The Hamiltonian description of the gravitational field and gauge symmetries

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A Hamiltonian theory of an (Einsteinian) gravitational field superposed on an arbitrary fixed space-time is constructed. Reasons for the appearance of internal (gauge) symmetries are indicated. It is shown that, in contrast to an arbitrary field theory on a fixed background, both internal and external symmetries in this theory introduce the same constraints. The theory is considered in application to the problem of the determination of the total energy and other conserved quantities for an asymptotically flat space-time and a closed universe. Three-covariant expressions are found for evaluating the integrals of motion of these systems.

1. INTRODUCTION

Many problems in the general theory of relativity (GTR) are solved under the assumption that the gravitational field, together with the fields of matter sources, is superposed on a given space-time. The concept of a flat background is used for the study of such problems as the propagation of weak gravitational waves, the quantization of a weak gravitational field, and the interpretation of the solutions of the equations of the GTR. Curved background universes, such as cosmological solutions and the geometry of black holes, are used to study vacuum polarization and the propagation and amplification of disturbances, or to investigate particle-creation effects.

To obtain rigorous results, a complete and exact formulation of the theory of a gravitational field superposed on an external background is necessary. The Hamiltonian formulation, which opens up the path to a quantum version of the theory, is especially desirable. In the present paper such a Hamiltonian theory is constructed. More precisely, we give a new Hamiltonian description of the GTR, possessing a number of advantages to be discussed below.

Studies of gravitation in a Hamiltonian description have been carried out over many years, from the pioneering papers of Dirac[1] and Arnowitt, Deser, and Misner (ADM)[2] up to the present time[3–5] (see also the recent paper Ref. 21). The reviews in Refs. 6–8 and the numerous references in them give an idea of the problem, difficulties, and achievements of the Hamiltonian approach to the GTR over the period during which this approach has been taken.

The difference between the theory constructed here and the standard approach is that here the dynamical elements are not the components of the metric tensor $g_{\mu\nu}$, but the field $h^{\mu\nu}$ specified in a fixed space-time with metric $\gamma_{\mu\nu}$. In this approach the gravitational field is treated equally with all the other dynamical fields. This makes it possible to compare an Einsteinian gravitational field with other physical fields and to look at the GTR from a new point of view.

First we describe briefly the $(3+1)$-splitting of an arbitrary space-time and of the reduction of four-covariant field theories to Hamiltonian form. We shall give the standard (geometrical) formulation of the GTR in the Hamiltonian description, in which the metric $g_{\mu\nu}$ is a dynamical field. The main result is formulated in Secs. 3 and 4, where the Hamiltonian interpretation of the GTR as a theory of dynamical fields on a fixed background is given, and the gauge freedoms of the theory are discussed.

In the proposed approach, three-covariant expressions are obtained for the conserved quantities of arbitrary physical systems. In relation to this, we consider asymptotically vanishing gravitational fields, and discuss the advantages of our approach and its differences from the standard approach. The question of the possibility of a theoretical description of the quantum creation of the universe has now been posed (see, e.g., Ref. 9). For this reason, the energy characteristics of cosmological models are important. The proposed formulation of the GTR makes it possible to determine these characteristics. For a closed Friedman universe the present approach implies that all integrals of motion are equal to zero (see also Refs. 10 and 11).

In the paper we use the space-time signature $-+++$.

2. GENERAL PRINCIPLES OF THE CONSTRUCTION OF THE HAMILTONIAN THEORY IN A FIXED AND A DYNAMICAL SPACE-TIME

To go over from the Lagrangian formulation of a field theory to the Hamiltonian formulation one usually uses a $(3+1)$-splitting of space-time. Here, space-time is interpreted as a set of nonintersecting spacelike hypersurfaces S_t, such that to each value of the continuously varying parameter t there corresponds a single surface S_t.

In this paper we take as the parameter t the time coordinate, labeled by the index 0, $x^0 = ct$. The space coordinates on the $t = \text{const}$ slices will be the space coordinates of the space-time, and have lower-case Latin indices. Greek indices are used for the space coordinates and the time coordinate. With these assumptions the four-metric $g_{\alpha\beta}$ can be represented in the form[12]

$$g_{\alpha\beta} = \left\| \begin{matrix} N_c N^c - N^2 & N_a \\ N_b & g_{ab} \end{matrix} \right\|, \tag{2.1}$$

where g_{ab} is the three-metric on the surfaces S_t, and the lapse function $N = (-g^{00})^{-1/2}$ and shift vector $N^a = -g^{0a}/g^{00}$ determine the displacement of the section $S_{t+\Delta t}$ relative to the section S_t as we go from t to $t+\Delta t$ for small Δt.

Next, at each point on the sections S_t we define a basis

$$\{n^\alpha, e_a{}^\alpha\}, \tag{2.2}$$

where n^α is a unit timelike vector field ($n^\alpha = \{1/N, -N^a/$

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$N\}$) orthogonal to the S_t slices, and the vector fields e_a^α are tangent to the sections S_t ($n^\alpha e_{\alpha a} = 0$) and form a coordinate basis on them.

