The complete probability functional and the intrinsic structure of the ultrametric topology in spin glasses

S. L. Ginzburg

B. P. Konstantinov Leningrad Institute for Nuclear Physics
(Submitted 19 May 1986)
Zh. Eksp. Teor. Fiz. **91**, 2171–2189 (December 1986)

We show that the ultrametric space in spin glasses has a continuum structure. We find a representation of this space in the form of a time space with a logarithmic metric which is directly connected with the long-time logarithmic relaxation oberved experimentally. We show that all finite-dimensional relaxation and susceptibility distribution functions can be expressed in quadratures through the transition probability of a random Markov process. We evaluate the complete probability functional explicitly.

I. INTRODUCTION

The recent progress in the study of the molecular field theory of spin glasses is connected basically with the observation of the existence of a hierarchical structure of states which is usually called an ultrametric topology.[1–3] It is very clear that for further development of the theory one needs to have the most detailed information possible about the actual structure of the ultrametric space, for instance, one might wish to be able to enumerate all objects contained in this space. On the other hand, it is well known that the basic quantitites characterizing spin glases are the various distribution functions. The complete probability functional which is defined on the ultrametric space is the comprehensive characteristic of the spin glass. It turns out that one can completely determine the structure of the ultrametric space, define on it a complete probability functional, and find an explicit expression for this functional. These problems are the main topics of the present paper.

There is another set of questions for which an answer will be given in the present paper. One of them consists in the physical interpretation of the objects of the ultrametric space. In particular, one might wish to have a rather simple representation of this space. It turns out that one can accomplish that using the time approach proposed by Sompolinsky.[4] The present author showed in Ref. 5 that the time approach can be formulated for spin glasses by introducing a representation of the correlation function in terms of logarithmic variables. It turns out that such a representation is in the most direct way connected with the ultrametric topology. The usual time space with a metric determined in terms of the logarithmic variables defined in Ref. 5 is an ultrametric space. The topology of this space is completely equivalent to the topology of the corresponding space in a spin glass. We thus obtain a very simple representation of the space we need. As the time space can easily be enumerated, we obtain thus also an enumeration of the objects of the original space.

Moreover, as the time space is a continuum, this means that the space of the states in a spin glass is a continuum ultrametric space. We note that one can obtain the structure of the ultrametric space in spin glasses, and derive a method for enumerating the objects occurring in it and an explicit expression for the complete probability functional in it by using directly Parisi's approach.[6–8]

2. ULTRAMETRICITY OF THE TIME SPACE WITH A LOGARITHMIC METRIC

We consider the space of normal time t and define in it the distance $z(t_1 - t_2)$ between two points t_1 and t_2 through the formula

$$z(t_1-t_2)=\alpha \ln (|t_1-t_2|/\tau),$$
$$z\sim 1, \quad \alpha\to 0, \quad \tau\to 0. \tag{1}$$

In (1) τ is a microscopic time which we shall call paramagnetic. We shall in what follows be interested only in the macroscopic time. The second line in (1) gives the conditions that the distance between the times t_1 and t_2 is macroscopic. We note that our conditions are completely analogous to the usual situation with the main logarithms in field theory, while τ plays the role of cutoff at small distances. We get clearly from (1) for macroscopic times t_1, t_2, and t_3

$$z(t_1-t_3)=\mathrm{Sup}\{z(t_1-t_2), z(t_2-t_3)\}. \tag{2}$$

From (2) it follows at once that the main property of an ultrametric space, viz., that only equilateral and isosceles triangles can exist in that space, is satisfied. We see thus that the t-space with the metric (1) is ultrametric. We now prove that in order to exhibit explicitly the hierarchical structure of the t-space with logarithmic metric we must eumerate the different moments of time.

However, we first consider a general method for enumerating an ultrametric space. It is well known[1,2] that one can represent an ultrametric space in the form of a heirarchical tree. The simplest tree of this type is shown in Fig. 1. It is clear that one characterize a hierarchical tree by the number of the level of the hierarchy, k, and by the branching number, j (in the figure $k = 2, j = 3$). It is then clear from the figure that each element of the space can be characterized by k numbers a_l ($l = 0, 1, \ldots, k-1$) varying from 0 to $j-1$. As a result one obtains k-valued numbers in a j-row system of enumeration. Moreover, the distance between different elements in the ultrametric space is defined as the number of

 0038-5646/86/121291-10$04.00

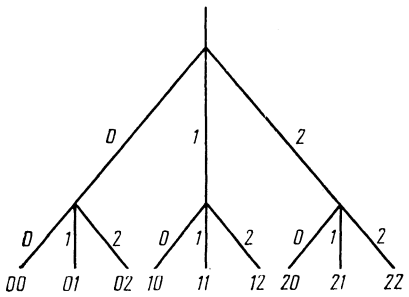

FIG. 1.

steps to the common progenitor.[1,2] For instance, the distance between the elements 10 and 12 is unity, and the distance between 01 and 12 is two. It is at once clear from the figure that, indeed, only equilateral and isosceles triangles occur in the ultrametric space. For instance, the elements 01, 12, and 20 form an equilateral triangle, and the elements 01, 11, and 12 an isosceles one. From the examples given here it is clear that if two elements are numbered by two sets of numbers a_l and b_l the distance between them depends solely on which of the a_l and b_l are the first to differ one from the other. For instance, if $a_0 \neq b_0$, the distance is zero, and if $a_0 = b_0$, but $a_1 \neq b_1$ the distance equals unity. The situation remains completely the same for arbitrary k and j. If all $a_n = b_n$, $n = 0, 1, \ldots, l-1$, but $a_l \neq b_l$ the distance between such elements equals $k - l$ ($l = 0, 1, \ldots, k-1$).

We now turn to the problem of the enumeration of the various moments of time in the time space with a logarithmic metric. It is completely clear that in such a space one must perform the enumeration in a logarithmic scale. Let us have a time interval $[t_0, t_1]$. After all calculations we consider the limit as $t_0 \to -\infty$, $t_1 \to +\infty$. We put

$$t_1 - t_0 = \tau j^{k+1}, \tag{3}$$

where j and k are integers. We now introduce the following representation of an arbitrary moment of time t_a:

$$t_a = t_0 + \tau(a_0 j^k + a_1 j^{k-1} + \ldots + a_l j^{k-l} + \ldots + a_k), \quad a_l = 0 \ldots j-1, \quad l = 0 \ldots k. \tag{4}$$

It is clear that (4) gives a representation of $(t_a - t_0)/\tau$ in a j-row system of enumeration. We see that t is determined by $k+1$ numbers a_l. We shall assume now that $j \to \infty$. As $a_l \sim j$ we get in this case clearly

$$z(t_a - t_b) = \alpha(k - l + 1)\ln j. \tag{5}$$

The number l in (5) is determined by which of the a_l and b_l are the first to differ from one another. For instance, if $a_0 \neq b_0$, we have $l = 0$, if $a_0 = b_0$, but $a_1 \neq b_1$, we have $l = 1$, and so on. As $\alpha \to 0$, z is independent of the actual values of the a_l and b_l and depends merely on l. We now put

$$z(t_1 - t_0) = z_{max} = (k+1)\Delta z, \quad \Delta z = \alpha \ln j, \quad z(t_a - t_b) = z_l = (k - l + 1)\Delta z, \quad k \to \infty, \quad j \to \infty, \quad \Delta z \to 0, \quad \alpha \to 0, \quad \Delta z/\alpha \to \infty, \quad z_{max} \to \infty. \tag{6}$$

The meaning of the representation (4) is completely clear from (6). We simply divide $z_{\max}$ into $k+1$ identical intervals Δz; it is then important that $j \to \infty$ as only in that limit will $z(t_a - t_b)$ depend solely on which of the a_l and b_l are the first to differ from one another. And this, as we have seen, defines the ultrametric topology. One sees easily that the definitions (6) and (1) of the distances are completely equivalent. On the other hand, it follows from (4) and (6) that in our definition of the ultrametric distance each moment of time is defined by an infinite set of numbers a_l in an infinite-row system of enumeration with base $j \to \infty$. This set a_l also determines the structure of our ultrametric space. The corresponding hierarchical tree has an infinite number of hierarchy levels ($k \to \infty$) and at each level of the hierarchy each branch divides into an infinite number ($j \to \infty$) of branches. The ends of this tree are numbered by the complete set of indexes a_l and determine the moments of time t_a. One can show that such a structure of a hierarchical tree can be obtained directly from Parisi's theory for objects which are called valleys in that theory. This means that the topology of the space of valleys in Parisi's theory and the topology of the time space with the metric (1) are identically the same.

