shown by Shafarenko and the present authors.[7] In the present paper we report a further study of highly nonlinear processes in dispersive media. The results show that the nonlinear evolution is determined by the dispersion law. If the dispersion is negative, neither collapse nor three-dimensional solitons occur, and the initial distribution undergoes a diffractive spreading. A wave collapse classification is proposed. This classification is based on the transformation properties of the integrals of motion under gauge transformations. We study the fundamental question of how the emission of small-amplitude wave affects the formation of a singularity in the wave field.

The general results derived here are then used to determine the structure of weak collisionless waves. In the frequency region in which we are interested here ($\omega < \omega_{Hi}$), we offer a classification of nonlinear processes. We show that the primary interaction is the three-wave interaction of fast magnetosonic waves. The collapse of fast magnetic sound gives rise to a fundamental change in the structure of the shock waves. The plane front breaks up into clusters; the long, leading, oscillating train disappears, etc. We derive the front width and the effective rate of energy transfer to the particles which result from the collapse.

§ 1. INSTABILITY OF SOLITONS

Let us examine the propagation of sound waves in a weakly dispersive medium, for which the expansion of the dispersion law in the long-wave region begins with the term linear in k:

$$\omega(k)=kc_s(1+\chi(\theta)k^2\lambda_D^2). \qquad (1)$$

Sound waves of this type are characteristic primarily of isotropic media, but they are also sometimes seen in anisotropic media. In a magnetized plasma with $\beta = 8\pi nT/H_0^2 \ll 1$, at frequencies below the ion cyclotron frequency, we know that fast magnetosonic waves have this type of dispersion.

We will assume here that the dispersion of the waves, which is a function of the propagation angle θ and which is characterized by the length λ_D, is weak. By this we mean that for small-amplitude waves the most important nonlinear process is the three-wave process, for which, in the approximation of zero dispersion, the decay conditions.

$$c_s|\mathbf{k}|=c_s|\mathbf{k}_1|+c_s|\mathbf{k}_2|, \quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$$

are satisfied for collinear vectors $\mathbf{k}_i$. Nonlinearity or dispersion (if the dispersion is positive) causes the wave vectors of the interacting waves to be noncollinear. It is clear that if the nonlinearity and the dispersion are both weak the angle between the interacting waves will be small. Such an interaction allows us to describe the evolution of a sound wave packet with a narrow angular distribution by means of the universal Kadomtsev-Petviashvili equation[8]

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+c_s\frac{\partial u}{\partial x}-c_s\chi\lambda_D^2\frac{\partial^3 u}{\partial x^3}+\mu u u_x\right)=-\frac{c_s}{2}\Delta_\perp u. \quad (2)$$

 0038-5646/86/110947-09$04.00

The main term in this equation, $c_s u_x$, describes propagation along the x axis at the sound velocity. All the other terms, which are responsible for the dispersion, the nonlinearity, and the diffraction, describe slow variations of the sound field superposed on the fast motion at the velocity c_s. A consequence of this adiabatic situation is the conservation of the integral $\frac{1}{2}\int u^2 d\mathbf{r}$, which is the momentum component P_x for Eq. 2 and which is equal to the energy of the sound waves, to within a constant factor.

A point of fundamental importance is that the interaction of the sound waves in a region of weak dispersion is of the form $\mu u u_x$. This form results from a renormalization of the sound velocity and reflects the quasi-one-dimensional nature of the three-wave interaction. Accordingly, the probabilities for other nonlinear processes, caused by the so-called vector nonlinearity, are small because of the small wave interaction angle. Nonlinearities of this sort may be important only because of some large additional factor which cancels out this small factor.

Transforming to a coordinate system which is moving along the x axis at velocity c_s, and carrying out some simple manipulations, we can put Kadomtsev-Petviashvili equation (2) in the standard form:

$$\frac{\partial}{\partial x}(u_t+u_{xxx}+6uu_x)=-\varkappa\Delta_\perp u.$$

This equation can be written in canonical form:

$$u_t=\frac{\partial}{\partial x}\frac{\delta\mathscr{H}}{\delta u}, \qquad (3)$$

where the Hamiltonian is

$$\mathscr{H}=\int\left\{\frac{u_x^2}{2}+\frac{\varkappa}{2}(\nabla_\perp w)^2-u^3\right\}dx\,d\mathbf{r}_\perp, \qquad w_x=u. \qquad (4)$$

In these expressions, the sign of the coefficient $\varkappa$ determines the type of dispersion. If the dispersion is positive, we have $\varkappa=1$, while if it is negative we have $\varkappa=-1$. Positive dispersion is characteristic of gravity-capillary waves at the surface of a liquid and, under certain conditions, phonons in liquid helium. For fast magnetosonic waves in a plasma with $\beta\ll1$, the dispersion is positive for angles with respect to the external magentic field which are not close to $\theta=0$ and $\theta=\pi/2$. For nearly longitudinal and transverse propagation, the dispersion of the fast magnetosonic waves undergoes changes, and Kadomtsev-Petviashvili equation (2) is no longer valid for these angles. Waves with positive dispersion, in contrast with those with negative disperison, have nontrivial dynamics. Consequently, if the dispersion is positive one-dimensional solitons

$$u(x,t)=2\nu^2\,\mathrm{ch}^{-2}\,\nu(x-4\nu^2 t)$$

are unstable against transverse perturbations with a growth rate[9]

$$\Gamma(k_\perp)=3^{-1/2}\cdot4k_\perp(\nu^2-3^{-1/2}\cdot4k_\perp)^{1/2} \qquad (5)$$

but at the same time they are stable if the dispersion is negative ($\varkappa=-1$). The growth rate $\Gamma(k_\perp)$ turns out to be positive in a finite region $k_\perp<k_\perp^*=3^{1/2}\nu^2/4$, as in the case of Langmuir waves (plasma waves) in a plasma. The reason for this instability is that the velocity of a soliton decreases with increasing amplitude. If the modulation of the soliton along the transverse coordinate is small, regions with a lower amplitude will overtake regions with a higher amplitude. The result is a self-focusing instability.[8,10] Two-dimensional solitons are unstable in an analogous way to variations in the transverse direction.[11]

We should point out that this instability may be thought of as a particular case of the decay instability of a steady-state periodic wave. The growth rate (5) corresponds to the case in which the separation between the solitons making up the stationary wave becomes large in comparison with the dimensions of a soliton. If the dispersion is negative, periodic stationary waves are stable according to the Kadomtsev-Petviashvili equation[1)] (Ref. 12).