If physical fields are specified in the space-time, then each four-covariant quantity $\Phi_{\{\alpha\}}$ ($\{\alpha\}$ is a four-dimensional collective index) can be written in the coordinates of the basis (2.2) as $\Phi_{\{A\}}$, where the set of indices corresponding to the basis (2.2) is denoted by an upper-case Latin letter. In this way the quantity $\Phi_{\{\alpha\}}$ is represented in the form of a set of three-covariant fields. For example, the four-vector A^α is decomposed into the three-scalar $A^\perp = A^\alpha n_\alpha$ and the three-vector $A^a = A^\alpha e_\alpha^a$

Thus, the Lagrangian of the four-covariant theory is reduced to a three-covariant form and one goes over to the Hamiltonian formulation of the theory as a theory of three-covariant fields defined on the surfaces S_t. A detailed description of this procedure can be found in Refs. 13, and also in the review Ref. 6.

Hamiltonian theories constructed using a $(3+1)$ space-time splitting and physical fields differ substantially depending on whether the space-time metric is regarded as a dynamical field or not.

We shall consider first the Hamiltonian theory of four-covariant fields $\varphi\{\alpha\}$, specified on a fixed background, i.e., with the assumption that the metric is not a dynamical field. The components $\varphi_{\{A\}}$ are defined as generalized coordinates. The generalized momenta $\pi^{\{A\}}$ conjugate to them are defined in the standard way (see, e.g., Ref. 6). Then the Hamiltonian action for the fields $\varphi_{\{\alpha\}}$ is

$$S = \int dt \int_{S_t} d^3x \{\pi^{\{A\}}\dot{\varphi}_{\{A\}} - NH^\varphi - N^a H_a{}^\varphi - \mathrm{div} + \dot{A}\}, \quad (2.3)$$

where div denotes the three-dimensional divergence, and the dot denotes a total time derivative.

Now suppose, as in the ordinary Hamiltonian formulation of the GTR, that the four-metric $g_{\alpha\beta}$ is a dynamical field. Then, as well as the components of the fields $\varphi_{\{A\}}$ of the matter sources, the components of the three-metric g_{ab} ar also assigned the meaning of generalized coordinates. After determination of the generalized momenta π^{ab} conjugate to them, the Hamiltonian action for the GTR can be written in the general form

$$S = \int dt \int_{S_t} d^3x \{\pi^{ab}\dot{g}_{ab} + \pi^{\{A\}}\dot{\varphi}_{\{A\}} - NH^{g+\varphi} - N^a H_a^{g+\varphi} - \mathrm{div} + \dot{A}\}. \quad (2.4)$$

In the following, the terms $\dot{A}$ will be ignored, since taking them into account leads only to canonical transformations. The structure of div is important, but will interest us only in Sec. 5.

The difference between the theories with the actions (2.3) and (2.4) is as follows. In both cases the functions N and N^a are components of the four-metric $g_{\alpha\beta}$ (2.1). However, since $g_{\alpha\beta}$ in the latter case is a dynamical field, and dynamical equations for the components N and N^a do not exist, these functions will be undetermined Lagrange multipliers. It is necessary to vary the action (2.4) with respect to them, and this leads to constraints. On the other hand, in a field theory on a fixed background the four-metric $g_{\alpha\beta}$ is specified, and so the functions N and N^a are not Lagrange multipliers in the action (2.3), and there is no variation with respect to them. Consequently, the GTR in the Hamiltonian interpretation always has the constraints

$$H^{g+\varphi}=0, \quad H_a^{g+\varphi}=0,$$

while the theory with the action (2.3) has no such constraints.

We have discussed the constraints that arise as a result of treating the metric as a dynamical field. Of course, constraints can also arise for another reason, namely, because of the structure possessed by the function H^φ or $H^{g+\varphi}$ on account of the internal symmetries of the theory. For example, the constraints in the theory of the electromagnetic field on a fixed background (i.e., for specified N and N^a) are manifested in the fact that H^φ consists of a sum of two terms, one of which has a Lagrange multiplier. The role of this multiplier is played by a four-potential component for which there are no dynamical equations. Our aim in Sec. 3 is to bring the theory of the gravitational field to approximately the same form.

3. HAMILTONIAN FORMULATION OF THE GTR ON A FIXED BACKGROUND

The theory of the gravitational field on an arbitrary background (the field interpretation of the GTR) in the Lagrangian formulation has been developed previously.[14] Here we shall represent this theory in Hamiltonian form for the simple case when the background space-time is Ricci-flat, i.e., $R_{\mu\nu}^{(0)} = 0$, and (in particular) flat, i.e., $R_{\alpha\mu\beta\nu}^{(0)} = 0$. (The results obtained can also be extended to more-general cases.)