We would like to note here the following fact. Usually an ultrametric space is representated in the form of a discrete set. On the other hand, the t-space in (1) is a continuum set and the way to number it in (4) to (6) is simply the usual way of going from a discrete to a continuous variable. We have thus in (1) in fact defined a continuum ultrametric space. As the ultrametric space in Parisi's theory is isomorphous with ours it is also a continuum.

3. EFFECTIVE HAMILTONIAN

We shall consider a standard Ising spin glass. However, there arise in the derivation of the effective Hamiltonian in the time approach immediately difficulties for an Ising glass which are connected with the use of a Glauber dynamics. The way out of this position is well known.[4,5] We must to start with write down the equations of the dynamics for a weak spin glass model. These equations have a rather simple structure and one can easily derive the effective Hamiltonian from them. After this is will be clear that the equation obtained turns out to be valid also for the usual Ising glass. The weak model Hamiltonian has the following form:

$$H = -\sum_{ik} J_{ik} m_i m_k + \sum_i U(m_i), \quad U(m) = m^2/2b + um^4/8, \quad \langle J_{ik}^2 \rangle = I_{ik}. \tag{7}$$

The m_i in (7) are classical fields, and the J_{ik} are random exchange integrals with a Gaussian distribution. In the time approach one must write down the Langevin equations for the m_i, write down the stochastic functional for these equations, and average it over the J_{ik}. We shall in what follows consider only a molecular field theory. After averaging over the J_{ik} we must in that theory perform the standard decoupling in the stochastic functional.[9-11] If after this decoupling we turn to the equations of motion for the m_i we get an equation for the motion of a single spin m in a field of Langevin forces and in the self-consistent field of all other spins. This equation has the following form:

$$\frac{1}{\Gamma T}\frac{\partial m}{\partial t}-\frac{4I_0}{T^2}\int_{-\infty}^{t}dt'\,G(t-t')m(t')$$
$$=\frac{1}{T}\frac{\partial U}{\partial m}+\varepsilon(t)+\frac{1}{T}h(t),$$
$$\langle h(t)h(t')\rangle=4I_0D(t-t'),$$
$$\langle\varepsilon(t)\varepsilon(t')\rangle=\frac{2}{\Gamma T}\delta(t-t'),\quad I_0=\sum_k I_{ik}. \tag{8}$$

In (8) $G(t)$ is a retarded Green function, and $D(t)$ is the correlator of the spins $m(t)$. Equation (8) determines the $m(t)$ in terms of $G(t)$ and $D(t)$ and the self-consistency conditions consist in that these $G(t)$ and $D(t)$ are in the standard way determined in terms of $m(t)$. Equation (8) is valid both in the ergodic region and in the non-ergodic region. However, in the latter there arises a long-time, singular part in $G(t)$ and $D(t)$. This singular part is just the characteristic of the non-ergodicity of spin glasses.[4,5,9-11] In the present paper we consider only the non-ergodic region. We split off in (8) the long-term contribution. We then get

$$\frac{1}{\Gamma T}\frac{\partial m}{\partial t}-\frac{4I_0}{T^2}\int_{-\infty}^{t}dt'\,G_0(t-t')m(t')$$
$$=\frac{1}{T}\frac{\partial U}{\partial m}+\varepsilon(t)+\frac{1}{T}h_0(t)+\frac{1}{T}\xi(t),$$
$$\xi(t)=h(t)+\frac{4I_0}{T}\int_{-\infty}^{t}dt'\,G_s(t-t')m(t'),$$
$$\langle h_0(t)h_0(t')\rangle=4I_0D_0(t-t'),$$
$$\langle h(t)h(t')\rangle=4I_0D_s(t-t'),\quad \langle h(t)h_0(t')\rangle=0,$$
$$G(t)=G_0(t)+G_s(t),\quad D(t)=D_0(t)+D_s(t). \tag{9}$$

We split off from $G(t)$ and $D(t)$ the ergodic quantities $G_0(t)$ and $D_0(t)$ which satisfy the fluctuation-dissipation theorem and describe the usual thermodynamic fluctuations and the non-ergodic, singular quantities $G_s(t)$ and $D_s(t)$ describing the long-time correlations. The present author[5] has shown that $G_s(t)$ and $D_s(t)$ have the following form:

$$D_s(t)=q(z),$$
$$G_s(t)=\frac{\alpha}{t}\Delta'(z)\vartheta(t), \tag{10}$$

where z and α are defined in (1) while $q(z)$ and $\Delta(z)$ are standard order parameters which characterize non-ergodicity in spin glasses.

It is clear from (9) that all quantities characterizing non-ergodicity are included in $\xi(t)$. It follows from (1), (9), and (10) that Eq. (9) is a generalized Langevin equation for $m(t)$. Of most importance in (9) is then the appearance of a magnetic field $\xi(t)$ which is time-dependent and changes over macroscopic times. We defined macroscopic times in (1). It is very clear that the macroscopicity condition is the condition for the adiabatic approximation for solving Eq. (9). If we use this condition we get easily an explicit expression for the complete probability functional for the field $m(t)$. It has th form of a Gibbs distribution in the slowly varying field $\xi(t)$. The fact that there are the terms with G_0 and D_0 present in (9) which are usual for Langevin equations does not change the structure of our equation, as one shows easily. The only consequence of these quantities is the renormalization of the function[11] $U(m)$ which in the Gibbs distribution must be replaced by the function

$$U_1(m)=U(m)-2I_0T^{-1}gm^2,\quad g=G_0(\omega=0)=D_0(t=0). \tag{11}$$

Since the condition that there be an equilibrium which is local in time is satisfied in our case, the probability functional has the form

$$P\{m(t)\}=\prod_t\left\{\int dm(t)\exp\left\{-\frac{1}{T}H[m(t),t]\right\}\right\}^{-1}$$
$$\times\exp\left\{-\frac{1}{T}H[m(t),t]\right\},$$
$$H[m(t),t]=U_1[m(t)]-\xi(t)m(t),$$
$$H[s(t),t]=-\xi(t)s(t),\quad s^2(t)=1. \tag{12}$$

We note first of all that we can now replace in (12) $m(t)$ from the weak model by the Ising spin $s(t)$, and we need then simply use the last line in (12) for $H[s(t),t]$ and replace the normalization integral over m in (12) by the corresponding sum over s. We shall in what follows consider only the Ising case. Moreover, when using (12) it is necessary to bear in mind that $m(t)$ (or $s(t)$) occurs in H not only directly, but also through $\xi(t)$. We must take this fact into account not only in the numerator in (12), but also when calculating the normalization integral in the denominator. For what follows we use the well known connection between the retarded G_- and the advanced G_+ Green functions:

$$G_-(t-t')=G_+(t'-t). \tag{13}$$

We can symmetrize the expression for $P\{s(t)\}$. As a result we get the following expression for the complete probability functional $P\{s(t)\}$:

$$P\{s(t)\}=\frac{P_0\{s(t)\}}{\sum_{s(t)}P_0\{s(t)\}},$$
$$P_0\{s(t)\}=\exp\left\{\frac{1}{T}\int\frac{dt}{\tau}h(t)s(t)\right.$$
$$\left.+\frac{2I_0}{T^2}\int\frac{dt\,dt'}{\tau^2}p(t-t')s(t)s(t')\right\},$$
$$p(t)=\frac{\tau}{2}[G_{-s}(t)+G_{+s}(t)]=\frac{\alpha}{2}\frac{\tau}{|t|}\Delta'(z),$$
$$p(t)=p(-t). \tag{14}$$