The nonlinear stage of the instability should result in the breakup of the front into distinct regions, because of the conservation of P_x (the total energy of the sound waves). The dynamics of the system in the next stage will be determined by the behavior of each such region. The evolution of these regions may take several paths. We first consider the possibility that three-dimensional solitons form. A three-dimensional soliton corresponds to a solution of the form $u_0(x-vt, r_\perp)$, which is determined by virtue of (3) from the variational problem

$$\delta(\mathscr{H}+vP_x)=0. \qquad (6)$$

The relationship between the values of $\mathscr{H}$ and P_x can be found quite simply for the soliton solution of Refs. 7 and 11. We first consider the case $\varkappa=1$. Multiplying the equation

$$\frac{\delta}{\delta u}(\mathscr{H}+vP_x)=0$$

by u and integrating over r, we find

$$2P_x v+I_1+I_2-3I_3=0, \qquad (7)$$

where

$$I_1=\int u_x^2\,d\mathbf{r}, \quad I_2=\int(\nabla_\perp w)^2 d\mathbf{r}, \quad I_3=\int u^3\,d\mathbf{r}.$$

In order to derive two other relations among I_1, I_2, and I_3, we consider trial functions of the form $u_0(\lambda x, \mu r_\perp)$, for which we have two equalities by virtue of (4):

$$\frac{\partial}{\partial\lambda}(\mathscr{H}+vP_x)\big|_{\lambda=\mu=1}=\frac{\partial}{\partial\mu}(\mathscr{H}+vP_x)\big|_{\lambda=\mu=1}=0 \qquad (8)$$

or

$$\begin{gathered}-P_x v+{}^1/_2 I_1-{}^3/_2 I_2+I_3=0,\\ (d-1)(P_x v+{}^1/_2 I_1-I_3)+{}^1/_2(d-3)I_2=0.\end{gathered} \qquad (9)$$

If follows from (7)–(9) that the velocity of the soliton is positive, while the value of the Hamiltonian for the soliton solution,

$$\mathscr{H}_s=vP_x(2d-5)/(7-2d) \qquad (10)$$

is negative for one-dimensional and two-dimensional solitons and positive for three-dimensional ($d=3$) solitons. For waves with a negative dispersion, relations (7)–(9) turn out to be in consistent for $d = 2$ and 3: I_1 and I_2 have different signs, although they are positive by assumption. This is the simplest proof that in a system with a negative dispersion there are no solutions corresponding to multidimensional solitons. Relation (10), which we have derived for waves with a positive dispersion, yields a simple conclusion regarding the nonlinear stage of the instability (5). As the instability occurs, and the front of the soliton breaks up into distinct clusters, it is not possible for three-dimensional solitons to form because $\mathscr{H}$ is negative in the initial state. We have an analogous situation for the nonlinear Schrödinger equation and for the Zakharov equations describing the interaction of Langmuir waves and ion acoustic waves in a plasma.

§ 2. COLLAPSE OF SOUND WAVES

The development of instability (5) therefore cannot end in the formation of three-dimensional solitons. It is also important to note that three-dimensional solitons are unstable, as can be shown with the help of the Lyapunov theorem. A given stationary point will be stable if it corresponds to a minimum or maximum the Hamiltonian (possibly a local minimum or maximum). If the stationary point is a saddle point, the state is unstable.

Let us consider the simplest scaling transformations which leave the momentum projection P_x invariant:

$$u_0(x, r_\perp) \to \alpha^{-1/2}\beta^{(1-d)/2}u_0(x/\alpha, r_\perp/\beta).$$

Under these transformations, the Hamiltonian becomes a function of the transformation parameters α and β:

$$\mathscr{H}(\alpha, \beta) = \frac{I_1}{2\alpha^2} + \varkappa\frac{\alpha^2 I_2}{2\beta^2} - \alpha^{-1/2}\beta^{(1-d)/2}I_3. \qquad (11)$$

In the case $\varkappa = 1$, in two dimensions, this function has a minimum at a soliton solution. It can be shown[11] that this minimum coincides with an absolute minimum of the Hamiltonian, so that the two–dimensional solition is stable with respect to two-dimensional perturbations. On the other hand, at $d = 3$ the Hamiltonian $\mathscr{H}$ has, as a function of α and β, a saddle point instead of a minimum, which is the basis for the conclusion that a three-dimensional soliton is unstable.

Yet another important fact follows from the form of the function $\mathscr{H}(\alpha,\beta)$ in (11). The function $\mathscr{H}(\alpha,\beta)$ is not bounded from below in the limit $\alpha, \beta \to 0$. The fact that this function is not bounded is a consequence of nonlinear terms, whose relative role increases with as the x and $r_\perp$ scales decrease. Of fundamental importance is the positive definiteness of the quadratic terms in $\mathscr{H}$, which represent a "kinetic" energy, while the nonlinear term serves as a "potential" energy. If the dispersion is negative, the quadratic part of the Hamiltonian does not have a fixed sign; we can say that the masses of the quasiparticles have different signs in different directions. As we will see below, this radically changes the nonlinear dynamics of the system.

The fact that the Hamiltonian is not bounded when the secondary integrals ("secondary" relative to the Hamiltonian) are fixed and when the quadratic terms in $\mathscr{H}$ are positive definite is characteristic of all systems in which a collapse can occur (self-focusing[14,15] and the collapse of various types of plasma waves[1–6]). We can therefore say that the process by which a singularity forms—the wave collapse—is energetically favored. In other words, wave collapse is equivalent to the reflection of a particle from an unbounded potential. Wave collapse is distinguished from the reflection of a mechanical particle in one important way. The difference stems from the wave nature of the collapse; specifically, waves may be emitted from cavities, i.e., regions of an elevated field concentration. Clearly, this would be a dissipation process for the cavities, and in principle, because of the propagation of the waves away from the cavities, the result might be slowing or complete stopping of the collapse. This possibility exists for cavities with $\mathscr{H} > 0$. For cavities with a negative Hamiltonian, however, the emission of waves would promote the collapse.