The action for such a theory has the form (Ref. 14)[1)]

$$S = -\frac{1}{2\varkappa c}\int d^4x\, L^g + \frac{1}{c}\int d^4x\, L^m, \quad (3.1)$$

$$L^g = (-\gamma)^{1/2}\Big[h^{\mu\nu}{}_{;\alpha}\Big(-K_{\mu\nu}{}^\alpha + \frac{1}{2}\delta_\mu{}^\alpha K_{\rho\nu}{}^\rho + \frac{1}{2}\delta_\nu{}^\alpha K_{\rho\mu}{}^\rho\Big) + (h^{\mu\nu}+\gamma^{\mu\nu})(K_{\mu\nu}{}^\alpha K_{\rho\alpha}{}^\rho - K_{\mu\alpha}{}^\beta K_{\nu\beta}{}^\alpha)\Big], \quad (3.2)$$

where $\varkappa = 8\pi G/c^4$, a semicolon indicates a covariant (with respect to the four-metric $\gamma_{\mu\nu}$) derivative, and $\gamma = \det\gamma_{\mu\nu}$. A comma in the following formulas will indicate a partial derivative. The action (3.1) is written in the first-order formalism, in which the components of the symmetric tensor field $h^{\mu\nu}$ [in fact, $(-\gamma)^{1/2}h^{\mu\nu}$ is regarded as the dynamical variable] and of the tensor field $K_{\mu\nu}^\alpha$, which is symmetric in its subscripts, are treated as independent dynamical gravitational variables. Here the second-order formalism will also be used; for this it is necessary to regard the $K_{\mu\nu}^\alpha$ as known functions of $h^{\mu\nu}$ and $h^{\mu\nu}_{,\alpha}$, determined from the first-order equations obtained by variation with respect to $K_{\mu\nu}^\alpha$.

From the universality of the coupling of the gravitational field with other fields it follows that the Lagrangian of the matter sources $\varphi_{\{\alpha\}}$ has the form (for more details, see Ref. 14)

$$L^m = L^m[(-\gamma)^{1/2}(h^{\mu\nu}+\gamma^{\mu\nu});\ \varphi_{\{\alpha\}};\ \varphi_{\{\alpha\},\beta}]. \quad (3.3)$$

For simplicity, we confine ourselves to the case when L^m does not depend on the derivatives $[(-\gamma)^{1/2}(h^{\mu\nu}+\gamma^{\mu\nu})]_{,\beta}$.

Now, as was done in Sec. 2, in the background space-time with four-metric $\gamma_{\mu\nu}$, we select slices S_t and define for them the three-metric γ_{ab} ($\lambda \equiv \det\gamma_{ab}$), the lapse function N, the shift N^a, and the basis (2.2). We then project the quantities appearing in the Lagrangian of the action (3.1) onto the surfaces S_t.

After rather cumbersome calculations, using the technique of Refs. 6, 7, and 13, we bring L^g to the three-covariant form

$$
\begin{aligned}
-(1/2\varkappa)L^g = {} & \lambda^{1/2}K_{ab}\{2h^{\perp a}(\lambda^{1/2}h^{\perp b})^{\cdot} - (h^{\perp\perp}+\gamma^{\perp\perp}) \\
& \times[\lambda^{1/2}(h^{ab}+\gamma^{ab})]^{\cdot} - (h^{ab}+\gamma^{ab})[\lambda^{1/2}(h^{\perp\perp}+\gamma^{\perp\perp})]^{\cdot}\} \\
& -N(H^h+h^{\perp a}H_a{}^h)/(h^{\perp\perp}+\gamma^{\perp\perp}) - N^a H_a{}^h - \mathrm{div},
\end{aligned}
\tag{3.4}
$$

$$
\lambda^{1/2}H^h \equiv -2\varkappa q^{ab}q^{cd}(K_{ab}K_{cd}-K_{ac}K_{bd}) - (1/2\varkappa)q^{ab}R_{ab}{}^{(3)}, \tag{3.5}
$$

$$
\begin{aligned}
H_a{}^h &\equiv q^{bc}{}_{,a}K_{bc} + 2(q^{bc}K_{ab})_{,c} - 2(q^{bc}K_{bc})_{,a}, \\
q^{ab} &\equiv \lambda[h^{\perp a}h^{\perp b} - (h^{\perp\perp}+\gamma^{\perp\perp})(h^{ab}+\gamma^{ab})], \\
K_{ab} &\equiv (K_{ab}{}^{\perp}+C_{ab})/2\varkappa\lambda^{1/2}(h^{\perp\perp}+\gamma^{\perp\perp}).
\end{aligned}
\tag{3.6}
$$

Here $R_{ab}^{(3)}$ is a three-tensor, constructed as a Ricci tensor from q^{ab} (the "metric" density of weight $+2$), and C_{ab} is the external curvature of the surfaces S_t. We assume that the Lagrangian L^m, like L^g, has also been reduced to a three-covariant form.