In (14) $P\{s(t)\}$ is expressed in terms of the external field $h(t)$ and the symmetrized susceptibility $p(t)$. In that case $h(t)$ is a random function with a correlator defined in (9) which is expressed in terms of the characteristics of the random field $s(t)$. This also closes the set of equations. It is very important that the correlator of the external fields $h(t)$ and $p(t)$ also changes over macroscopic times. It is just this which enables us to write down $P\{s(t)\}$ explicitly. However, the expression (14) for P is not the final one as it must be averaged explicitly over $h(t)$. As we must in (14) average a fraction over $h(t)$ we must use the standard replica method

(see, e.g., Ref. 12) which is used in such cases. To do this we must introduce n quantities $s_\mu(t)$ ($\mu = 1,...,n$) and after all calculations we must put $n = 0$. Using standard manipulations for the averaging over $h(t)$ and using (10) we get

$$\tilde{P}\{s_\mu(t)\} = \exp\left\{\frac{2I_0}{T^2}\int\frac{dt\,dt'}{\tau^2}\sum_{\mu\nu}[p(t-t')s_\mu(t)s_\mu(t')\delta_{\mu\nu} + q(t-t')s_\mu(t)s_\nu(t')]\right\}. \quad (15)$$

We shall call the expression in the argument of the exponential in (15) the effective Hamiltonian. The connection between (15) and (14) is well known. If we must evaluate the average of some functional Φ of $s(t)$ over $P\{s(t)\}$ it is equal to

$$\langle\Phi\rangle = \langle \mathrm{Sp}_{s(t)}[\Phi\{s(t)\}P\{s(t)\}]\rangle_h = \lim_{n\to 0}\mathrm{Sp}_{s_\nu(t)}[\Phi\{s_\mu(t)\}\tilde{P}\{s_\nu(t)\}], \quad (16)$$

where the $s_\mu(t)$ in (16) are any of the n replicas. A very important particular case is when

$$\Phi\{s(t)\} = s(t)s(t'). \quad (17)$$

The average in (16) then determines the correlator of spins at different times. Equations (14) to (16) solve in principle the problem of the averaging of any functional $\Phi(t)$. However, in reality the calculation of the traces in (16) is not at all a simple problem.

It turns out that one can solve this problem explicitly by using the ultrametric structure of $q(t)$ and $p(t)$ and that one can thus write down an explicit expression for $P\{s(t)\}$. The next section is devoted to this problem.

In concluding this section we note that using $\tilde{P}\{s(t)\}$ in (15) we can calculate not only averages of functionals of a single replica $s_\mu(t)$, but also averages of different replicas $s_\mu(t)$, $s_\nu(t)$. These averages have a very well defined physical meaning as in the usual case.[12] We shall not discuss this in detail. We give only a single example which is important for what follows. There occur two functions in (15): $p(t)$ and $q(t)$, which are order parameters in spin glass theory. One shows easily that they can be expressed in terms of average quantities as follows:

$$q(t-t') = \langle s_\mu(t)s_\nu(t')\rangle_{\mu\neq\nu} = \langle\langle s(t)\rangle_P\langle s(t')\rangle_P\rangle_h,$$
$$p(t-t') = \langle s_\mu(t)s_\mu(t')\rangle - \langle s_\mu(t)s_\nu(t')\rangle_{\mu\neq\nu} = \langle s(t)s(t')\rangle_P - \langle s(t)\rangle_P\langle s(t')\rangle_P\rangle_h,$$
$$\langle\Phi\rangle_P = \mathrm{Sp}_{s(t)}[\Phi\{s(t)\}P\{s(t)\}]. \quad (18)$$

In (18) we denoted by $\langle...\rangle_p$ the trace over $s(t)$ with the functional distribution $P\{s(t)\}$ similar to what was done in (16), and by $\langle...\rangle_h$ averaging over the random field $h(t)$.

We note here the following very important fact. Comparing the expression for $p(t)$ in (14) with the corresponding expression in (18) we see that the susceptibility $G(t)$ can be expressed simply in terms of a correlator of the fields $s(t)$. This expression is indeed a variant of the fluctuation-dissipation theorem and is completely analogous to the corresponding expression of the static susceptibility in terms of a correlator. In our case the fluctuation-dissipation theorem thus connects the susceptibility not at all with the correlator $q(t)$ but with the correlator $p(t)$.

4. EVALUATION OF AVERAGES

One investigates the structure of the probability functional by evaluating the simplest averages. We shall evaluate these averages using the discrete time representation (4). To do this we must first find the discrete representation for the effective Hamiltonian (15). For what follows we add one more term to (15) with an external magnetic field h which we write in the exponent of (15) in the form

$$\exp\left\{\frac{h}{T}\int\frac{dt}{\tau}\sum_\mu s_\mu(t)\right\}. \quad (19)$$

Changing from the continuous times t, t' to the discrete times t_a, t_b (see (4)) we get

$$\tilde{P}\{s_\mu(t)\} = \exp\left\{\frac{h}{T}\sum_{\mu a}s_{\mu a} + \frac{2I_0}{T^2}\sum_{ab\mu\nu}(q_{ab}s_{\mu a}s_{\nu b} + p_{ab}\delta_{\mu\nu}s_{\mu a}s_{\nu b})\right\},$$
$$s_{\mu a} = s_\mu(t_a), \quad q_{ab} = q(t_a - t_b), \quad p_{ab} = p(t_a - t_b). \quad (20)$$

It is perfectly clear from (6) and (10) that q_{ab} depends solely on the ultrametric distance z_l between t_a and t_b, i.e., on which of the a_l and b_l are the first to differ from one another. It is clear from (6) that $z(t_a - t_b)$ runs through $k+1$ values corresponding to a change in l from 0 to k and z from $z_{\max}$ to Δz. According to (10) q_{ab} also runs through the same $k+1$ values. Using the ultrametric structure of q_{ab} we then get from (20)

$$\sum_{ab\mu\nu}q_{ab}s_{\mu a}s_{\nu b} = q_0\Big(\sum_{\mu a_0...a_k}s_{\mu a_0...a_k}\Big)^2 + q_1'\sum_{a_0}\Big(\sum_{\mu a_1...a_k}s_{\mu a_0...a_k}\Big)^2 + ... + q_k'\sum_{a_0...a_{k-1}}\Big(\sum_{\mu a_k}s_{\mu a_0...a_k}\Big)^2,$$
$$q_l' = q_l - q_{l-1}, \quad q_l = q(z_l). \quad (21)$$

A similar formula also arises for the term with p_{ab}. However, there is here an important difference from (21). The fact is that, as is clear from (10), g_s and hence also p_{ab} decreases for large times as a power law and not logarithmically like q_{ab}. We must thus parametrize this term in such a way as to take that fact into account. One shows easily that the corresponding expression has the following form:

$$\sum_{ab\mu}p_{ab}s_{\mu a}s_{\mu b} = -\frac{\Delta_0'}{p_0}\sum_\mu\Big(\sum_{a_0...a_k}s_{\mu a_0...a_k}\Big)^2 - \frac{\Delta_1'}{p_1}\sum_{\mu a_0}\Big(\sum_{a_1...a_k}s_{\mu a_0...a_k}\Big)^2 - ... - \frac{\Delta_k'}{p_k}\sum_{\mu a_0...a_{k-1}}\Big(\sum_{a_k}s_{\mu a_0...a_k}\Big)^2,$$
$$\Delta_l' = \Delta_{l+1} - \Delta_l < 0, \quad \Delta_l = \Delta(z_l), \quad \Delta_{k+1} = 0, \quad p_l = j^{k-l+1}. \quad (22)$$