Let us assume that at $t = 0$ we have a field distribution $u_0(r)$ with length scales $l_\parallel$ and $l_\perp$ and with a negative Hamiltonian. In this case we evidently have an integral

$$I_3 = \int u^3\, d\mathbf{r} \geqslant |\mathscr{H}|.$$

Hence, by virtue of the mean value theorem, the maximum value $u_{\max}$ is bounded from below by the conserved quantity:

$$u_{max} \geqslant |\mathscr{H}|/2P_x. \qquad (12)$$

It is easy to understand that this inequality holds for an arbitrary region, in particular, for a contracting cavity with $\mathscr{H} < 0$ emission from such cavities will increase the ratio $|\mathscr{H}|/P_x$, since P_x decreases as a result of the removal of the waves, while $|\mathscr{H}|$, on the contrary, increases. The reason for the latter effect is that for the emitted waves the dispersive effects exceed the nonlinear effects; i.e., the emission carries off the positive part of the Hamiltonian, reducing $\mathscr{H}$ ($\mathscr{H} < 0$) of the collapsing cavity. Consequently, the maximum field increases as a result of the emission according to (12).

Estimates similar to (12) have been made for Langmuir collapse,[16] and they can also be written for other systems. For Hamiltonians which have a lower bound (when the secondary integrals are fixed) estimates of this type mean that solitons should form as a result of the evolution of a system with $\mathscr{H} < 0$, while for unbounded Hamiltonians this process should always terminate in collapse.

The emission thus has a strong effect on the collapse dynamics. Let us assume that the asymptotic behavior of the field amplitude near a singularity has the self-similar form

$$u(r, t) = \frac{1}{(t_0 - t)^a} f\left(\frac{x}{(t_0 - t)^b}, \frac{r_\perp}{(t_0 - t)^c}\right). \qquad (13)$$

With this distribution, the value of P_x for the waves trapped in a cavity has a power-law behavior $(t_0 - t)^{-2a+b+2c}$. From the requirement that P_x not increase we find a first restriction on the indices a, b, and c:

$$2a \leqslant b + 2c.$$

Another condition—the condition for collapse—follows from the requirement that the ratio of the values of $|\mathscr{H}|/2P_x$ taken over the cavity increase:

$$a \geqslant \max(2b, -2b+2c).$$

The index a thus lies in the interval

$$c+b/2 \geqslant a \geqslant \max(2b, -2b+2c). \quad (14)$$

Accordingly, the parameter c lies between $3b/2$ and $5b/2$. The maximum value of a is reached under the condition $a = c + b/2$. This is the so-called strong regime of wave collapse,[17] in which a finite amount of energy, i.e., P_x, is trapped at a singularity, and there is no emission. Among these regimes the fastest occurs, when the index b is fixed, in the case with $a = 3b$ and $c = 5b/2$. The behavior of the field amplitude near the singularity is

$$u(r,t) \approx \frac{1}{(t_0-t)^{3b}} f\left(\frac{x}{(t_0-t)^b}, \frac{r_\perp}{(t_0-t)^{5b/2}}\right).$$

Curiously, inequality (14) becomes an equality in this case: The nonlinear and diffractive terms in $\mathscr{H}$ behave identically, while the dispersive terms lag behind. The weak regime of wave collapse occurs in the case $c + b/2 > a$ and is accompanied by the emission of waves from the cavity. The emission from the cavity is greatest when $\delta = c - a + b/2$ is at a maximum. Using inequalities (14), we can easily show that for a fixed value of b, the value $\max \delta = b/2$ is reached at $a = c = 2b$. This regime corresponds to the following asymptotic behavior of the field amplitude:

$$u(x,r_\perp,t) \approx \frac{1}{(t_0-t)^{2b}} f\left(\frac{x}{(t_0-t)^b}, \frac{r_\perp}{(t_0-t)^{2b}}\right), \quad (15)$$

for which the dispersive, diffractive, and nonlinear terms in $\mathscr{H}$, taken over the cavity, behave identically.

On the basis of the analysis above we can identify restrictions on the indices and construct a very simple collapse classification. This analysis will of course not generate any specific values of the indices; that would require working directly with equation of motion (3).

We substitute (15) into (3). It is easy to see that a self-similar substitution of the type in (15) is permissible only if

$$a=2b=c={}^2/_3. \quad (16)$$

In this case the structure function f is found from the solution of the equation

$$\frac{\partial}{\partial \eta_x}\left(\frac{2}{3}f + \frac{1}{3}\eta_x\frac{\partial f}{\partial \eta_x} + \frac{2}{3}\eta_\perp\frac{\partial f}{\partial \eta_\perp} + \frac{\partial^3 f}{\partial \eta_x^3} + f\frac{\partial f}{\partial \eta_x}\right) = \Delta_{\perp\eta} f.$$

The solution of this equation has a power-law asymptotic behavior in the limit $\eta_x, \eta_\perp \to \infty$:

$$f \to \eta_x^{-1+2\alpha}\,\eta_\perp^{-1/2-\alpha}\, g(\eta_x^2/\eta_\perp).$$

Here the function $g(\xi)$ is regular at the points $\xi = 0$ and $\xi = \infty$ and has nonzero values there.

This asymptotic behavior shows that in the limit $t \to t_0$ a singularity of the following type develops at the point $r = 0$:

$$u^2 = x^{-2+4\alpha} r_\perp^{-1-2\alpha}\, g^2(x^2/r_\perp).$$

Of these solutions, the only ones which are physically meaningful are those for which the given singularity is integrable. We thus have the following restriction on the index α:

$${}^1/_4 < \alpha < {}^1/_2.$$

The specific value of α is found from the requirement that the function g be regular.

It should be noted that on this distribution the integral $\int f^2 d\eta_x d\eta_\perp$ diverges in the limit $|\eta| \to \infty$. The divergence is a direct consequence of the conservation of the momentum projection

$$P_x = \frac{1}{2}\int u^2\, d\mathbf{r}.$$

For this reason, a given solution cannot be realized over the entire space; it can exist only locally, near a cavity, merging with a non-self-similar region, contracting more slowly or, alternatively, spreading out.

We should add that the presence of a given self-similar solution or, more precisely, form generally does not rule out the existence of other solutions with a self-similar asymptotic behavior different from the given behavior. Furthermore, known examples of the solution of the Cauchy problem for the Korteweg–de Vries equation[18] and for the two-dimensional Kadomtsev-Petviashvili equation[19] demonstrate that in no sense are the self-similar solutions close to the self-similar asymptotic expressions for the nonsoliton part in the limit $t \to \infty$.