Taking $\lambda^{1/2}h^{AB}$ and $\varphi_{\{A\}}$ as generalized coordinates, we determine the generalized momenta conjugate to them:

$$
P_{AB} \equiv -(1/2\varkappa)\partial L^g/\partial(\lambda^{1/2}h^{AB})^{\cdot}, \tag{3.7}
$$

$$
\pi^{\{A\}} \equiv \partial L^m/\partial\dot{\varphi}_{\{A\}}. \tag{3.8}
$$

These systems of equations are used to determine the generalized velocities $(\lambda^{1/2}h^{AB})^{\cdot}$ and $\dot{\varphi}_{\{A\}}$ and to replace them by the generalized momenta in the definition of the Hamiltonian theory. However, the system (3.7), represented in the second-order formalism, cannot be solved for $(\lambda^{1/2}h^{AB})^{\cdot}$. Consequently, there are restrictions on the canonical variables P_{AB} and $\lambda^{1/2}h^{AB}$—first class constraints (Ref. 1, p. 326). The number of such constraints—four, i.e., to the difference between the dimensionality of the field configuration space $\lambda^{1/2}h^{AB}$ and the rank of the kinetic matrix[6]

$$
\partial^2 L^g/\partial(\lambda^{1/2}h^{AB})^{\cdot}\partial(\lambda^{1/2}h^{CD})^{\cdot}.
$$

From the system (3.7), written in the explicit form

$$
\begin{aligned}
P_{\perp\perp} &= -\lambda^{1/2}(h^{ab}+\gamma^{ab})K_{ab}, \\
P_{\perp a} &= 2\lambda^{1/2}h^{\perp b}K_{ab}, \\
P_{ab} &= -\lambda^{1/2}(h^{\perp\perp}+\gamma^{\perp\perp})K_{ab},
\end{aligned}
\tag{3.9}
$$

these constraints are easily obtained:

$$
\Phi_{\perp} \equiv P_{\perp\perp} - \frac{h^{ab}+\gamma^{ab}}{h^{\perp\perp}+\gamma^{\perp\perp}}P_{ab} = 0, \quad \Phi_a \equiv P_{\perp a} + \frac{2h^{\perp b}}{h^{\perp\perp}+\gamma^{\perp\perp}}P_{ab} = 0. \tag{3.10}
$$

To simplify the account, we assume that the system (3.8) can be solved for $\dot{\varphi}_{\{A\}}$.

We now construct the complete Hamiltonian of the system (3.1). For this, in the standard defining relation

$$
H = \int_{S_t} d^3x\left\{P_{ab}(\lambda^{1/2}h^{ab})^{\cdot} + P_{\perp A}(\lambda^{1/2}h^{\perp A})^{\cdot} + \pi^{\{A\}}\dot{\varphi}_{\{A\}} - \left(-\frac{1}{2\varkappa}L^g + L^m\right)\right\} \tag{3.11}
$$

[where we use the three-covariant expression for L^g (3.4) and L^m] we replace the generalized velocities by the generalized momenta. It would appear that there remains some arbitrariness in the solution of the system (3.9) for the functions K_{ab} and in their replacement by the generalized momenta P_{AB} in the Hamiltonian (3.11). However, because the constraints (3.10) are fulfilled, any choice of K_{ab} will do.

Next, in order to obtain a Hamiltonian with independent canonical variables,[15] we add to H (3.11) the first-class constraints Φ_A (3.10), multiplied by undertermined Lagrange multipliers $u^A(t,x)$. After this the Hamiltonian of the system (3.1) takes the form

$$
H = \int_{S_t} d^3x\left\{N\left[\frac{H^{h+m}+h^{\perp a}H_a^{h+m}}{h^{\perp\perp}+\gamma^{\perp\perp}} + \Psi + u^A\Phi_A\right] + N^a(H_a^{h+m} + \Psi_a) + \mathrm{div}\right\}, \tag{3.12}
$$

where

$$
\begin{aligned}
& H^{h+m} \equiv H^h + H^m, \qquad H_a^{h+m} \equiv H_a{}^h + H_a{}^m, \\
& \Psi \equiv \lambda^{1/2}(P_m{}^m C_n{}^n - 2P_{mn}C^{mn} - P_{\perp\perp}C_m{}^m), \\
& \Psi_a \equiv \lambda^{1/2}(P^m_{m\,|a} - 2P^m_{a\,|m} - P_{\perp\perp|a}),
\end{aligned}
$$

in which a vertical line indicates a derivative is covariant with respect to the three-metric γ_{ab}. For definiteness we assume that H^h and H_a^h in (3.12) have been obtained by substitution into Eqs. (3.5) of functions $K_{ab} = K_{ab}(P,h)$ found just from the last equation of the system (3.9).

In the matter part of the Hamiltonian (3.12) the function H^m and H_a^m depend on the canonical variables $\pi^{\{A\}}$, $\varphi_{\{A\}}$ and their spatial derivatives. Owing to the choice of L^m in the form (3.3) only H^m depends on γ^{AB} and $\lambda^{1/2}h^{AB}$, and in such a way that

$$
H^m = H^m[\pi^{\{A\}}, \varphi_{\{A\}}, q^{ab}(\gamma^{AB}, h^{AB})], \tag{3.13}
$$

where q^{ab} is defined in (3.6).