Substituting (21) and (22) into (20) and performing a Stratonovich–Hubbard transformation for all quadratic terms, we obtain the following expression for

$$\tilde{P}\{s_\mu(t)\} \equiv \tilde{P}(s_{\mu a}) = \int \frac{dh^{(0)}}{(8\pi I_0 q_0)^{1/2}} \exp\left\{-\frac{(h^{(0)})^2}{8I_0 q_0}\right\}$$
$$\times \prod_\mu \int \frac{d\xi_\mu^{(0)}}{[8\pi I_0(-\Delta_0'/p_0)]^{1/2}} \exp\left\{-p_0 \frac{(\xi_\mu^{(0)})^2}{8I_0(-\Delta_0')}\right\}$$
$$\times \prod_{a_0} \int \frac{dh_{a_0}^{(1)}}{[8\pi I_0 q_1']^{1/2}} \exp\left\{-\frac{(h_{a_0}^{(1)})^2}{8I_0 q_1'}\right\}$$
$$\times \prod_{\mu a_0} \int \frac{d\xi_{\mu a_0}^{(1)}}{[8\pi I_0(-\Delta_1'/p_1)]^{1/2}} \exp\left\{-p_1 \frac{(\xi_{\mu a_0}^{(1)})^2}{8I_0(-\Delta_1')}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\times \prod_{\mu a_0 \ldots a_{k-1}} \int \frac{d\xi_{\mu a_0 \ldots a_{k-1}}^{(k)}}{[8\pi I_0(-\Delta_k'/p_k)]^{1/2}} \exp\left\{-p_k \frac{(\xi_{\mu a_0 \ldots a_{k-1}}^{(k)})^2}{8I_0(-\Delta_k')}\right\}$$
$$\times \prod_{a_0 \ldots a_{k-1}} \int \frac{dh_{a_0 \ldots a_{k-1}}^{(k)}}{[8\pi I_0 q_k']^{1/2}} \exp\left\{-\frac{h_{a_0 \ldots a_{k-1}}^{(k)}}{8I_0 q_k'}\right\}$$
$$\times \exp\left\{\frac{1}{T} \sum_{\mu a_0 \ldots a_k} \zeta_{\mu a_0 \ldots a_{k-1}}^{(k)} s_{\mu a_0 \ldots a_k}\right\},$$
$$\zeta_{\mu a_0 \ldots a_{k-1}}^{(k)} = h + h^{(0)} + h_{a_0}^{(1)} + \ldots + h_{a_0 \ldots a_{k-1}}^{(k)} + \xi_\mu^{(0)} + \xi_{\mu a_0}^{(1)} + \ldots + \xi_{\mu a_0 \ldots a_{k-1}}^{(k)}. \quad (23)$$

Expression (23) for $\tilde{P}(s_{\mu a})$ has already a rather simple structure which enables us to use it to calculate various actual averages. There arises, though, in the calculation of these averages, a difficulty which is totally characteristic for any spin glass theory at the present time. Using (23) to calculate averages one must assume the number of replicas n to be a finite quantity and to evaluate all integrals over h and ξ which occur in (23) for finite n and only after that one can let n tend to zero. One checks easily that when one proceeds in such a way one cannot obtain from (23) any simple recurrence relations which we, naturally, aim at. We are thus obliged when calculating averages to take into account in the intermediate calculations that $n \to 0$. This means an interchange of limiting transitions. A similar interchange of limiting transitions is, for instance, characteristic for Parisi's theory.[6-8] By all appearances the necessity for such an interchange of limits is not accidental but connected with the occurrence of non-ergodicity. As we are just interested in the non-ergodic region this necessity arises also in our theory. We demonstrate the way to calculate averages using the example of the calculation of the free energy. The free energy φ per spin can be expressed in terms of $\tilde{P}(s_{\mu a})$ as follows:

$$\sum_{s_{\mu a}} \tilde{P}(s_{\mu a}) = 1 + n p_0 \varphi. \quad (24)$$

As ζ in (23) is independent of a_k, the trace over $s_{\mu a}$ gives the following expression:

$$\prod_{\mu a_0 \ldots a_{k-1}} \exp\{p_k \varphi_k(\zeta_{\mu a_0 \ldots a_{k-1}}^{(k)})\},$$
$$\varphi_k(h) = \ln\left(2 \operatorname{ch} \frac{h}{T}\right). \quad (25)$$

One must integrate the expression in (25) over all h and ξ which occur in (23) with the appropriate weight. We consider the integral over one of these variables ξ_μ, h. It is equal to

$$\int \frac{dh_{a_0 \ldots a_{k-1}}^{(k)}}{(8\pi I_0 q_k')^{1/2}} \exp\left\{-\frac{(h_{a_0 \ldots a_{k-1}}^{(k)})^2}{8I_0 q_k'}\right\}$$
$$\times \prod_\mu A_k[\zeta_{\mu a_0 \ldots a_{k-2}}^{(k-1)} + h_{a_0 \ldots a_{k-1}}^{(k)}],$$
$$A_k(h) = \int \frac{d\xi}{[8\pi I_0(-\Delta_k'/p_k)]^{1/2}}$$
$$\times \exp\left\{p_k\left[-\frac{\xi^2}{8I_0(-\Delta_k')} + \varphi_k(h+\xi)\right]\right\},$$
$$\zeta_{\mu a_0 \ldots a_{k-2}}^{(k-1)} = h + h^{(0)} + h_{a_0}^{(1)} + \ldots + h_{a_0 \ldots a_{k-2}}^{(k-1)} + \xi_\mu^{(0)} + \xi_{\mu a_0}^{(1)} + \ldots + \xi_{\mu a_0 \ldots a_{k-2}}^{(k-1)}. \quad (26)$$

There occurs a product over μ from 1 to n in the integrand in (26). If we assume n to be finite the integration over h mixes up all ζ_μ. If we, on the other hand, at once take into account that we take $n \to 0$, we can put in (26)

$$\prod_\mu A_k(h+\zeta_\mu) = 1 + \sum_\mu \ln A_k(h+\zeta_\mu). \quad (27)$$

After this the integration over h in (26) can be performed elementarily and afterwards, using a formula such as (27), we collect everything in the exponent. As a result the integral in (26) turns out to equal

$$\prod_\mu \exp\{p_k \varphi_{k-1}(\zeta_\mu)\},$$
$$\varphi_{k-1}(\zeta) = \frac{1}{p_k} \int \frac{dh}{(8\pi I_0 q_k')^{1/2}} \exp\left(-\frac{h^2}{8I_0 q_k'}\right) \ln A_k(\zeta+h). \quad (28)$$

Since in the calculation of the integrals in (23) over $h^{(k)}$ and $\xi^{(k)}$ we have on each branch $j = p_{k-1}/p_k$ terms, we get from (28) on each branch similarly to (25) an expression

$$\prod_{\mu a_0 \ldots a_{k-2}} \exp\{p_{k-1} \varphi_{k-1}(\zeta_{\mu a_0 \ldots a_{k-2}}^{(k-1)})\}. \quad (29)$$

We have thus again obtained an expression of exactly the same structure as the initial one. Repeating the iteration we get at any level of iteration l a recurrence relation connecting φ_{l-1} with φ_l. This recurrence relation has exactly a form such as (28), but with k replaced by l (q_k') by q_l', and so on). After all iterations we are at the last level left with integrals over $h^{(0)}$ and $\xi_\mu^{(0)}$. Instead of ζ_μ we have then simply the external magnetic field h for which there is, of course, no index μ. We can therefore at the last level explicitly split off the factor n. We get then for the total free energy, using (24), the following expression:

$$\varphi(h) = \frac{1}{p_0} \int \frac{dh_1}{(8\pi I_0 q_0)^{1/2}} \exp\left(-\frac{h_1^2}{8I_0 q_0}\right) \ln A_0(h+h_1). \quad (30)$$

Moreover, it is clear from (22) that $p_l \to \infty$ as $j \to \infty$ so that we can use the saddle-point method for the integrals

over ξ in (28) and (30). It is convenient to express the final answer in terms of the function $R_l(h)$ which is defined as the solution of the equation

$$R_l(h)=h-4I_0\Delta_l' T^{-1}M_l[R_l(h)],$$
$$M_l(h)=\varphi_l'(h)\equiv T\partial\varphi_l(h)/\partial h. \tag{31}$$

We shall indicate in (31) and in other formulae in what follows $T\partial/\partial h$ by a prime. If we take the saddle point in ξ in (28) and (30) we get the following final expression for $\varphi(h)$ and the recurrence relation for the $\varphi_l(h)$:

$$\varphi(h)=\int\frac{dh_1}{(8\pi I_0q_0)^{1/2}}\exp\left(-\frac{h_1^2}{8I_0q_0}\right)\left\{\varphi_0[R_0(h+h_1)]+\frac{2I_0}{T^2}\Delta_0'M_0^2[R_0(h+h_1)]\right\},$$
$$\varphi_{l-1}(h)=\int\frac{dh_1}{(8\pi I_0q_l')^{1/2}}\exp\left(-\frac{h_1^2}{8I_0q_l'}\right)\left\{\varphi_l[R_l(h+h_1)]+\frac{2I_0}{T^2}\Delta_l'M_l^2[R_l(h+h_1)]\right\},$$
$$\varphi_k(h)=\ln\left(2\operatorname{ch}\frac{h}{T}\right). \tag{32}$$

One shows easily that (32) reproduces Sompolinsky's functional[4] for the free energy but we shall not discuss this in detail.