Note also that solution (15) with indices (16) corresponds according to our classification to the weak wave collapse regime, in which the emission is a maximum ($b = 1/3$).

§ 3. RESULTS OF NUMERICAL SIMULATIONS

To study the nonlinear dynamics of sound waves, we have carried out a numerical simulation of the Kadomtsev-Petviashvili equation in the axisymmetric case. The diffractive term in (2) is nonlocal, so that the standard methods for "breaking up" the problem into several one-dimensional problems in the numerical solution of the Cauchy problem cannot be used for Eq. (2). Projection (spectral) methods involving an expansion in spatial harmonics are also unwieldy because of this particular form of the dispersion law for small perturbations of (2): $k_x(\omega + k_x^3) = -k_\perp^2$, This dispersion law leads to very large values of the group velocities for the lower harmonics of the solution. A new differencing technique, employing an iterative splitting, has accordingly been developed. It is described, its convergence is proved, its errors are estimated, etc., in Ref. 20. For a numerical implementation of this technique, we used a difference scheme with a high order of accuracy $O((\Delta x)^4, (\Delta t)^2, (\Delta r_\perp)^2)$, which causes only a slight distortion of the dispersion properties of (2). One criterion for the efficacy of this scheme was a test of the planar form of a two-dimensional soliton: retention of its shape, conservation of the invariants, etc., up to $t \sim 1$. To incorporate possible effects of emission from the system, we selected "penetrable" boundary conditions, which do not conserve $\mathscr{H}$ or P_x. To monitor the calculations we calculated the fluxes of $\mathscr{H}$ and P_x at the boundaries of the region, and we tested the conservation of the invariants with allowance for these fluxes. In all the calculations $\mathscr{H}$ was conserved at the end of the calculation to better

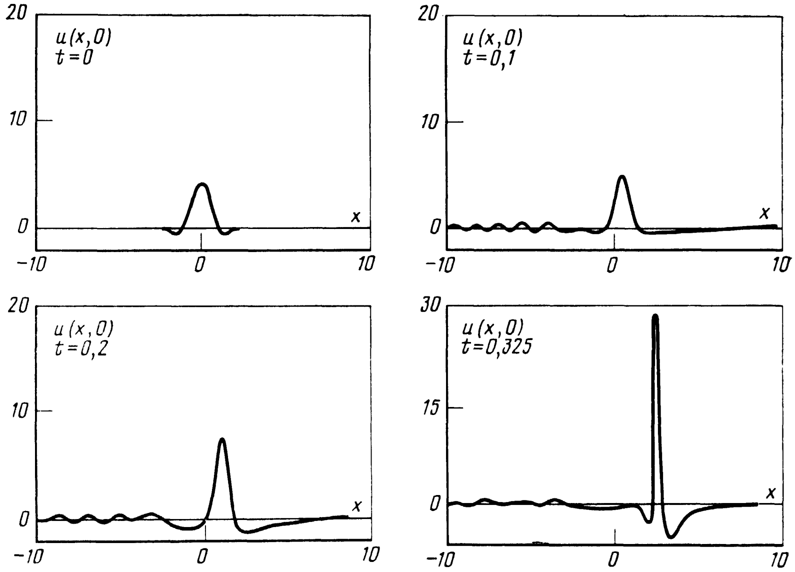

FIG. 1. The distribution $u(x,0)$ at successive times.

than 5%; the conservation of P_x was an order of magnitude better.

In a first series of simulations, corresponding to the case of positive dispersion ($\varkappa = 1$), with an initial value of the Hamiltonian $\mathscr{H}$ below the critical value [which was found to be positive, in accordance with (10)], collapse of the sound waves occurred (Fig. 1). The central part lagged behind the periphery, and a characteristic U-shaped profile formed (Fig. 2). The amplitude of the wave field in the cavities which formed increased by an order of magnitude over a time $t = 0.3$; i.e., the intensity increased by two orders of magnitude. We observed emission from both the system itself and the cavity. In the calculation region, $\mathscr{H}$ decreased from 30 to -80 over this time, while the change in P_x was small, $\sim 10\%$. Significant changes occurred in the cavity, whose boundary we took to be the line $u = 0$. Specifically, P_x decreased by a factor of about 3, and $\mathscr{H}$ by an order of magnitude (Figs. 3 and 4). It follows from Fig. 3 that at $t \sim 0.3$ a trend toward a self-similar time dependence of the quantities sets in. Analysis of the results of the numerical simulation shows that, for example, the time evolution of the amplitude of the wave field at the cavity axis is approximately $(t_0 - t)^{-2/3}$. This result means that, in accordance with §2, there is a collapse regime with maximum emission in this case.

It follows from the results of §1 that when the dispersion is negative there are no solutions in the form of multidimensional solitons, so that we need to consider the question of the asymptotic behavior of the solutions in the limit $t \to \infty$. The different signs of the quadratic terms of the Hamiltonian correspond in the case

$$I_3 = \int u^3 \, d\mathbf{r} > 0$$

to an attraction along the longitudinal coordinate and a repulsion along the transverse coordinate. (In the case $I_3 < 0$, the directions are switched.) In this situation, if the waves are of sufficiently large amplitude, the nonlinear evolution of the perturbations may in principle give rise to caustics. The numerical simulations with the Kadomtsev-Petviashvili equation show, however, that wave collapse does not occur in the case $\varkappa = -1$. In the initial stage, partial focusing of the wave field occurs and the wave amplitude increases (Fig. 4). The peripheral region lags behind the central region, and the picture which appears is qualitatively the same as in the positive dispersion case (Fig. 2). In a later stage, the con-

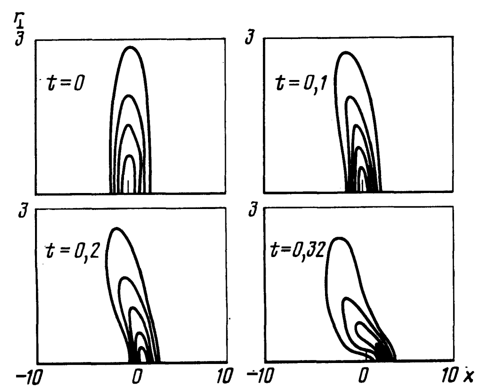

FIG. 2. Contour map of the function $u(x,r_\perp)$.