The equations of motion for any dynamical variables $g(P,h,\pi,\varphi,t)$, including each canonical variable in the Hamiltonian (3.12), are written using standard Poisson brackets[6]: $\dot{g} = \partial g/\partial t + \{g,H\}$. The field theory is consistent if the equations of motion preserve the constraints. Therefore, fulfillment of the relations $\dot{\Phi} = 0$ is necessary, which leads to the second-class constraints (Ref. 1, p. 326)

$$
H^{h+m}(P,h,\pi,\varphi) = 0, \qquad H_a^{h+m}(P,h,\pi,\varphi) = 0. \tag{3.14}
$$

For consistency it is necessary that these constraints too be conserved, i.e., $\dot{H}^{h+m} = 0$ and $\dot{H}_a^{h+m} = 0$. But these conditions do not give any new restrictions on the variables $\lambda^{1/2}h^{AB}$ and P_{ab}, and the functions u^A remain undetermined.

As a result, for the theory with Hamiltonian (3.12) we have eight Dirac first-class constraints[15] (3.10) and (3.14). This follows from the fact that

$$
\{\Phi_A, \Phi_B\} = \{\Phi_A, H^{h+m}\} = \{\Phi_A, H_a^{h+m}\} = 0,
$$

and the Poisson brackets of the constraints (3.14) in all combinations vanish by virtue of (3.14).

The Hamiltonian (3.12) contains arbitrary functions, the undetermined Lagrange multipliers u^A, which suggests the existence of gauge (nonphysical) degrees of freedom in the theory. All the eight first-class constraints displayed are effective generators of gauge transformations. Each constraint and each effective generator can decrease the number of degrees of freedom by a factor 1/2 (see Ref. 6). Since the dimensionality of the field configuration space is $10 + n$ (n is the number of degrees of freedom of the sources), the number of physical degrees of freedom is equal to

$10+n-(1/2)8-(1/2)8=2+n$ (this follows from the previously introduced restrictions that all degrees of freedom of the sources be physical).

We shall confine ourselves to eliminating only four gauge degrees of freedom. For this we change to new variables by means of a canonical transformation:

$$q^{ab}=\lambda[h^{\perp a}h^{\perp b}-(h^{\perp\perp}+\gamma^{\perp\perp})(h^{ab}+\gamma^{ab})],\quad q^A=h^{\perp A}/(h^{\perp\perp}+\gamma^{\perp\perp}),$$
$$K_{ab}=-P_{ab}/\lambda^{1/2}(h^{\perp\perp}+\gamma^{\perp\perp}),\quad K_A=\lambda^{1/2}\Phi_A(P,h). \tag{3.15}$$

Then, in the Hamiltonian (3.12), the new constraints will be $K_A=0$. In this case the generalized momenta K_A can be eliminated entirely from the analysis, and the generalized coordinates q^A conjugate to them can be regarded as undetermined Lagrange multipliers. After this, the action with the Hamiltonian (3.12) is written as

$$S=\int dt\int_{\Sigma_t} d^3x\{K_{ab}\dot{q}^{ab}+\pi^{\{A\}}\dot{\varphi}_{\{A\}}$$
$$-N[(-1+q^\perp)H^{q+m}+q^aH_a^{q+m}]-N^aH_a^{q+m}-\text{div}\},$$
$$H^{q+m}\equiv H^q+H^m,\quad H_a^{q+m}\equiv H_a{}^q+H_a{}^m. \tag{3.16}$$

Here the dependence of H^q and H_a^q on the canonical variables q^{ab} and K_{ab} (3.15) coincides with the dependence of H^h and H_a^h (3.5) on the functions (3.6). But the dependence of H^m and H_a^m on the gravitational variables is determined entirely by the relation (3.13). Thus, H^{q+m} and H_a^{q+m} in the action (3.16) do not depend on q^A. Consequently, four first-class contraints hold:

$$H^{q+m}(K,q,\pi,\varphi)=0,\quad H_a^{q+m}(K,q,\pi,\varphi)=0. \tag{3.17}$$

No other constraints follow from the conditions for the consistency of (3.17).

A theory with the action (3.16) can also be obtained starting from the standard Hamiltonian formulation of the GTR with the action (2.4). For this it is necessary to decompose the four-metric $g_{\alpha\beta}$ into a background part $\gamma_{\mu\nu}$ and a dynamical part $h^{\mu\nu}$ by means of the relation

$$(-g)^{1/2}g^{\mu\nu}=(-\gamma)^{1/2}(\gamma^{\mu\nu}+h^{\mu\nu}) \tag{3.18}$$

and redefine the procedure of the $(3+1)$-splitting for the background space-time with metric $\gamma_{\mu\nu}$. [We recall that by virtue of the identifications (3.18), and theory developed in Ref. 14 is equivalent to the GTR in the usual formulation. On the subject of the relation between the "field" and "geometrical" formulation of the GTR, see also Ref. 22.]

Thus, the action for the gravitational field has been brought to a form different from that the standard Hamiltonian action (2.4) in the GTR, but with the same physical content. The coefficient of N in the action (3.16) has the meaning of H^φ in the action (2.3), and the Lagrange mutlipliers q^A appear in the function H^φ but are not coefficients of $H^{g+\varphi}$ and $H_a^{g+\varphi}$. Then the constraints (3.17), as, e.g., in the theory of the electrogmagnetic field on a fixed background, should follow from the internal (gauge) symmetries of the theory. We shall show this in the following section.