In deriving (32) we managed to integrate explicitly over all $\xi_{\mu a}$, but, it is true, at the price of interchanging limits as we have already mentioned. It would appear that one can in exactly the same way integrate in (23) over all $\xi_{\mu a}$ and obtain an explicit expression for $\tilde{P}(s_{\mu a})$ in the form of integrals only over the physical fields h_a. However, it turns out that this cannot be done directly for the following reason. We managed to integrate over $\xi_{\mu a}$ only after evaluating the trace over all $s_{\mu a}$ but we cannot do this before calculating this trace and this is connected with the fact that the replicas are only symbols and not at all physical objects and they require a well defined limiting transition. To derive the required expressions we must therefore proceed similarly to what we did when deriving (32) and just investigate the average of various functions over $s_{\mu a}$. When evaluating these averages we first of all calculate all traces over $s_{\mu a}$, then integrate over all $\xi_{\mu a}$ and only after that as the last step we recover the expression for $\tilde{P}(s_{\mu a})$ in which we are interested in terms of integrals over the physical fields h_a.

We demonstrate the proposed plan of action by the example of the calculation of the average of a spin function $f(s_{\mu b})$ which depends only on a single time t_b while $f(s_{\mu b})$ may depend on different replicas, for instance, we may have

$$f(s_{\mu b})=s_{\mu b},$$
$$f(s_{\mu b})=s_{\mu b}s_{\nu b},\quad \mu\neq\nu \tag{33}$$

and so forth. Let $f(s_{\mu b})$ depend on m different replicas $s_{1b},\ldots,s_{mb}$. We shall then assume, of course, that m is finite everywhere, for instance also in the limit as $n\to 0$. The necessity to take the limit as $n\to 0$ under the condition that m is finite was just the reason that it was impossible to evaluate directly in Eq. (23) the integrals over the $\xi_{\mu a}$, without calculating beforehand the averages over the $s_{\mu a}$.

For what follows it is convenient for us to introduce two functions:

$$\Psi_k(h_1\ldots h_m)=\sum_{s_1\ldots s_m}\left\{\prod_{\mu=1}^{m}\frac{\exp(h_\mu s_\mu/T)}{2\operatorname{ch}(h_\mu/T)}\right\}f(s_1\ldots s_m),$$
$$\rho_k(h)=\Psi_k(h\ldots h)=\sum_{s_1\ldots s_m}\left\{\prod_{\mu=1}^{m}\frac{\exp(hs_\mu/T)}{2\operatorname{ch}(h/T)}\right\}f(s_1\ldots s_m)$$
$$\equiv\lim_{n\to 0}\sum_{s_1\ldots s_n}\left\{\prod_{\mu=1}^{n}\exp(hs_\mu/T)\right\}f(s_1\ldots s_m). \tag{34}$$

The last line in (34) is obtained by direct calculation and is the main limit we need.

The further calculations are very similar to the derivation of the expression for the free energy and we reproduce them therefore only briefly. Exactly as in the derivation of (25) one sees easily that the trace over all $s_{\mu a}$ leads to us obtaining instead of (25) the following expression:

$$\prod_{\mu a_0\ldots a_{k-1}}\Psi_k(\xi^{(k)}_{1b_0\ldots b_{k-1}}\ldots\xi^{(k)}_{mb_0\ldots b_{k-1}})\exp\{p_k\varphi_k(\xi^{(k)}_{\mu a_0\ldots a_{k-1}})\}. \tag{35}$$

Acting in exactly the same way as in going from (25) to (29) we get from (35) the expression

$$\prod_{\mu a_0\ldots a_{k-2}}\Psi_{k-1}(\xi^{(k-1)}_{1b_0\ldots b_{k-2}}\ldots\xi^{(k-1)}_{mb_0\ldots b_{k-2}})\exp\{p_{k-1}\varphi_{k-1}(\xi^{(k-1)}_{\mu a_0\ldots a_{k-2}})\},$$
$$\Psi_{k-1}(\xi_1\ldots\xi_m)=\int\frac{dh}{(8\pi I_0q_k')^{1/2}}\times\exp\left(-\frac{h^2}{8I_0q_k'}\right)\frac{B_k(\xi_1+h,\ldots\xi_m+h)}{A_k(\xi_1+h)\ldots A_k(\xi_m+h)},$$
$$B_k(h_1\ldots h_m)=\prod_{\mu=1}^{m}\left\{\int\frac{d\xi_\mu}{[8\pi I(-\Delta_k'/p_k)]^{1/2}}\right.$$
$$\left.\times\exp\left[-p_k\frac{\xi_\mu^2}{8I_0(-\Delta_k')}+p_k\varphi_k(h_\mu+\xi_\mu)\right]\right\}\Psi_k(h_1+\xi_1\ldots h_m+\xi_m). \tag{36}$$

We have thus again obtained an iteration process for the quantity $\Psi_k(h_1,\ldots,h_m)$. Iterating (36) further and evaluating the integrals over ξ_μ by steepest descent we get

$$\langle f(s_{\mu b})\rangle=\Psi(h,\ldots h),$$
$$\Psi(h_1\ldots h_m)=\int\frac{dh}{(8\pi I_0q_0)^{1/2}}\exp\left(-\frac{h^2}{8I_0q_0}\right)\times\Psi_0[R_0(h_1+h)\ldots R_0(h_m+h)],$$
$$\Psi_{l-1}(\xi_1\ldots\xi_m)=\int\frac{dh}{(8\pi I_0q_l')^{1/2}}\exp\left(-\frac{h^2}{8I_0q_l'}\right)\times\Psi_l[R_l(\xi_1+h)\ldots R_l(\xi_m+h)], \tag{37}$$

where $R_l(h)$ was defined in (31). It is clear from (37) that in the last stage of the iteration in all arguments of Ψ the external magnetic field h which is independent of μ appears rather than the ξ_μ. The average of $f(s_{\mu a})$ can thus be ex-

pressed in terms of the function Ψ with coincident arguments. On the other hand, it is clear from (37) that for this function of a single variable there is exactly the same iteration scheme as for a Ψ of m arguments. We therefore get finally

$$\langle f(s_{\mu b})\rangle=\rho(h),$$
$$\rho(h)=\int \frac{dh_1}{(8\pi I_0 q_0)^{1/2}}\exp\left(-\frac{h_1^2}{8I_0q_0}\right)\rho_0[R_0(h+h_1)],$$
$$\rho_{l-1}(\zeta)=\int \frac{dh}{(8\pi I_0 q_l')^{1/2}}\exp\left(-\frac{h^2}{8I_0q_l'}\right)\rho_l[R_l(\zeta+h)],$$
$$\rho_l(h)=\Psi_l(h\ldots h). \qquad (38)$$

We have written down the boundary condition for $\rho_l(h)$ for $l=k$ in (34).

Equations (38) solve in general form the problem of evaluating any average of a quantity which depends on a single time t_a. We note here one important particular case. If $f=s_{\mu a}$, we obtain an iteration process for $M_l(h)$ which determines the average moment $M(h)$. This iteration process is obtained simply by replacing in (38) ρ_l and ρ by M_l and M with the boundary condition

$$M_k(h)=\operatorname{th}\frac{h}{T}\equiv\lim_{n\to 0}\sum_{s_\mu}s_\nu\left\{\prod_{\mu=1}^{n}\exp\left(\frac{h}{T}s_\mu\right)\right\}. \qquad (39)$$

The iteration scheme for $M_l(h)$ and Eqs. (31) and (32) close the set of equations for $R_l(h)$ and $\varphi_l(h)$, $M_l(h)$ which determines the free energy and the average moment of the system.