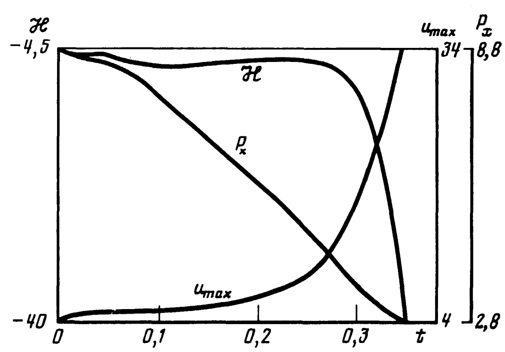

FIG. 3. Time evolution of $\mathscr{H}$, P_x for a cavity, and u_{max}.

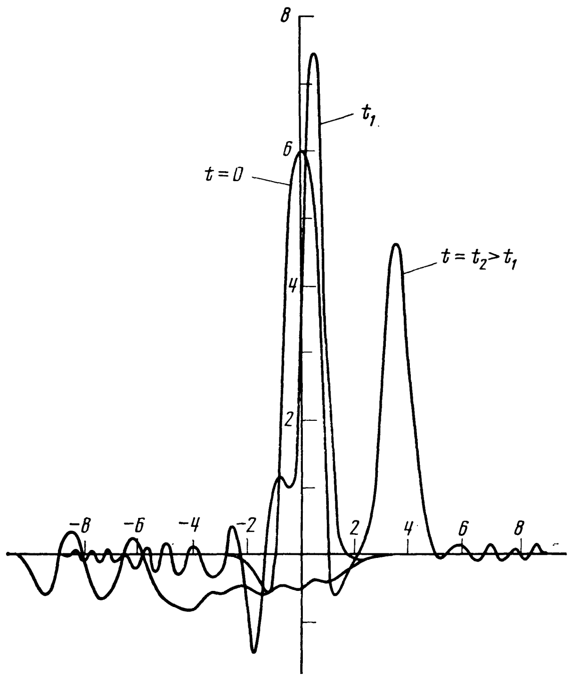

FIG. 4. Curves of the field u for the case of a negative dispersion at the axis ($r_\perp = 0$) at successive times.

traction of the perturbations comes to a halt, and in all cases a transition to a defocusing regime takes place. Interestingly, the picture drawn here of the evolution of sound waves in media with a negative dispersion is analogous to the nonlinear interaction of electromagnetic waves with media in which the derivatives $\partial^2\omega/\partial k_x^2$, $\partial^2\omega/\partial k_y^2$ have different signs (ion cyclotron waves, whistlers, etc.).[21]

§ 4. OBLIQUE SHOCK WAVES IN A STRONGLY MAGNETIZED PLASMA

In this section we will study how the wave collapse affects the structure of oblique shock waves in a magnetized plasma with $\beta = 8\pi nT/H_0^2 \ll 1$. At frequencies below the ion cyclotron frequency, $\omega \ll \omega_{Hi}$, fast magnetosonic waves are excited in such a plasma; their dispersion law is (for $\beta > m/M$); (cf. Ref. 22)

$$\omega_{MS} = kv_A\left\{1 + \frac{k^2c^2}{2\omega_{pi}^2}\operatorname{ctg}^2\theta - \frac{1}{2}k^2\rho^2\left(3 - \frac{11}{4}\sin^2\theta\right)\right\}. \quad (17)$$

Here $v_A = H_0(4\pi nM_i)^{-1/2}$ is the Alfvén velocity, and θ is the angle between the magnetic field and the wave propagation direction. The plasma is quasineutral in such motions; this case is valid for $\omega \ll \omega_{pi}$. It follows from (17) that except in small angular neighborhoods of $\theta = 0$ and $\theta = \pi/2$ the dispersion is positive and has effects at distances $\sim c/\omega_{pi}$. For nearly transverse propagation, $|\theta - \pi/2| < (\beta/8)^{1/2}$ the dispersion is negative and is determined by effects stemming from the finite ion Larmor radius $\rho = v_{Ti}/\omega_{Hi}$. Expression (17) does not apply at angles $\theta < (kc/\omega_{pi})^{1/2}$, where there is a pronounced change in the wave dispersion mechanism. At $\beta < m/M_i$ the dispersion is determined over the entire angular range from the hydrodynamic equations:

$$\omega_{MS} = kv_A\left\{1 + \frac{k^2c^2}{2\omega_{pi}^2}\left(\operatorname{ctg}^2\theta - \frac{m}{M_i}\right)\right\}.$$

For nearly transverse propagation, $|\pi/2 - \theta| \leqslant (m/M_i)^{1/2}$, the dispersion of fast magnetosonic waves is negative. Outside this cone the dispersion is positive. We will consider here only the region of angles with positive dispersion, for which the dispersion length satisfies

$$\lambda_D^2 = \frac{c^2}{2\omega_{pi}^2}\left(\operatorname{ctg}^2\theta - \frac{m}{M_i}\right) > 0.$$

In this case the Kadomtsev-Petviashvili equation, (2), is valid for describing small-amplitude fast magnetosonic waves with a narrow angular distribution:

$$\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial t} + \frac{3}{2}v_A\sin\theta hh_x - v_A\lambda_D^2h_{xxx} + \hat{\gamma}h\right) = -\frac{v_A}{2}\Delta_\perp h. \quad (18)$$

Here h is the dimensionless amplitude of the fast magnetosonic waves, in terms of which the total magnetic field can be expressed:

$$H = H_0(\cos\theta, 0, \sin\theta + h).$$

The coefficient of the nonlinear term in (18) can be found with the help of the results given in Karpman's book.[23]

In Eq. (18), in contrast with (2), we have incorporated a damping described by the operator $\hat{\gamma}$. The transform of the operator $\hat{\gamma}$ is the damping rate for a monochromatic wave, $\gamma(k)$. It is governed by both collisional and collisionless mechanisms:

$$\gamma = \gamma_{col} + \gamma_L.$$

Most of the collisionless damping is caused by electrons. In the region $\omega/k_z \ll v_{Te}$ [i.e., at angles $\cos\theta > (m/M_i\beta)^{1/2}$] we have[24]

$$\gamma_L \approx \frac{\pi^{1/2}}{4}v_{Ti}\left(\frac{m}{M_i}\right)\frac{k_\perp^2}{k_z}.$$

For $\omega/k_zv_{Te} \gg 1$ the damping γ_L by electrons is exponentially small:

$$\gamma_L = kv_{Ti}\left(\frac{m}{M}\right)^{1/2}\frac{\pi^{1/2}k}{k_z}\exp\left(-\frac{k^2v_A^2}{k_z^2v_{Te}^2}\right).$$

The collisionless damping is anomalously strong for

$$\cos\theta \approx (m/M_i\beta)^{1/2}.$$

We now discuss the roles are played by other waves in this frequency range ($\omega < \omega_{Hi}$); Alfvén waves and slow magnetosonic waves, whose natural frequencies are, respectively, $\omega_A = |k_z|v_A$ and $\omega_s = |k_z|c_s$ ($c_s^2 = T_e/M_i$). In a collisionless plasma, the slow magnetosonic waves exist only if the plasma is nonisothermal, i.e., only if $T_e \gg T_i$.