4. GAUGE INVARIANCE AND CONSTRAINTS

The theory with the action (3.1) is invariant under the gauge transformations[14]

$$(-\gamma)^{1/2}h_{(2)}^{\mu\nu}=(-\gamma)^{1/2}h_{(1)}^{\mu\nu}+\sum_{n=1}^{\infty}\frac{1}{n!}L_\xi{}^n[(-\gamma)^{1/2}(h_{(1)}^{\mu\nu}+\gamma^{\mu\nu})],$$
$$\varphi_{(2)}^{\{\alpha\}}=\varphi_{(1)}^{\{\alpha\}}+\sum_{n=1}^{\infty}\frac{1}{n!}L_\xi{}^n\varphi_{(1)}^{\{\alpha\}}. \tag{4.1}$$

Here L_ξ^n is the Lie derivative along an arbitrary vector field ξ^α, taken n times, i.e., the series in the transformations (4.1) can be represented by means of the operator exponential

$$\sum_{n=1}^{\infty}\frac{1}{n!}L_\xi{}^n\equiv\exp L_\xi-1.$$

First we clarify how this invariance is connected with the covariance in the usual formulation of the GTR. For the two variables $h_{(1)}$ and $h_{(2)}$ connected by the relation (4.1) we make the identification (3.18). Then, in the metric density $[(-g)^{1/2}g^{\mu\nu}]_{(2)}$ we replace the coordinates x^α by x'^α, satisfying the transformation

$$x'^\alpha=x^\alpha+\xi^\alpha(x)+\frac{1}{2!}\xi^\beta\xi^\alpha{}_{,\beta}+\frac{1}{3!}\xi^\rho(\xi^\beta\xi^\alpha{}_{,\beta})_{,\rho}+\dots.$$

After this it is found that $[(-g)^{1/2}g^{\mu\nu}]_{(1)}$ and $[(-g)^{1/2}g^{\mu\nu}]_{(2)}$ are the same metric density, but written in two different systems of coordiantes x^α and x'^α. Having established this fact, using the identifications (3.18) we can easily show that the invariance of the geometrical formulation of the GTR under coordinate transformations induces gauge invariance in the field interpretation of the GTR, and vice versa.

We turn now to the question of the origin of the constraints (3.17). The existence of first-class constraints in field theory corresponds to invariance of the action under particular transformations, the parameters of which are functions of space-time (see, e.g., Ref. 3a). In our case the invariance of the action (3.1) under the transformations (4.1) is associated with four strong identities (for more detail, see Ref. 14). Using the definitions (3.7) and the technique of Ref. 16, we can write them in the form

$$P_{\perp\perp}-\frac{h^{ab}+\gamma^{ab}}{h^{\perp\perp}+\gamma^{\perp\perp}}P_{ab}\equiv f_\perp(h),\quad P_{\perp a}+\frac{2h^{\perp b}}{h^{\perp\perp}+\gamma^{\perp\perp}}P_{ab}\equiv f_a(h). \tag{4.2}$$

The derivation of these identities is such that the structure of the functions $f_A(h)$ is unknown. We can assert, however, that $f_A(h)$ depends only on $\lambda^{1/2}h^{AB}$ and does not depend on P_{AB}, $(\lambda^{1/2}h^{AB})^{\cdot}$, or $\dot{P}_{AB}$. Thus, the identities (4.2), obtained without the use of equations of motion, will be first-class constraints. It has been shown that all the first-class constraints (3.10) have already been found: They are the left-hand sides of (4.2). This means that the right-hand sides of (4.2) are equal to zero, since otherwise, in addition to (3.10), there would exist the constraints $f_A(h)=0$. Thus, the constraints (3.10), and, consequently, the second-class constraints (3.14) [which are the same as (3.17)], correspond to internal symmetries of the theory of the gravitational field on a fixed background.

For an arbitrary four-covariant field $\varphi_{\{\alpha\}}$ propagating in a fixed space-time, it is not necessary at all that the coordinate invariance give rise to any constraints when we go over to the Hamiltonian form. Therefore, it is interesting to note that the constraints (3.10) follow also from the four-coordinate invariance of the theory with the action (3.1).

Finally, we shall discuss briefly the question of how the gauge freedoms manifest themselves in the Hamiltonian description of the theory. The Hamiltonian H of the action (3.16) contains the arbitrary functions q^A. Therefore, the equations of motion $\dot{g} = \{g,H\}$ for any particular dynamical variable g describe the same evolution for all admissible choices of the functions q^A, if the initial data on the starting surface S_t are the same. We shall say that a choice of definite values of q^A fixes the gauge freedoms. Then changing from one set of functions $q^A_{(1)}$ to another set of functions $q^A_{(2)}$ signifies gauge transformations (in the Dirac definition,[15] contact transformations). We shall construct such a transformation for the variable g.