We have thus shown that the calculation of the simplest averages reduces to iteration schemes of the kind (32), (37), and (38).

We now formalize the iteration scheme (38). We introduce the operator $\hat{P}_{l-1,l}$ which changes one function $\Psi(h)$ into another function h through the following formula:

$$\hat{P}_{l-1,l}\{\Psi(h)\}=\int \frac{dh_1}{(8\pi I_0 q_l')^{1/2}}\exp\left(-\frac{h_1^2}{8I_0q_l'}\right)\Psi[R_l(h+h_1)],$$
$$\hat{P}\{\Psi(h)\}=\int \frac{dh_1}{(8\pi I_0 q_0)^{1/2}}\exp\left(-\frac{h_1^2}{8I_0q_0}\right)\Psi[R_0(h+h_1)]. \qquad (40)$$

As in Eqs. (32) and (38) one must distinguish the last iteration. We note that the quantities $\hat{P}_{l-1,l}$ and $\hat{P}\{\Psi(h)\}$ are functions of the magnetic field h which occurs as a parameter in (40). We now define operators for a finite number of iterations:

$$\hat{P}_{l,n}\{\Psi(h)\}=\hat{P}_{l,l+1}\{\hat{P}_{l+1,l+2}[\ldots\hat{P}_{n-1,n}(\Psi(h))]\},$$
$$\hat{F}_l\{\Psi(h)\}=\hat{P}\{\hat{P}_{0,l}[\Psi(h)]\}. \qquad (41)$$

In these notations Eq. (38) with the boundary condition (34) have the following simple form:

$$\langle f(s_{\mu b})\rangle=\hat{F}_k\left\{\lim_{n\to 0}\sum_{s_\mu}\exp\left[\frac{h}{T}\sum_{\mu=1}^{n}s_\mu\right]f(s_\mu)\right\}. \qquad (42)$$

We shall in what follows the transition to the limit n will be implied. We note that the field h which occurs as a parameter in the expression for the operators $\hat{P}$, $\hat{F}_l$, and $\hat{F}_k$ is the external magnetic field h.

We have thus learned to evaluate in a general form averages of a single-time function $f(s_{\mu b})$. It is easy to calculate by the same method also averages of functions of several times. Let us need, for instance, to average the function $f_1(s_{\mu a})f_2(s_{\mu b})$ where the common ancestor of the times a and b is at level l, i.e., the ultrametric distance between them equals $z_{ab}=z_l$. It is convenient to associate a picture with all these words; for instance, we associate the picture of Fig. 2a with two times a and b. In exactly the same way as we derived Eq. (42) we get then

$$\langle f_1(s_{\mu a})f_2(s_{\mu b})\rangle=\hat{F}_l\left\{\hat{P}_{l,k}\left[\sum_{s_\mu}\exp\left(\frac{h}{T}\sum_{\mu=1}^{n}s_\mu\right)f_1(s_\mu)\right]\right.$$
$$\left.\times\hat{P}_{l,k}\left[\sum_{s_\mu}\exp\left(\frac{h}{T}\sum_{\mu=1}^{n}s_\mu\right)f_2(s_\mu)\right]\right\},\quad z_l=z_{ab}. \qquad (43)$$

As an example we write down yet another average of a function of four times: a, b, c, d. For the many-time case the distances between different times often coincide. We shall associate with each time distribution its own picture. For instance, for four times there are different figures, two of which are topologically non-equivalent and those are depicted in Figs. 2b and 2c, while the others are obtained from them by a permutation of the indexes. The ultrametric distances for these figures b and c are, respectively, equal to

$$z_{ac}=z_{ad}=z_{bc}=z_{bd}=z_1,\quad z_{ab}=z_2,\ z_{cd}=z_3$$
$$z_{ad}=z_{bd}=z_{cd}=z_1,\quad z_{ac}=z_{bc}=z_2,\ z_{ab}=z_3. \qquad (44)$$

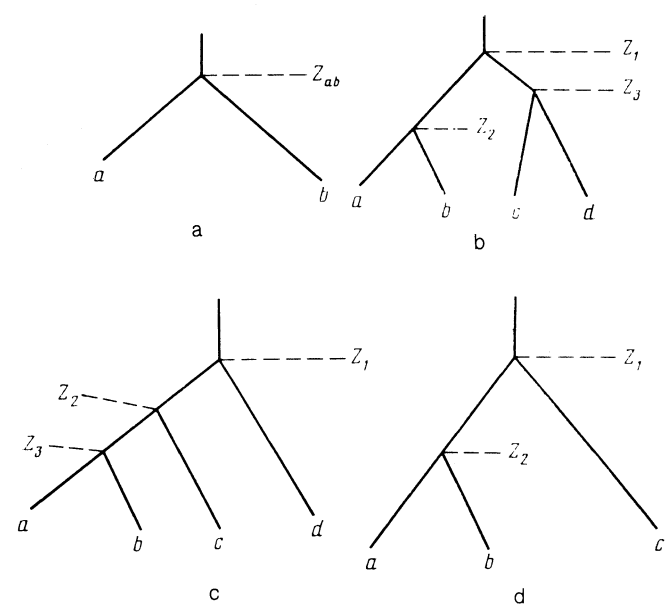

FIG. 2.

We write down the average only for the case of Fig. 2b:

$$\langle f_1(s_{\mu a}) f_2(s_{\mu b}) f_3(s_{\mu c}) f_4(s_{\mu d})\rangle$$
$$= \bar{F}_{l_1}\{\hat{P}_{l_1 l_2}[\hat{P}_{l_2 k}(\rho_1(h))\hat{P}_{l_2 k}(\rho_2(h))]$$
$$\times \hat{P}_{l_1 l_3}[\hat{P}_{l_3 k}(\rho_3(h))\hat{P}_{l_3 k}(\rho_4(h))]\},$$
$$z_{lt}=z_i,$$
$$\rho_i(h)=\sum_{s_\mu}\exp\left[\frac{h}{T}\sum_{\mu=1}^{n}s_\mu\right]f_i(s_\mu). \tag{45}$$

In (43) and (45) we evaluated for the sake of simplicity only the averages of products of $f_i(s_\mu)$, but, as we shall see below, this is unimportant and one can easily lift that restriction.

We have thus learned how to calculate in general form the mean values of arbitrary functions of type (42), (43), and (45). These expressions, however, are not very helpful. It turns out that they can be represented in a rather simple and helpful form by recognizing that we have an infinite number of levels of hierarchy levels k, while q_l' and Δ_l' are infinitesimally small quantities. We deal with this in the next section.

5. DISTRIBUTION FUNCTIONS AND COMPLETE PROBABILITY FUNCTIONAL

We now consider the continuum limit of our expressions. It is clear from (31) and (40) that in the continuum limit when q_l', $\Delta_l'\to 0$ the operators $\hat{P}_{l-1,l}$ and $\hat{P}$ become linear integral operators:

$$\hat{P}_{l-1,l}\{\Psi(h)\}\to\int P(z_{l-1},h,z_l,h')\Psi(h')dh',$$
$$\hat{P}\{\Psi(h)\}\to\int P(h,h')\Psi(h')dh',$$
$$P(z_{l-1},h,z_l,h')$$
$$=\delta(h-h')+\frac{4I_0}{T^2}\left\{\frac{1}{2}q_l'\delta''(h-h')-\Delta_l' M_l(h)\delta'(h-h')\right\},$$
$$P(h,h')=\frac{1}{(8\pi I_0 q_0)^{1/2}}\exp\left\{-\frac{(h-h')^2}{8I_0 q_0}\right\}$$
$$\delta'(h)=T\frac{\partial}{\partial h}\delta(h). \tag{46}$$