In the case $\beta \ll 1$, the most important nonlinear processes for fast magnetosonic waves which involve Alfvén waves and slow magnetosonic waves are the decays

$$\omega_{MS}(\mathbf{k}_0) = \omega_{MS}(\mathbf{k}_0 - \boldsymbol{\varkappa}) + \omega_s(\boldsymbol{\varkappa}), \quad (19)$$

$$\omega_{MS}(\mathbf{k}_0) = \omega_A(\mathbf{k}_0 - \boldsymbol{\varkappa}) + \omega_s(\boldsymbol{\varkappa}). \quad (20)$$

To analyze these processes it is sufficient to use the growth rate of the decay instability for a monochromatic wave. The specific expressions for the growth rates can be found, for example, by the Hamiltonian approach of Refs. 25 and 26 or by a direct method, as was first used by Galeev and Oraevskiĭ[27] for the decay instability of Alfvén waves. For the decay (19) the growth rate γ is

$$\gamma=(W/nT)^{1/2}(\omega_{MS}\omega_s)^{1/2}|(\mathbf{n}_{k0},\mathbf{n}_{k0-\varkappa})|.$$

Here W is the energy density of the fast magnetosonic wave, and $\mathbf{n}_{k0}$, $\mathbf{n}_{k0-\varkappa}$ are the linear polarization vectors of the initial and scattered fast magnetosonic waves. Here

$$\mathbf{n}_k=[\mathbf{k}[\mathbf{kn}_0]]/|[\mathbf{k}[\mathbf{kn}_0]]|,\quad \mathbf{n}_0=\mathbf{H}_0/H_0.$$

Since the relative change in the frequency of the fast magnetosonic wave is small in Eq. (19), $\Delta\omega/\omega\sim c_s/v_A\sim\beta^{1/2}\ll 1$, the maximum of the growth rate corresponds to the maximum value of the frequency of the slow magnetosonic wave, ω_s. Consequently, this factor favors a z projection of the wave vector of the excited fast magnetosonic wave which is directed opposite that of the initial wave and which is equal to k_0. Because of the scalar product of the polarization vectors, however, the maximum of the growth rate is shifted slightly in angle. Nevertheless, the tendency toward scattering with a large change in wave vector remains.

In order of magnitude, we have

$$\gamma_{max}\approx\omega_{MS}(k_0)(W/nT)^{1/2}(c_s/v_A)^{1/2}. \tag{21}$$

The instability growth rate for decay (20) is given by a similar expression:

$$\gamma=(W/nT)^{1/2}(\omega_{MS}(k_0)\omega_s)^{1/2}|(\mathbf{n}_{k0},\mathbf{n}^A_{k0-\varkappa})|,$$

where $\mathbf{n}^A_k=[\mathbf{kn}_0]/|[\mathbf{kn}_0]|$ is the polarization vector of the Alfvén wave. The maximum value of the growth rate for this process is of the same order of magnitude as (21); the Alfvén waves which are excited make angles of order unity with respect to the initial fast magnetosonic wave. The situation which arises here is extremely reminiscent of decay processes involving Langmuir, electromagnetic, and ion acoustic waves in an isotropic plasma.

If the plasma is nonisothermal, fast magnetosonic waves will be converted into fast magnetosonic waves or Alfvén waves involving ions, instead of decaying according to (19) and (20). The nature of the interaction, however, changes markedly. The time scale τ for these processes is significantly longer:

$$\tau^{-1}\sim\omega_{MS}(W/nT)(c_s/v_A)^2.$$

In the first place, this process is proportional not to the amplitude, as in (21), but to its square; second, there is an additional small factor in this case, because of the differential pumping along the frequency scale, $\Delta\omega/\omega\sim c_s/v_A$. Let us examine the decay process within the fast magnetosonic branch. As we mentioned back in §1, the waves participating in this process have nearly parallel wave vectors. In order of magnitude, the maximum value of the growth rate,

$$\gamma_{max}\sim\omega_{MS}(k_0)(W/\rho v_A^2)^{1/2}. \tag{22}$$

is the same as the characteristic reciprocal of the nonlinear time for Eq. (18). The time scales of the decay processes

$$\begin{aligned}\omega_{MS}(\mathbf{k}_0)&=\omega_{MS}(\mathbf{k}_0-\varkappa)+\omega_A(\varkappa),\\ \omega_{MS}(\mathbf{k}_0)&=\omega_A(\mathbf{k}_0-\varkappa)+\omega_A(\varkappa),\end{aligned} \tag{23}$$

are of the same order of magnitude. For these processes, as for (19) and (20), the maximum growth rate does not coincide in direction with the initial fast magnetosonic wave. For angles far from 0 and $\pi/2$, secondary waves are separated from the original fast magnetosonic wave by an angle $\Delta\theta\sim 1$.

We thus see that only the nonlinear interaction of fast magnetosonic waves is of a quasi-one-dimensional nature. All other processes lead to a large-angle scattering, and the so-called vector nonlinearity is unimportant in this case.