We shall consider the change of g that occurs in passing from S_t to $S_{t+\Delta t}$ for different choices of the functions q^A and the same initial data on S_t. If, in H, functions $q^A_{(1)}$ are fixed, then g changes by the amount $\dot{g}\Delta t = \{g,H_{(1)}\}\Delta t$ while if functions $q^A_{(2)}$ are fixed, g changes by $\dot{g}\Delta t = \{g,H_{(2)}\}\Delta t$. Since the physical situation on the surface $S_{t+\Delta t}$ will be the same in both cases, the transformation

$$g \to g + \{g,\ (H_{(2)} - H_{(1)})\}\Delta t, \tag{4.3}$$

is obviously a purely gauge transformation. Its generators are the constraints (3.17), and the quantities $N(q^A_{(2)} - q^A_{(1)})\Delta t$ serve as the gauge functions.

We note that when we go over from the Lagrangian formulation of the theory to the Hamiltonian formulation the linear part of the transformation (4.1) goes over into the transformation (4.3). And, by successive application of the transformation (4.3), we can obtain the Hamiltonian version of the complete transformation (4.1).

5. CONSERVED INTEGRAL QUANTITIES IN THE FIELD INTERPRETATION OF THE GTR

With the development of the Hamiltonian formulation of the GTR, many authors,[2,16–18] using the advantages of this approach, have studied questions concerning the determination of the energy and other integrals of motion in an asymptotically flat space-time. Rigorous mathematical forms for these quantities were obtained in Ref. 18.

The field formulation of the GTR in the Hamiltonian representation makes it possible to give a definition of the conserved quantities from other standpoints. Here, as in any field theory, integrals of motion are defined if the background space-time possesses symmetries, and are conserved if the dynamical fields fall off sufficiently rapidly at spatial infinity. On the other hand, the approach developed in Sec. 3 is three-covariant, i.e., invariant with respect to the choice of the spatial coordinates on the surfaces S_t. As a consequence of this, the integrals of motion should also be three-covariant quantities. For example, it will be shown that the standard (for field theory) definition of the Hamiltonian action in the field formulation of the GTR leads automatically to three-covariant conserved integrals for an asymptotically vanishing gravitational field.

We shall define an asymptotically vanishing gravitational field in the same way as in Ref. 14. We assume that the background space-time is flat, i.e., $R^{(0)\alpha}_{\mu\beta\nu} = 0$. We then require that the gravitational variables in the Lorentz coordinates t, x, y, z satisfy in the limit $r \to \infty$ the relations

$$h^{\mu\nu} = O\left(\frac{1}{r}\right), \quad h^{\mu\nu}{}_{,\alpha} = O\left(\frac{1}{r^2}\right), \quad K_{\mu\nu}{}^{\alpha} = O\left(\frac{1}{r^2}\right), \tag{5.1}$$

where $r^2 \equiv x^2 + y^2 + z^2$. With these conditions the Lagrangian (3.2) $L^g = 0(1/r^4)$. We also assume that the gravitating matter is effectively localized, i.e., L^m is also of order $0(1/r^4)$. Thus, the system does not radiate. This definition coincides with the usual definition of an asymptotically flat space-time in the GTR. This is easily seen by making the identification (3.18) in the field formulation of the GTR.

Next, for a system of fields defined in this way, we fix the choice of the slices S_t: In the Hamiltonian of the action (3.16) we set $N = 1$ and $N^a = 0$. Then the generator of the passage from one surface S_t to another (i.e., the Hamiltonian of the theory) takes the form

$$H = \int_{S_t} d^3x\{(-1+q^\perp)H^{q+m} + q^a H_a^{q+m} + \mathrm{div}\}. \tag{5.2}$$

The definition of the canonical variables q^{ab}, K_{ab}, and the Lagrange multipliers q^A in (3.15), with allowance for the conditions (5.1) on the potentials of the gravitational field, leads to the result that, as $r \to \infty$ in (5.2) in cartesian coordianates on the surfaces S_t,

$$q^{ab} = -1 + O\left(\frac{1}{r}\right), \quad \dot{q}^{ab} \sim q^{ab}{}_{,c} \sim K_{ab} = O\left(\frac{1}{r^2}\right), \tag{5.3}$$

$$q^A = O\left(\frac{1}{r}\right), \quad \dot{q}^A \sim q^A{}_{,b} = O\left(\frac{1}{r^2}\right). \tag{5.4}$$

The divergence $\mathrm{div} \equiv B^a{}_{|a}$ in the Hamiltonian (5.2) contains only gravitational variables:

$$B^a = -\frac{1}{2\varkappa}\left\{\frac{1}{\lambda^{1/2}}(-1+q^\perp)q^{ab}{}_{|b} + \frac{\lambda^{1/2}}{-1+q^\perp}[(q^a q^b{}_{|b} - q^b q^a{}_{|b})(-1+q^\perp) + q^a q^\perp]\right\} + 2K_{bc}(q^b q^{ac} - q^a q^{bc}). \tag{5.5}$$

This expression has been obtained without contributions on the asymptotic behavior of the variables and can be used for arbitrary physical systems. We note that, with the conditions (5.3) and (5.4), the surface integral in (5.2) has a finite value.