In the expression for $P(h,h')$ in (46) we neglected infinitesimal terms as $\Delta_0'\to 0$ as compared to q_0 and in the expression for $P(z_{l-1},h,z_l,h')$ we changed from a numbering of the hierarchy levels by the index l to a numbering, corresponding to that level, by the quantity z_l defined in (6). We then get from (41) and (46)

$$\hat{P}_{l,n}\{\Psi(h)\}\to\int P(z_l,h,z_n,h')\Psi(h')dh',$$
$$\hat{F}_l\{\Psi(h)\}\to\int F(z_l,h')\Psi(h')dh',$$
$$P(z_l,h,z_n,h')=\int dh_1\,dh_2\ldots dh_{l-n-1}P(z_l,h,z_{l+1},h_1)P(z_{l+1},h_1,$$
$$z_{l+2},h_2)\ldots P(z_{n-1},h_{l-n-1},z_n,h'),$$
$$F(z_l,h')=\int dh_1 P(h,h_1)P(\infty,h_1,z_l,h'). \tag{47}$$

We defined in (47) two very important quantities: $P(z,h,z',h')$ and $F(z,h')$ where F depends on the external magnetic field h as on a parameter. From the equations obtained it is clear that P satisfies the equation

$$P(z,h,z',h')=\int dh_1 P(z,h,z_1,h_1)P(z_1,h_1,z',h'), \tag{48}$$

which, as is well known, is the basic equation for the transition probability in the theory of Markov processes while the variable z plays here the role of the time. It is completely clear that $F(z,h)$ is the single-particle distribution function for the appropriate Markov process. We have a Fokker-Planck equation for $F(z,h)$. The same equation is satisfied by $P(z,h,z',h')$ as function of h', but as function of h this function satisfied the adjoined equation. We write down these two equations which we easily obtain from (46) and (47):

$$-\frac{\partial P}{\partial z}=4I_0\left\{\frac{1}{2}q'(z)\frac{\partial^2 P}{\partial h^2}-\frac{1}{T}\Delta'(z)M(z,h)\frac{\partial P}{\partial h}\right\},$$
$$\frac{\partial P}{\partial z'}=4I_0\left\{\frac{1}{2}q'(z')\frac{\partial^2 P}{\partial h'^2}+\frac{1}{T}\Delta'(z')\frac{\partial}{\partial h'}[PM(z',h')]\right\},$$
$$\frac{\partial F}{\partial z}=4I_0\left\{\frac{1}{2}q'(z)\frac{\partial^2 F}{\partial h^2}+\frac{1}{T}\Delta'(z)\frac{\partial}{\partial h}[FM(z,h)]\right\},$$
$$M(z,h)=\int P(z,h,0,h')\,\mathrm{th}\frac{h'}{T}dh',$$
$$P(z,h,z,h')=\delta(h-h'),$$
$$F(\infty,h')=P(h,h'),$$
$$q'(z_l)=q_l',\quad \Delta'(z_l)=\Delta_l'. \tag{49}$$

Equations (47) to (49) give us the transition from a discrete representation which we numbered by the index l to a continuum one which is determined by the variable z.

One sees easily that Eq. (49) are the same as the analogous equations in Parisi's theory which were obtained in Refs. 13, 14. The only difference consists in that in Parisi's theory there is a parameter changing in the range $[0,1]$ while we have a logarithmic variable z which changes in the range $[\infty,0]$. One can show that, by virtue of gauge invariance,[4,5] x may be any monotonic function of z, for instance, we may assume that

$$x(z)=\frac{1}{1+z}=\frac{1}{1+\alpha\ln(|t-t'|/\tau)}. \tag{50}$$

We have thus also established the connection between the time theory and Parisi's approach. All results, therefore, which we obtain (in the continuum limit, of course) can be completely extended also to Parisi's theory.

We now turn to a consideration of many-particle distribution functions which we did not consider earlier. We show that any many-particle distribution function can be expressed in quadratures in terms of the transition probability $P(z,h,z',h')$. We first of all consider in detail the single-particle distribution function. From (42) we have

$$\langle f(s_{\mu b})\rangle=\int dh_1 F_1(h_1)\sum_{s_{\mu b}}\exp\left[\frac{h_1}{T}\sum_{\mu=1}^{n}s_{\mu b}\right]f(s_{\mu b}),$$
$$F_1(h)=F(0,h). \tag{51}$$

It is at once clear from (51) that the single-time spin distri-

bution function $\tilde{P}(s_{\mu b})$ obtained from (15) by integrating over all times except t_b is equal to

$$\tilde{P}(s_{\mu b}) = \int dh_1 F_1(h_1) \exp\left[\frac{h_1}{T}\sum_{\mu=1}^{n} s_{\mu b}\right]. \tag{52}$$

If $f(s_\mu)$ depends only on a single replica we get by evaluating the trace over all other replicas in the limit as $n \to 0$ from (51) and (52) an explicit expression already for the spin distribution function $P(s_b)$ which is obtained from (14) like (52) from (15):

$$P(s_b) = \int dh_1 F_1(h_1) \frac{\exp(h_1 s_b/T)}{2\operatorname{ch}(h_1/T)}. \tag{53}$$

It is clear from (51) to (53) that the single-time spin distribution function is obtained by averaging the usual distribution functions in a magnetic field $\tilde{K}(h,s_\mu)$ or $K(h,s)$:

$$\tilde{K}(h, s_\mu) = \exp\left\{\frac{h}{T}\sum_{\mu=1}^{n} s_\mu\right\},$$

$$K(h, s) = \frac{\exp(hs/T)}{2\operatorname{ch}(h/T)} \tag{54}$$

over the single-particle molecular field distribution function $F_1(h)$.

Exactly the same situation arises also for the many-particle distribution functions. One shows easily, using equations such as (43), (45) that we have for the arbitrary r-particle distribution functions $\tilde{P}_r$ and P_r

$$\tilde{P}_r(s_{\mu_1}\dots s_{\mu_r}) = \int dh_1 \dots dh_r F_r(h_1 \dots h_r) \times \exp\left\{\frac{1}{T}\sum_{m=1}^{r} h_m\left[\sum_{\mu_m=1}^{n} s_{\mu_m}\right]\right\},$$

$$P_r(s_1 \dots s_r) = \int_r dh_1 \dots dh_r F_r(h_1 \dots h_r) \prod_{m=1}^{r} \frac{\exp(h_m s_m/T)}{2\operatorname{ch}(h_m/T)}, \tag{55}$$

where the $F_r(h_1,\dots,h_r)$ are r-particle molecular field distribution functions. We have here marked the different moments in time t_m by the index m. Therefore, F_r depends not only on the variables h_m, but also on the ultrametric distances $z(t_m - t_k)$, $m, k = 1,\dots,r$ between these times. As we already discussed when deriving Eqs. (43) to (45) it is convenient to represent the picture of the ultrametric distances by figures such as 2a to 2c. We write down $F_2(z_{zb},h_a,h_b)$ corresponding to Fig. 2a and $F_4(z_1,z_2,z_3,h_a,h_b,h_c,h_d)$ corresponding to Fig. 2b (the definition of the z_i is given in (44)):

$$F_2(z_{ab}, h_a, h_b) = \int dh_1 F(z_{ab}, h_1) P(z_{ab}, h_1, 0, h_a) P(z_{ab}, h_1, 0, h_b),$$

$$F_4(z_1, z_2, z_3, h_a, h_b, h_c, h_d)$$

$$= \int dh_1\, dh_2\, dh_3 F(z_1, h_1) P(z_1, h_1, z_2, h_2) \times P(z_1, h_1, z_3, h_3) P(z_2, h_2, 0, h_a) P(z_2, h_2, 0, h_b) \times P(z_3, h_3, 0, h_c) P(z_3, h_3, 0, h_d). \tag{56}$$

The way to write down expression such as (56) is clear. With each figure there is associated a well defined expression. The upper line is associated with the function $F(z,h)$ and the other ones with $P(z,h,z',h')$ and one must integrate over the free h_i variables. The variables z_{ik} are determined, as in (44), by the ultrametric distances between the different times t_i.

We see thus that any r-particle function $F_r(h_1,\dots,h_r)$ for any ultrametric distances between the times $z(t_{ik})$ $(i,k = 1,\dots,r)$ can be completely expressed in terms of the distribution function $F(z,h)$ and the transition probability $P(z,h,z',h')$. Since, as can be seen from (47), $F(z,h)$ can also be expressed in terms of $P(z,h,z',h')$ we need only know the transition probability.