We consider a one-dimensional solution of Eq. (18) in the form of a shock wave, which has the asymptotic behavior

$$h\to h_0 \text{ as } x\to-\infty,$$
$$h\to 0 \text{ as } x\to\infty.$$

The state $h=h_0$ behind the shock front corresponds to a density jump

$$\delta\rho/\rho_0=h_0\sin\theta$$

and a plasma velocity

$$v_0=(4\pi\rho_0)^{-1/2}H_0h_0\sin\theta.$$

As a result, the change in the Alfvén velocity is

$$\Delta v_A={}^1/_2(4\pi\rho_0)^{-1/2}H_0h_0\sin\theta,$$

which, along with the velocity

$$v_0=(4\pi\rho_0)^{-1/2}H_0h_0\sin\theta$$

gives us $3/2(4\pi\rho_0)^{-1/2}H_0h_0\sin\theta$, in complete accordance with Eq. (18). Since Kadomtsev-Petviashvili Equation (18) describes small wave amplitudes, this solution represents a weak shock wave. At this point we adopt the assumption that the damping is weak—weaker than dispersion effects. In this case, according to the Sagdeev theory,[28] a collisionless shock wave arises with a front of oscillatory structure. The amplitude of the oscillations decays with distance from the jump. The oscillatory structure may be thought of as a set of one-dimensional solitons. The first has a size $l_s\sim\lambda_D(M-1)^{-1/2}$ and an amplitude of order h_0 where M is the Mach number; the amplitudes of the successive solitons fall off. If the primary dissipative mechanism is the ohmic loss due to electron-ion collisions, the size of the oscillatory structure can be estimated to be[28]

$$l_d\sim v_A\lambda_D{}^2\omega_{pe}{}^2/c^2\nu_{ei}. \tag{24}$$

It is easy to understand that the laminar oscillatory structure of a shock wave is unstable against processes (19), (20), and (23). This question was first taken up by Galeev and Karpman,[29] who studied the effect of the decay of a fast magnetosonic wave into fast magnetosonic and Alfvén waves. In a collisionless plasma with $\beta\ll 1$ (but in which β is not identically zero), the primary nonlinear process with $T_e\gg T_i$ is a decay involving slow magnetosonic waves. For such processes, the waves which are excited propagate at large angles with respect to the wavefront; because of the finite width of this front, the removal of the excited waves may saturate the instability. A necessary condition here is that the growth rate satisfy

$$\Gamma \sim \max\left[\gamma l_d/(v_1 v_2)^{1/2}\right] \leqslant \Lambda, \tag{25}$$

where Λ is the Coulomb logarithm, and $v_{1,2}$ are the group velocities of perturbations along the x axis. In the case of an isothermal plasma, Γ represents, roughly speaking, the number of solitons in the structure, N. In a nonisothermal plasma, (25) would be multiplied by a factor $\beta^{-1/2}$ because of processes (19) and (20).

This reason for the saturation of the instability is not present for a nonlinear interaction between fast magnetosonic waves which have a small angular spread with respect to the front and thus a small spread in group velocities. This nonlinear interaction is of an absolute nature, while all the other processes are of a drift nature. Consequently, under condition (25) the mechanism which primarily determines the front structure of weak oblique shock waves is the nonlinear interaction of waves within the fast magnetosonic branch.

Let us take a more detailed look at the changes caused in the front structure by instability (5). We recall that a laminar oscillatory structure consists of "rarefaction" solitons and that the maximum magnetic field in each soliton is weaker than the field ahead of the wavefront. In the nonlinear stage of the instability, when the front is modulated, those regions of the soliton which have a smaller amplitude lead regions with a larger amplitude. As a result, in those parts of the front where the magnetic field decreases the soliton slows down, and its trailing edge becomes steeper. A decrease in the magnetic field in the soliton itself should be accompanied by a repulsion of the field from the center of the soliton and should cause an increase in the magnetic field at the wings; this is what is observed in a numerical simulation of the collapse of a single cavity[2)] (Figs. 1 and 2). A rough comparison of the time scales for the onset of this instability, τ, and for the formation of the oscillatory structure, $\tau_{\rm osc}$, indicates that they are on the same order of magnitude (actually, $\tau_{\rm osc}$ should be longer than τ because of the large number of solitons). This result means that the front structure should consist of one or two collapsing cavitons. We should thus replace criterion (25) by

$$1 < \Lambda \begin{cases} \beta^{1/2}, & T_e \gg T_i, \\ 1, & T_e \approx T_i. \end{cases} \tag{26}$$

Since the Coulomb logarithm Λ has a typical value on the order of 10, criterion (26) is far from being stringent; it is satisfied for broad ranges of parameter values.

In each collapsing caviton, as the magnetic field increases, there comes a time when the ions begin to be reflected from the front. In the case $\beta \ll 1$, according to (30), breaking occurs at Mach numbers $M = 1.5$–2.5, depending on the angle between the propagation direction and the magnetic field. It can thus be suggested that the amplitude grows to a value of order unity. At a larger wave amplitude, the energy in the transcritical region should be transferred to ions. As a result, the plasma acquires a group of fast ions, which in turn excite high-frequency waves (ion cyclotron and lower hybrid waves) with a wavelength far smaller than the size of the original structure.

Let us find the effective $\nu_{\rm eff}$ of this process. We can use the energy balance condition:

$$\nu_{\rm eff}\varepsilon_k \sim \gamma \Delta\varepsilon,$$

where ε_k is the energy stored in one cavity, γ is the rate at which energy is pumped from scale to scale, and $\Delta\varepsilon$ is the energy transferred to the ions.

The initial size of a cavity, l_x, is on the order of the size of a soliton, $l_s \sim \lambda_D (M-1)^{-1/2}$. The transverse dimension found from the maximum of growth rate (5) is given in order of magnitude by $l_\perp \sim l_x (l_x/\lambda_D)$. Initially, the amplitude at the center of the cavity is $\sim (M-1)$. We thus find

$$\varepsilon_k \sim \rho v_A^2 (M-1)^2 l_x l_\perp^2 \sim \rho v_A^2 \lambda_D^3 (M-1)^{-1/2}.$$

For γ we must take the reciprocal of the typical nonlinear time:

$$\tau^{-1} \sim v_A (M-1) l_x^{-1} \sim v_A (M-1)^{3/2}/\lambda_D.$$

Assuming that the collapse regime and the growth of the wings are determined by self-similar behavior (15), (16), we easily find

$$\Delta\varepsilon \sim \rho v_A^2 l_x l_\perp^2 (M-1)^{-5/2}.$$

Hence

$$\nu_{\rm eff} \sim \gamma (M-1)^{1/2} \sim v_A (M-1)/\lambda_D.$$

We must stress that this effective rate is a measure of the energy transfer to ions, but it does not determine the turbulent width of the front. We can associate with $\nu_{\rm eff}$ an energy length

$$l_h \sim v_A/\nu_{\rm eff} \sim \lambda_D (M-1)^{-1}.$$

The front width should be estimated from $l_f \sim v_A/\gamma \sim \lambda_D (M-1)^{-1/2}$. The turbulent width of the front is determined specifically by the value of γ. If the collapse is strong, then we have $l_f \sim l_h$; in our case, there is a collapse regime with maximum emission (§3), and the relation $l_h \gg l_f$ holds. The collapse of sound waves for the case of collisionless shock waves is a mechanism which can transfer energy of directed motion into other degrees of freedom: lower hybrid noise, fast ions, and transverse modulations of the front. The energy transfer rate is γ.