The Hamiltonian defined in (5.2) is the generator of translations along a timelike Killing vector in Minkowski space. Consequently, for a system of fields with the asymptotic behavior (5.3), (5.4), on solutions of the equations of motion [and hence on the constraints (3.17)] the numerical value of H gives the total conserved energy P^0 of the system:

$$P^0 = -H = -\int_{S_t} d^3x\, B^a{}_{|a} = -\lim_{r\to\infty} \oint_{\partial S_t} dS_a B^a. \tag{5.6}$$

We note that, as we should expect, the value of P^0 does not depend on the coordinate system chosen on S_t.

Next, the system of the asymptotically vanishing gravitational field is defined in Minkowski space, and therefore, in addition to (5.6), there exist a further nine integrals of motion. In the integraion over the volume, they all have three-covariant integrands. After use of the constraints, all the integrals of motion are reduced to three-covariant surface integrals.

In order that the total angular and Lorentz momenta of the asymptotically vanishing gravitational field have finite values (and also in order to derive the conservation laws for these integrals), it is necessary to make the conditions (5.3) more precise—to impose restrictions on the even and odd parts of the canonical variables (see Ref. 18).

We now compare the proposed approach with the usual one, and discuss certain distinctive features.

For an asymptotically flat space-time Regge and Teitelboim[18] proved that it is necessary to include the surface integral in the Hamiltonian of the unreduced theory. This integral determines the total energy of the system. It coincides with the energy introduced by ADM but is not a three-covariant quantity. This definition requires the use, at spatial infinity, of an asymptotically Lorentz system of coordinates.[2,7,18,19]

It is true that, after the introduction of the auxiliary structure (the metric tensor of Minkowski space), it is possible to replace the integrand by a three-covariant integrand without changing the value of the ADM energy.[6,20] In the approach developed in this paper, three-covariant expressions arise naturally and inevitably, and not as a result of constructions in which the conserved integrals are brought to covariant form only after they have been determined in an asymptotically Lorentz system of coordinates. In addition, in our approach the gauge freedoms and the constraints that stem from them are exhibited at once, while the constructions in Refs. 6 and 20 do not have this feature.

Next, if in the standard formulation of the GTR we start from the Hilbert Lagrangian $(-g)^{1/2}R$, the Hamiltonian defined in the standard way:

$$H=\int_{S_t} d^3x\left\{\pi^{ab}\dot{g}_{ab}+\pi^{(A)}\dot{\varphi}_{(A)}-\left[-\frac{1}{2\varkappa}(-g)^{1/2}R+L^m\right]\right\}, \quad (5.7)$$

with all surface terms conserved and constraints fulfilled, does not coincide with the value of the ADM energy. But using the truncated Einstein Lagrangian to define a Hamiltonian in the manner of (5.7) gives an energy integral that coincides with the ADM integral but is not three-covariant.

In comparison with this, the field formulation of the GTR has advantages. The standard definition (3.11) of the Hamiltonain yields for the energy P^0 a definition (5.6) that is three-covariant, automatically satisfies the variational principle of Regge and Teitelboim,[18] and coincides with the value of the ADM energy. We note that in the definition (5.6) of the energy for an asymptotically vanishing gravitational field we can make a simplification. In place of the cumbersome integrand (5.5) in P^0 we can use, e.g., $(1/2\varkappa\lambda^{1/2})q^{ab}_{|a|b}$. This replacement does not destroy any of the advantages mentioned.

In Ref. 14 it was shown that the symmetric total energy-momentum tensor $T^{\rm tot}_{\mu\nu}$ for a system with the action (3.1) is not invariant under the gauge transformations (4.1). However, the total conserved integrals of motion determined by means of $T^{\rm tot}_{\mu\nu}$ do not change if the behavior of the gauge functions ξ^α in (4.1) ensures invariance of the action.

In the Hamiltonian case the situation is analogous. Any gauge transformation of the canonical variables is constructed by means of the relation (4.3). But the asymptotic behavior of the Lagrange multipliers q^A (5.4) ensures conservation of all the integrals of motion under such transformations for a system with an asymptotically vanishing gravitational field.

In Ref. 11 a Friedmann closed universe is described in terms of the field interpretation of the GTR in the Lagrangian form. However, in the development of the quantum version of the theory the determination of the integrals of motion in the Hamiltonian formulation is especially important. The present approach makes it possible to determine the integrals of motion not only for isolated systems [in which the gravitational-field potentials $h^{\mu\nu}$ tend to zero as $r\to\infty$, as in (5.1)], but also, in particular, for a configuration of field that represents a closed universe [for which $h^{\mu\nu}$ tend to constant values as $r\to\infty$ (Ref. 11)]. Thus, for a closed Friedmann universe, the integrals of motion calculated using the formulas of the Hamiltonian GTR formulation developed in this paper are equal to zero. This coincides with the results obtained earlier in the Lagrangian description.

[1)] The Lagrangian (3.2) differs by an exact four-divergence from the $L^\varkappa$ that was used in Ref. 14.

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