It is now already completely clear that for our problem we can write down also the complete probability functional. To do this we must in (55) let $r \to \infty$. We then get

$$\tilde{P}\{s_\mu(t)\} = \int Dh(t) F\{h(t)\} \exp\left\{\frac{1}{T}\sum_{\mu}\int \frac{dt}{\tau} h(t) s_\mu(t)\right\},$$

$$P\{s(t)\} = \int Dh(t) F\{h(t)\} \prod_t \frac{\exp\{h(t)s(t)/T\}}{2\operatorname{ch}(h(t)/T)},$$

$$Dh(t) = \prod_t dh(t). \tag{57}$$

The functionals $\tilde{P}\{s_\mu(t)\}$ and $P\{s(t)\}$ are the complete probability functionals introduced in (14) and (15). They can be expressed explicitly in terms of the local Gibbs distribution $\tilde{K}(h,s_\mu)$ and $K(h,s)$ and in terms of the complete molecular field probability functional $F\{h(t)\}$. To write down an explicit expression for $F\{h(t)\}$ we must again change to discrete times t_a similarly as was done to derive (23). We then get

$$F\{h(t)\} \equiv F(h_a) = \int dh^{(0)} P(h, h^{(0)}) \prod_{a_0} \int dh_{a_0}^{(1)} P(z_0, h^{(0)}, z_1, h_{a_0}^{(1)}) \times \prod_{a_0 a_1} \int dh_{a_0 a_1}^{(2)} P(z_1, h_{a_0}^{(1)}, z_2, h_{a_0 a_1}^{(2)}) \prod_{a_0 \dots a_{k-1}} \int dh_{a_0 \dots a_{k-1}}^{(k)} \times P(z_{k-1}, h_{a_0 \dots a_{k-2}}^{(k-1)}, z_k, h_{a_0 \dots a_{k-1}}^{(k)}) \prod_{a_0 \dots a_k} \delta(h_{a_0 \dots a_k} - h_{a_0 \dots a_{k-1}}^{(k)}). \tag{58}$$

The presence of a δ-function in the last stage instead of P is completely analogous to the corresponding property of Eq. (23) and is connected with the fact that $z_k = \Delta z$ is the last term which corresponds to macroscopic times and in the continuum limit $\Delta z \to 0$. We note also that in that limit $z_0 \to \infty$.

All expressions in (57) and (58) are, of course, normalized. If we must evaluate the explicit expression for some r-

particle distribution function we must in (57) and (58) integrate over all $s(t)$ or $h(t)$ corresponding to all times bar t_r when we obtain expressions such as (52), (53), (55), (56), and similar ones.

The problem of calculating complete spin and molecular field probability functionals has thus been solved in a general form. One might wish to note that obtaining the explicit form of these functionals turned out to be possible only thanks to the ultrametric topology of the time space.

6. CORRELATORS AND SUSCEPTIBILITY

To close the molecular field equations one must, as can be seen from (18), calculate two-particle correlators. We get for $q(z)$ from (18) and (55)

$$q(z) = \int dh_1\, dh_2 F_2(z, h_1, h_2) \operatorname{th}\frac{h_1}{T} \operatorname{th}\frac{h_2}{T} = \int dh F(z, h) M^2(z, h). \tag{59}$$

The evaluation of the correlator $p(z)$ is somewhat more complicated The fact is that, as can be seen from (55) when $t_a \neq t_b$

$$\langle s_{\mu a} s_{\mu b}\rangle = \langle s_{\mu a} s_{\nu b}\rangle_{\mu\neq\nu}. \tag{60}$$

It is clear from (18) and (60) that $p(t) = 0$. If we recall that in deriving the expressions we need we used everywhere steepest descent for the ξ_μ and if we examine the corresponding expressions before evaluating the saddle points we see that only when the saddle points are calculated is (60) obtained. This means that $p(t) = 0$ only in the dominant order in the steepest descent parameter $j \to \infty$. One should have expected this as it is clear from (10) that $G_s(t) \sim p(t) \sim 1/t$ and it is clear from (22) that the point-law decrease in $p(t)$ at once leads to being small in the steepest descent parameter. It is rather complicated to calculate the corrections to the saddle point and we shall therefore use the following circuitous method. We turn to (23). We multiply (23) by $s_{\mu a}$, integrate the expression obtained by $h^{(l)}_{a_0 \ldots a_{l-1}}$ and sum over all $s_{\mu a}$. One easily sees that we then get

$$\sum_{\nu b} \langle s_{\mu a} s_{\nu b}\rangle, \tag{61}$$

where the sum over b in (61) extends over all b such that

$$z(t_a - t_b) < z_l. \tag{62}$$

We then get from (61) the following expression in the limit as $n \to 0$:

$$\int_{-t}^{t} dt_1 \langle s_\mu(t_1) s_\mu(0) - s_\mu(t_1) s_\nu(0)\rangle_{\mu\neq\nu} = 1 - q(0) + \Delta(z). \tag{63}$$

The last line is obtained from (10), (14), (18), and the boundary conditions for $q(z)$ and $\Delta(z)$. We can thus evaluate $\Delta(z)$. If after differentiating with respect to $h^{(l)}$ we carry out the whole procedure for obtaining the recurrence relations and afterwards go over the continuum limit we get

$$1 - q(0) + \Delta(z) = \int F(z, h) M'(z, h)\, dh. \tag{64}$$

Equations (59) and (64) close the set of molecular field theory equations as the functions $F(z,h)$ and $M(z,h)$ which occur in (59) and (64) are in turn determined from equations in which $q(z)$ and $\Delta(z)$ occur. The equation for $F(z,h)$ is written down in (49) and one easily obtains the equation for $M(z,h)$ from the definition in (49) of this quantity and the equation for $P(z,h,z',h')$ in the first variable. We write down explicitly the equation for $M(z,h)$:

$$-\frac{\partial M}{\partial z} = \frac{4I_0}{T^2}\left\{\frac{1}{2} q'(z) M''(z, h) - \Delta'(z) M(z, h) M'(z, h)\right\}, \quad M(0, h) = \operatorname{th}(h/T), \quad M'(z, h) = T\partial M/\partial h. \tag{65}$$

We can thus obtain an equation and an explicit expression for any correlation function. Concluding we write down explicitly the three-particle correlation function corresponding to Fig. 2d (it is, of course, non-vanishing only when $h \neq 0$):

$$\langle s_a s_b s_c\rangle = \int dh_1\, dh_2\, dh_3\, dh_4\, dh_5 F(z_1 h_1) P(z_1, h_1, z_2, h_2) \times P(z_1, h_1, 0, h_5) \times P(z_2, h_2, 0, h_3) P(z_2, h_2, 0, h_4) \operatorname{th}\frac{h_3}{T} \operatorname{th}\frac{h_4}{T} \operatorname{th}\frac{h_5}{T}. \tag{66}$$

[1] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).
[2] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, J. Physique **45**, 843 (1984).
[3] M. Mezard and M. A. Virasoro, J. Physique **46**, 1293 (1985).
[4] H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981).
[5] S. L. Ginzburg, Zh. Eksp. Teor. Fiz. **90**, 754 (1986) [Sov. Phys. **63**, 439 (1986)].
[6] G. Parisi, Phys. Rev. Lett. **50**, 1946 (1983).
[7] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).
[8] G. Parisi, J. Phys. **A13**, L 115 (1980).
[9] S. L. Ginzburg, Zh. Eksp. Teor. Fiz. **85**, 2171 (1983) [Sov. Phys. **56**, 1257 (1983)].
[10] H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **47**, 359 (1981).
[11] H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **B25**, 2065 (1982).
[12] S. F. Edwards and P. W. Anderson, J. Phys. **F5**, 965 (1975).
[13] J. R. de Almeida and E. J. S. Lage, J. Phys. **C16**, 939 (1983).
[14] H. J. Sommers and W. Dupont, J. Phys. **C17**, 5785 (1984).

Translated by D. ter Haar