CONCLUSION

The experimental data[31] available on shock waves in the auroral regions of the earth with $\beta < 1$ and with Mach numbers of order unity are evidence in favor of the theory presented here. These measurements demonstrate the existence at a shock front of pronounced MHD turbulence, of fast ion beams, and of ion cyclotron and lower hybrid waves; i.e., there is agreement in terms of this set of data. In order to make a quantitative comparison with experimental data, it will be necessary to carry out some numerical simulations dedicated to the purpose.

We wish to thank L. M. Zelenyĭ for calling our attention to Ref. 31. We also thank A. V. Shafarenko for participating in the numerical simulations and for the development of the numerical algorithm. We thank M. D. Spektor for a useful comment.

[1)]In an erroneous recent paper, Mikhaĭlovskiĭ *et al.*[13] derived an instability of stationary waves with respect to oblique perturbations for the Kadomtsev-Petviashvili equation with a negative dispersion. However, the dispersion relation (3.15 in Ref. 13) is cubic in ω and has only real roots, as can be verified by calculating the discriminant of the cubic equation and making use of the properties of elliptic functions.

[2)]It should be kept in mind that the switch from the Kadomtsev-Petviashvili equation in its standard form, (3), to the form (2) is made by means of the replacements $u \to -u$, $x \to -x$.

[1]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[2]E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **66**, 2037 (1974) [Sov. Phys. JETP **39**, 1003 (1974)].
[3]S. L. Musher and B. I. Sturman, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 537 (1975) [JETP Lett. **22**, 265 (1975)].
[4]V. V. Krasnosel'skikh and V. I. Sotnikov, Fiz. Plazmy **3**, 872 (1977) [Sov. J. Plasma Phys. **3**, 491 (1977)].
[5]A. G. Litvak and A. M. Sergeev, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 549 (1978) [JETP Lett. **27**, 517 (1978)].
[6]V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 479 (1975) [JETP Lett. **21**, 221 (1975)].
[7]E. A. Kuznetsov, S. L. Musher, and A. V. Shafarenko, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 204 (1983) [JETP Lett. **37**, 241 (1983)].
[8]B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR **192**, 753 (1970) [**15**, 539 (1970)].
[9]V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 7 (1975) [JETP Lett. **22**, 3 (1975)].
[10]B. B. Kadomtsev, Kollekivnye yavleniya v plazme (Collective Phenomena in Plasmas), Nauka, Moscow, 1976.
[11]E. A. Kuznetsov and S. K. Turitsyn, Zh. Eksp. Teor. Fiz. **82**, 1457 (1982) [Sov. Phys. JETP **55**, 844 (1982)].
[12]E. A. Kuznetsov, M. D. Spector, and G. E. Fal'kovich, Physica **10D**, 1603 (1984).
[13]A. B. Mikhaĭlovskiĭ, S. V. Makurin, and A. I. Smolyakov, Zh. Eksp. Teor. Fiz. **89**, 1603 (1985) [Sov. Phys. JETP **62**, 928 (1985)].
[14]A. M. Prokhorov and V. N. Lugovoĭ, Usp. Fiz. Nauk **111**, 203 (1973) [Sov. Phys. Usp. **16**, 658 (1974)].
[15]S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **12**, 1353 (1970).
[16]V. E. Zakharov, in: Basic Plasma Physics (ed. A. A. Galeev and R. N. Sudan), North-Holland, Amsterdam.
[17]V. E. Zakharov, E. A. Kuznetsov, and S. L. Musher, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 125 (1985) [JETP Lett. **41**, 154 (1985)].
[18]V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **71**, 203 (1976) [Sov. Phys. JETP **44**, 106 (1976)].
[19]S. V. Manakov, P. M. Santini, and L. A. Takhtajan, Phys. Lett. **75A**, 451 (1980).
[20]S. L. Musher and A. V. Shafarenko, J. Comp. Phys. (1986) (in press).
[21]A. G. Litvak, T. A. Petrova, and A. M. Sergeev, Fiz. Plazmy **9**, 495 (1983) [Sov. J. Plasma Phys. **9**, 325 (1975)].
[22]A. B. Macmahon, J. Geophys. Res. **73**, 7538 (1968); C. F. Kennel, J. Geophys. Res. **73**, 7539 (1968).
[23]V. I. Karpman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, Moscow, 1973.
[24]V. D. Shafranov, in: Voprosy teorii plazmy, Vol. 3, Atomizdat, Moscow, 1963 (Reviews of Plasma Physics, Vol. 3, Consultants Bureau, New York, 1967).
[25]V. E. Zakharov and E. A. Kuznetsov, Dokl. Akad. Nauk SSSR **194**, 1288 (1970) [Sov. Phys. Dokl. **15**, 913 (1971)].
[26]E. A. Kuznetsov, Dissertation, Institute of Nuclear Physics, Novosibirsk, 1974.
[27]A. A. Galeev and V. N. Oraevskiĭ, Dokl. Akad. Nauk SSSR **147**, 71 (1962) [Sov. Phys. Dokl. **7**, 988 (1963)].
[28]R. Z. Sagdeev, in: Voprosy teorii plazmy, Vol. 4, Atomizdat, Moscow, 1966 (Reviews of Plasma Physics, Vol. 4, Consultants Bureau, New York, 1967).
[29]A. A. Galeev and V. I. Karpman, Zh. Eksp. Teor. Fiz. **44**, 592 (1963) [Sov. Phys. JETP **17**, 403 (1963)].
[30]V. Formisano, in: Plasma Astrophys., ESA. SP-161. 313. 1981.
[31]M. Temerin, C. Cattell, R. Lysak, *et al.*, J. Geophys. Res. **A86**, 11, 278 (1981).

Translated by Dave Parsons